Statistical Mechanics of Lattice Systems, Part II: List of Topics

1 Introduction

1.1 Classical Statistical Mechanics as Probability

- Notion of probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and random variables.

- Single-spin space Ω_0, configuration space Ω, canonical projections, cylinder sets. Product topology, characterisation of product topology (with proof), product σ-algebra $\mathscr{F}$.

- Observables: space of measurables functions $B(\Omega)$, $C(\Omega)$. Local functions and the Doob–Dynkin lemma (no proof). Space of local functions $B_{\mathrm{loc}}(\Omega)$, $C_{\mathrm{loc}}(\Omega)$ and quasi-local functions $B_{\mathrm{ql}}(\Omega)$ and $C_{\mathrm{ql}}(\Omega)$.

- Finite signed measures $\mathscr{M}(\Omega)$ and probability measures $\mathscr{M}_1(\Omega)$. Riesz–Markov–Kakutani representation theorem when Ω is compact (no proof). Weak topologies on $\mathscr{M}(\Omega)$, standard weak topology wrt to $C(\Omega)$.

1.2 Infinite-volume Gibbs measures

- Marginal distributions via canonical projections and consistency condition, Hahn–Caratheodory and Kolmogorov's extension theorems (no proofs). Costruction of the product measure via Kolmogorov's extension theorem (with proof).

- Interaction, Hamiltonian, and examples of specific models. Locality lemma for Hamiltonians (with proof). Reference measure, partition function and Gibbsian specification.

- Action of Gibbsian specification on $B(\Omega)$ and composition of Gibbsian specifications. Characterisation of Gibbsian specification (with proof). Definition of the infinite-volume Gibbs measures $\mathscr{G}$.

2 Gaussian Free Field

2.1 Introduction

- Definition of the model, rescaling of the parameters, overview of the results for massless and massive case

2.2 Preliminaries

- Gaussian vectors, characterisation of Gaussian vectors by the density (no proof), by the characteristic function (no proof), Levy's continuity theorem (no proof).

- Gaussian fields, existence of the infinite-volume limit of Gaussian vectors via Kolmogorov's extension theorem (with proof).

- Lattice gradient ∇, lattice Laplacian Δ, lattice Green identities (with proof).

- Inversion of the (finite-volume) lattice Laplacian Δ_Λ via Neumann series (with proof), existence and uniqueness of a solution to the lattice Dirichlet problem.

2.3 Density of the GFF and SSRW representation

- Hamiltonian of the GFF in terms of the lattice Laplacian and the solution to the Dirichlet problem (with proof), density of the GFF in Gaussian form (with proof).
- Definition of the symmetric simple random walk (SSRW), notion of conditional probability, conditional expectation, Markov property (with proof).
- Definition of entrace times, SSRW representation of Δ_Λ (with proof), of G_Λ (with proof) and of the solution to the Dirichlet problem (with proof).
- Definition of recurrence and transience, their characterisation (no proof).

2.4 Infinite-volume massless GFF

- RW representation of the infinite-volume limit G and of $G - G_\Lambda$ (with proof). Infiniteness/finiteness of the entries of G (with proof). Rate of divergence (no proof).
- Absence of infinite-volume GFF in $d = 1, 2$ (with proof).
- Conditional expectation with respect to a σ-subalgebra, characterisation as projection (no proof). DLR condition as conditional expectation (proof skipped).
- RW representation of the conditional expectation of the GFF. Infinitely many infinite-volume GFF in $d \geqslant 3$ (with proof).

3 $O(N)$-symmetric Models

3.1 Introduction

- Definition of the model, group $O(N)$ and rigid $O(N)$ transformations, overview of results.
- Existence of infinite-volume measures based on compactness (with proof), $O(N)$-invariance of the set of infinite-volume measures (with proof)

3.2 Mermin–Wagner theorem

- Notion of continuous symmetry and symmetry breaking, statement of the Mermin–Wagner theorem and proposition about quantitative version in finite volume.
- Consequence of the Mermin–Wagner theorem for mean and correlation of spin variables. Proof: proposition $\Rightarrow$ Mermin–Wagner theorem.
- Relative entropy, Pinsker's inequality (with proof). Proof of the proposition via spin-wave transformation and minimisation Ansatz. Minimisation of the Dirichlet energy via RW representation (with proof).