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Scaling limit of stochastic 2D Euler to deterministic Navier-Stokes equations

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2D Euler equations and the vortex model

Incompressible Euler equations in velocity form are the following (no external forces):

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla p = 0 & \text{in } D \\ u\cdot n = 0 & \text{on } \partial D \end{cases}.$$

In 2D, introducing the vorticity $\omega = \nabla^\perp \cdot u$, $\nabla^\perp = (-\partial_2, \partial_1)$, the equations become

$$\begin{cases} \partial_t \omega + u\cdot\nabla\omega = 0 \\ u = K_D\,\omega \end{cases}, \qquad \text{(E)}$$

where K_D is the **Biot-Savart kernel**, i.e. $u = \nabla^\perp\Delta^{-1}\omega$. In particular there is no need to specify the boundary condition for u anymore as it is encoded in the kernel K_D.

$K_D(x,y)$ is always singular along the diagonal $x=y$ with $K_D(x,y)\sim|x-y|^{-1}$ for $x\sim y$.

Known invariants

- i) **Energy** $E(u) = |u|_{L^2}$;
- ii) **Enstrophy** $S(\omega) = |\omega|_{L^2}$ (which controls $|u|_{\dot H^1}$);
- iii) Casimir invariants of the vorticity: for any cts bdd g, $S_g(\omega) = \int_D g(\omega(x))\,\mathrm{d}x$ is invariant.

Main known results

- i) Global existence and uniqueness for $\omega_0\in L^\infty(D)$; smooth initial data stay smooth.
- ii) Global existence of solutions in $L^\infty(\mathbb{R}_{\geq 0}; L^p(D))$ for $\omega_0\in L^p(D)$ for any $p>1$; uniqueness is open.
- iii) Global existence of weak solutions (Schochet formulation) for ω_0 signed measure with a "sign preference"; uniqueness is open.

There is also another well studied family of solutions of (E), given by the so called **point vortices**. These are solutions of the form $\omega(t) = \sum_i \xi_i\delta_{x_i(t)}$, with ξ_i constant and $x_i(t)\in D$.

The vortices x_i form an **interacting particle system** given by the equation

$$\dot x_i = \sum_{\substack{j\in\{1,\dots,N\}\\ j\neq i}} \xi_j\,K_D(x_i,x_j) + \xi_i\,\Gamma_D(x_i) \qquad \forall i\in\{1,\dots,N\} \qquad \text{(N-VM)}$$

where Γ_D is a smooth function on D which models the interaction of the particle with ∂D.

Main difficulty: the **singularity of** K along the diagonal does not allow standard treatment. In particular **coalescence** of particle can occurr and explicit examples of pathological initial configurations are known.

We now restrict to the case $D=\mathbb{R}^2$ or $D=\mathbb{T}^2$, so that $\Gamma_D\equiv 0$ (no self-interaction). In this case the (N-VM) is a particle system of gradient type, which also admits an Hamiltonian formulation.

Theorem: Marchioro, Pulvirenti, [1]

Let $(\xi_i)_i\in\mathbb{R}^N$ be fixed. Then for $\mathcal{L}^{2N}$-a.e. initial configuration $(x_i)_i\in\mathbb{R}^{2N}$ (resp. $\mathbb{T}^2N$, the N-vortex dynamics is well-defined for $t\in[0,\infty)$ and no coalescence occurs.

Another remarkable property of (E) is the existence of a Gibbs-type (formally) invariant measure μ, called **enstrophy measure**, which is a Gaussian measure under which ω is white noise distributed: formally

$$\mathrm{d}\mu(\omega) = Z^{-1}e^{-\frac{1}{2}\|\omega\|^2_{L^2}}\,\mathrm{d}\omega.$$

Stochastic Euler equations and existing results

From now on we restrict ourselves to the case $D=\mathbb{T}^2$.

Theorem: Flandoli, [2]

Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of i.i.d. $\mathcal{N}(0,1)$ r.v., $\{X_0^n\}_{n\in\mathbb{N}}$ be a sequence of i.i.d. $\mathbb{T}^2$-valued, uniformly distributed r.v.; consider for each $N\in\mathbb{N}$ the rescaled N-point vortices

$$\frac{\mathrm{d}}{\mathrm{d}t}X_t^{i,N} = \frac{1}{\sqrt{N}}\sum_{\substack{j=1\\ j\neq i}}^N \xi_j K(X_t^{i,N}-X_t^{j,N}), \qquad X^{i,N}(0) = X_0^i \quad \forall i\in\{1,\dots,N\}$$

and the associated rescaled vorticities

$$\omega_t^N = \frac{1}{\sqrt{N}}\sum_{i=1}^N \xi_n\,\delta_{X_t^{i,N}}.$$

Then the family $\{\omega^N\}_N$ is tight in $C([0,T];H^{-1-}(\mathbb{T}^2))$ and it converges (up to subseq.) to a **white noise solution** ω of (E), i.e. a weak solution such that $\omega_t\sim\mu$ for all $t\in[0,T]$.

Central question in stochastic fluid dynamics: find which kind of noise, once introduced in the equation, preserves its structure but also allows to observe new phenomena (e.g. improves well-posedness).

Answer: the right noise to consider is a **multiplicative divergence free noise in Stratonovich form**. This is because it **preserves the variational structure** of the equation, Kelvin's circulation theorem and the invariants associated to the vorticity, as shown in [3]. This kind of noise however **has the drawback of destroying the energy conservation**.

Here we consider Stochastic Euler equations of the following form:

$$\begin{cases} \mathrm{d}\omega + u\cdot\nabla\omega\,\mathrm{d}t = \sum_k \theta_k\,\frac{k^\perp}{|k|}\,e_k\cdot\nabla\omega\circ\mathrm{d}W^k \\ u = K*\omega, \quad \omega(0)=\omega_0 \end{cases}. \qquad \text{(SE)}$$

where e_k denotes the standard Fourier basis ($k\in\mathbb{Z}^2\setminus\{0\}$) and $W^k = W^k(t)$ are independent BMs. We require $\theta=\{\theta_k\}_k$ to be a family of radially symmetric coefficients satisfying $\|\theta\|^2_{\ell^2} = \sum_k\theta_k^2<\infty$.

Theorem: Flandoli, Luo, [4]

For each N, define ε_N by

$$\varepsilon_N^{-2} = \sum_{k\in\mathbb{Z}^2:0<|k|\leq N}|k|^{-2}\sim\log N;$$

consider the family $(\xi_i, X_0^i)_{i\in\mathbb{N}}$ as above as well as a family $\{W^k\}_{k\in\mathbb{Z}^2}$ of independent standard Brownian motions. For each N consider the stochastic N-point vortices

$$\mathrm{d}X_t^{i,N} = \frac{1}{\sqrt{N}}\sum_{\substack{j=1\\ j\neq i}}^N \xi_j K(X_t^{i,N}-X_t^{j,N})\,\mathrm{d}t + 2\varepsilon_N\sum_{0<|k|\leq N}\frac{k^\perp}{|k|^2}e_k(X_t^{i,N})\circ\mathrm{d}W_t^k, \qquad X^{i,N}(0)=X_0^i \quad \forall i\leq N.$$

Then the associated vorticities are tight in $C([0,T];H^{-1-})$ and they converge in law to the unique white noise solution ω of the stochastic Navier–Stokes equation

$$\mathrm{d}\omega + u\cdot\nabla\omega\,\mathrm{d}t = \Delta\omega\,\mathrm{d}t + \sqrt{2}\nabla^\perp\cdot\mathrm{d}W \qquad (1)$$

where W is a space–time white noise.

$\rightsquigarrow$ Change in nature for the equation: **from hyperbolic to parabolic**!

Our result: Scaling limit to deterministic Navier-Stokes

A similar phenomenon is observed in [5] in the context of stochastic transport equations; the fundamental difference is however that here **more regular (namely L^2-valued) solutions are considered and the limit equation is deterministic**.

Theorem: G., Thm 1 from [5]

Consider a family $\{\theta^N\}_N\subset\ell^2$ such that $\|\theta^N\|_{\ell^\infty}=1$, $\|\theta^N\|_{\ell^2}\to\infty$ and a sequence $\{\varepsilon_N\}_N\subset\mathbb{R}$ s.t.

$$\lim_{N\to\infty}\varepsilon_N^2\,\|\theta^N\|^2_{\ell^2} = 4\,\nu\in(0,+\infty).$$

Then for any $v_0\in L^2$, any sequence of energy solutions $v^N\in C([0,T];L^2_w)$ of

$$dv = b\cdot\nabla v\,\mathrm{d}t + \varepsilon^N\sum_k\theta_k^N\,\frac{k^\perp}{|k|}\,e_k\cdot\nabla v\circ\mathrm{d}W^k \qquad \text{(STLE)}$$

converges in probability (in suitable topology, for suitable b) to the unique solution of

$$\partial_t v + b\cdot\nabla v - \nu\Delta v = 0, \quad v(0)=v_0.$$

This naturally leads to the question: what happens, in the same regularity regime, to solutions of the stochastic Euler equation?

Theorem: Flandoli, G., Luo, Thm 2.2 from [6]

For any $\omega_0\in L^2$ and any radially symmetric coefficients $\{\theta_k\}_k\in\ell^2$, there exists a weak (both in the analytic and probabilistic sense) solution of (SE), with $\mathbb{P}$-a.s. weakly continuous trajectories, satisfying

$$\sup_{t\in[0,T]}\|\omega_t\|_{L^2}\leq\|\omega_0\|_{L^2} \quad \mathbb{P}\text{-a.s.} \qquad (2)$$

Theorem: Flandoli, G., Luo, Thm 2.3 from [6]

Let $\omega_0\in L^2$; consider a family $\{\theta^N\}_N\subset\ell^2$ and a sequence $\{\varepsilon_N\}_N\subset\mathbb{R}$ as above. For any N, let Q^N denote the law of a weak solution ω^N of the (SE) associated to θ^N, satisfying the bound (2). Then the sequence $\{Q^N\}_N$ is tight in $C([0,T];H^{0-})$ and it converges to δ_ω, where ω is the unique solution of the deterministic 2D Navier-Stokes equations

$$\begin{cases} \partial_t\omega + u\cdot\nabla\omega - \nu\Delta\omega = 0 \\ u = K*\omega, \quad \omega(0)=\omega_0 \end{cases}.$$

Consequences of the scaling limit

Consequences

- Uniqueness for (SE) is open, but the limit is the same for any sequence $\rightsquigarrow$ **approximate uniqueness**;
- Asymptotic exponential decay of the energy $|u^N|_{L^2}$, similarly for $|u^N|_{H^{-\delta}}$ for $\delta\in(0,1)$; this is also important from the point of view of **mixing properties** of u^N;
- Existence for any initial data $u_0\in L^2$ of sequences of spatially smooth, enstrophy-preserving functions converging weakly to the unique solution of Navier-Stokes with initial data u_0;
- **Recovery sequences** for vanishing viscosity solutions of deterministic Euler equations;
- Convergence of the **passive scalars** advected by u^N to those advected by u.

Let us finally mention that a similar type of scaling limit has been employed in the recent work [7] to show that a similar kind of noise **improves the wellposedness theory for 3D Navier–Stokes equations**.

References

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