Bachelor topics SS2018

Onsager's conjecture and weak solutions of Euler's equation

▷ De Lellis, Camillo, and László Székelyhidi Jr. "Dissipative Euler Flows and Onsager's Conjecture." ArXiv e-print, May 16, 2012. http://arxiv.org/abs/1205.3626.

Onsager's conjecture states that energy conservation for weak solutions of Euler's equation for the motion of an inviscid perfect fluid

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{cases}$$

is linked with the Hölder regularity of the solutions. Namely if $|u(t,x) - u(t,y)| \lesssim |x-y|^\theta$ for $\theta > 1/3$ then energy is conserved and that there exists weak solutions with $\theta < 1/3$ which violates energy conservation. This issue is important in our understanding of weak solutions and in general of irregular solutions to fluid dynamical equations and also for discussion of turbulence. The above paper provide a partial confirmation of the negative side of Onsager's conjecture. Objective of the thesis is the investigation of weak solutions, conservation of energy with enough regularity and describe the construction of counterexamples given in the above paper.

Some results on the Yang–Mills quantum field theory

▷ Chatterjee, Sourav. "The Leading Term of the Yang–Mills Free Energy." *Journal of Functional Analysis* 271, no. 10 (November 15, 2016): 2944–3005. doi:10.1016/j.jfa.2016.04.032.

▷ http://www.claymath.org/millennium-problems/yang–mills-and-mass-gap

Yang-Mills theory is a generalization of Maxwell's theory of electromagnetism, in which the basic dynamical variable is a connection on a G-bundle over four-dimensional space-time. Its quantum version is the key ingredient in the Standard Model of the elementary particles and their interactions. Quantisation of Yang–Mills (QYM) theory is equivalent to the construction of a probability measure on the space of all connections where each configuration A is weighted according to the Yang-Mills action

$$\operatorname{Prob}(A) \propto e^{-S}$$

where

$$S = \frac{1}{4g^2} \int \operatorname{Tr} F \wedge *F,$$

and $F = \mathrm{dA} + A \wedge A$ is the curvature of the G-connection A. The understanding of this measure is still an open problem (and one of the Clay's Millennium Problems). The objective of the thesis is to reproduce the results contained in the paper above which give explicit formulas for the free energy F of the discrete versions of QYM partition function:

$$F = \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \log \int e^{-S_\Lambda} \mathrm{d}A.$$

Mathematical Theory of Deep Convolutional Neural Networks for Feature Extraction

▷ Wiatowski, Thomas, and Helmut Bölcskei. "A Mathematical Theory of Deep Convolutional Neural Networks for Feature Extraction." *ArXiv:1512.06293*, 2015. http://arxiv.org/abs/1512.06293.

Deep convolutional neural networks have led to breakthrough results in numerous practical machine learning tasks such as classification of images in the ImageNet data set, control-policy-learning to play Atari games or the board game Go, and image captioning. Many of these applications first perform feature extraction and then feed the results thereof into a trainable classifier. The mathematical analysis of deep convolutional neural networks for feature extraction was initiated by Mallat, 2012. Specifically, Mallat considered so-called scattering networks based on a wavelet transform followed by the modulus non-linearity in each network layer, and proved translation invariance (asymptotically in the wavelet scale parameter) and deformation stability of the corresponding feature extractor. This paper complements Mallat's results by developing a theory that encompasses general convolutional transforms, or in more technical parlance, general semi-discrete frames (including Weyl-Heisenberg filters, curvelets, shearlets, ridgelets, wavelets, and learned filters), general Lipschitz-continuous non-linearities (e.g., rectified linear units, shifted logistic sigmoids, hyperbolic tangents, and modulus functions), and general Lipschitz-continuous pooling operators emulating, e.g., sub-sampling and averaging. In addition, all of these elements can be different in different network layers. For the resulting feature extractor we prove a translation invariance result of vertical nature in the sense of the features becoming progressively more translation-invariant with increasing network depth, and we establish deformation sensitivity bounds that apply to signal classes such as, e.g., band-limited functions, cartoon functions, and Lipschitz functions.

Two-Dimensional Turbulence, Large Deviations and Jupiter's Red spot

▷ Boucher, Christopher, Richard S. Ellis, and Bruce Turkington. "Derivation of Maximum Entropy Principles in Two-Dimensional Turbulence via Large Deviations." *Journal of Statistical Physics* 98, no. 5–6 (March 1, 2000): 1235–78. https://doi.org/10.1023/A:1018671813486.

The continuum limit of lattice models arising in two-dimensional turbulence is analyzed by means of the theory of large deviations. In particular, the Miller–Robert continuum model of equilibrium states in an ideal fluid and a modification of that model due to Turkington are examined in a unified framework, and the maximum entropy principles that govern these models are rigorously derived by a new method. In this method, a doubly indexed, measure-valued random process is introduced to represent the coarse-grained vorticity field. The natural large deviation principle for this process is established and is then used to derive the equilibrium conditions satisfied by the most probable macrostates in the continuum models. The physical implications of these results are discussed, and some modeling issues of importance to the theory of long-lived, large-scale coherent vortices in turbulent flows are clarified.

Fast/slow diffusions, stochastic models and nonlinear PDEs

▷ Aronsson, Gunnar, Lawrence C. Evans, and Y. Wu. "Fast/slow diffusion and growing sandpiles." Journal of Differential Equations 131.2 (1996): 304-335.

▷ Evans, Lawrence C., and Fraydoun Rezakhanlou. "A stochastic model for growing sandpiles and its continuum limit." Communications in mathematical physics 197.2 (1998): 325-345.

This thesis consists of two interrelated projects. The first one studies the deterministic case of a mass-transfer model and the second part a stochastic version of the first one. Recall that the nonlinear p-Laplacian is

$$\Delta_p u := \operatorname{div}(|D u|^{p-2} D u).$$

For large values of p, the p-Laplacian could be considered as a model of a fast/slow diffusion operator in a suitable sense. Now consider the following evolution equation

$$\begin{cases} \partial_t u_p - \Delta_p u_p = f \text{ in } \mathbb{R}^n \times (0, \infty) \\ u_p = g \qquad \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

This equation can be used to understand the competing effects of fast and slow diffusions for large p. In this part, we study the solutions u_p as $p \to \infty$ and show that the limiting u solves a nonlinear PDE. In the second part, we consider a stochastic model for sandpile growth in the discrete case and show that in the macroscopic limit, the limiting dynamics of the height-function $u(x, t)$ is governed by the following nonlinear evolution equation(similar to the limiting PDE in the first part)

$$\begin{cases} f - \partial_t u \in \partial J[u] & \text{for } t > 0 \\ u = 0 & \text{for } t = 0, \end{cases}$$

where $f(x, t)$ is the macroscopic rate of adding particles, $J[\cdot]$ is a convex functional and the subdifferential notation $\partial J[\cdot]$ means that for a.e. time $t \geq 0$,

$$J[u(\cdot, t)] + (f - \partial_t u(\cdot, t), v - u(\cdot, t))_{L^2} \leq J[v] \text{ for each } v \in L^2(\mathbb{R}^n).$$
