Summary of the course

Semiclassical methods in statistical and quantum mechanics

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These notes are only for the use of the students in the class V5B3 at Bonn University, Summer term 2025.

Please send typos and corrections to `disertori@iam.uni-bonn.de`.

Bibliography

We will mostly follow the book *Semiclassical analysis, Witten Laplacians and statistical mechanics*, by B. Helffer (World Scientific)

See also Chapter 1.2 (Gaussian measures) and Chapter 3.1 (Laplace method) of the lecture notes on *Functional integrals involving commuting and anticommuting variables* from Winter Semester 2024/2025 available under the link: *https://www.iam.uni-bonn.de/users/disertori/teaching/functional-integrals-involving-commuting-and-anticommuting-variables-ws2025/25*

Contents

1 Preliminary notions

1.1 Introduction

Our goal is to study measures of the form $d\varphi^N e^{-\beta\mathcal{S}(\varphi)}$, where $d\varphi^N = \prod_{j=1}^N \varphi_j$ is a product of Lebesgue measures and $\mathcal{S}(\varphi) = \mathcal{S}(\varphi_1, \ldots, \varphi_N)$ is a real function.
Assuming $e^{-\beta\mathcal{S}} \in L^1(\mathbb{R}^N)$ we have

$$Z := \int_{\mathbb{R}^N} d\varphi^N e^{-\beta\mathcal{S}(\varphi)} < \infty$$

and hence we can define the probability measure

$$d\mu_N(\varphi) := \frac{1}{Z} d\varphi^N e^{-\beta\mathcal{S}(\varphi)}.$$

This measure has two parameters:

- N describes the number of particles or the volume of the system. The limit $N \to \infty$ is called the thermodynamic limit.
- $\beta > 0$ is the inverse temperature $\beta = T^{-1}$ or the inverse Plank constant $\beta = \hbar^{-1}$, depending on the context. The limit $\beta \to \infty$ is called the low temperature limit or semiclassical limit.

If we keep N fixed we can study the limit $\beta \to \infty$ by Laplace method, under some conditions:

$$\mathbb{E}[g] := \int_{\mathbb{R}^N} d\mu_N(\varphi) g(\varphi) = \sum_{k=0}^{M-1} \beta^{-k}\alpha_k + O(\beta^{-M})$$

The problem is to control this expansion in the limit $N \to \infty$. We will often compare to the following two reference cases.

Product measure. Let $S(\varphi) = \sum_{j=1}^N F(\varphi_j)$. In this case the variables φ_j are independent identically distributed

$$d\mu = \prod_{j=1}^N \frac{d\varphi_j}{Z_j} e^{-\beta F(\varphi_j)}$$

where $Z_j = \int_{\mathbb{R}} d\varphi \; e^{-\beta F(\varphi)}$. For any function $g = g(\varphi_{j_1}, \ldots, \varphi_{j_k})$ with $j_1, \ldots, j_k$ fixed and independent of N, the corresponding average

$$\mathbb{E}[g] = \int_{\mathbb{R}^k} d\varphi^k \prod_{l=1}^k \frac{1}{Z_j} e^{-\beta F(\varphi_{j_l})} g(\varphi_{j_1}, \ldots, \varphi_{j_k})$$

is independent of N and can be studied, for $\beta \gg 1$ via Laplace method.

Gaussian measure. Let $S(\varphi) = \frac{1}{2}(\varphi, A\varphi) = \sum_{i,j=1}^N \varphi_i A_{ij}\varphi_j$ where $A \in \mathbb{R}^{N\times N}_{sym,+}$. With this choice A is invertible and $A^{-1} > 0$ as a quadratic form. The measure

$$d\mu(\varphi) := \frac{1}{Z} d\varphi^N e^{-\frac{\beta}{2}(\varphi, A\varphi)}$$

is a centred normalized Gaussian measure with covariance $C = \frac{1}{\beta}A^{-1}$. The normalization constant is

$$Z = \int_{\mathbb{R}^N} d\varphi^N e^{-\frac{\beta}{2}(\varphi, A\varphi)} = \frac{(2\pi)^{\frac{N}{2}}}{\sqrt{\det A}}. \tag{1.1}$$

The corresponding Laplace transform is, for $t \in \mathbb{R}^N$,

$$\mathbb{E}[e^{\sum_{j=1}^N t_j\varphi_j}] = e^{\frac{1}{2\beta}(t, A^{-1}t)} = e^{\frac{1}{2\beta}(t, Ct)} \tag{1.2}$$

Using the Laplace transform one can compute the average of any polynomial function. In particular we compute

$$\mathbb{E}[\varphi_j] = 0, \qquad \mathbb{E}[\varphi_j\varphi_k] = A^{-1}_{jk} = C_{jk}.$$

General case. In general we try to approximate $d\mu$ with a product or Gaussian measure. In this lecture we will consider the following three strategies.

1: reduce to a low dimensional integral (I). Sometime one can reformulate the integral as

$$Z = \int_{\mathbb{R}^N} d\varphi^N e^{-\beta\mathcal{S}(\varphi)} = (f_l, T^N f_r)$$

where $T\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and $(.,.)$ is the scalar product in $L^2(\mathbb{R})$. The $N \to \infty$ limit corresponds to study high powers of the operator T. For $\beta \gg 1$ the operator T can be approximated by

$$Tf(\varphi) \sim \int_{\mathbb{R}} d\varphi\; e^{-m^2\varphi^2} e^{-(\varphi-\varphi')^2} e^{-m^2\varphi'^2} f(\varphi'),$$

which can be studied explicitely.
2: use convexity. If $\beta\mathcal{S}'' \geq C^{-1} > 0$ $\forall\varphi \in \mathbb{R}^N$, we will see (under some additional regularity requirements) that the following holds

$$\mathbb{E}[e^{\sum_{j=1}^N t_j(\varphi_j - \mathbb{E}[\varphi_j])}] \leq e^{\frac{1}{2}(t, Ct)}.$$

3: reduce to a low dimensional integral (II). As $N \to \infty$ we write Z a sum of integrals over a finite number of variables

$$Z = \sum_{X \subset \mathbb{N}} I_X.$$

1.2 Example 1: Curie-Weiss model and scalar Laplace principle

The measures above appear naturally even if the starting model deals only with Dirac measures. As an example we consider the Ising model in the mean field approximation (Curie-Weiss model). This is obtained by replacing

$$d\varphi_j \to [d\delta_{-1}(\sigma_j) + d\delta_1(\sigma_j)], \qquad \mathbb{R}^N \to \{-1, 1\}^N,$$

and defining the energy of the configuration $\sigma \in \Omega_N := \{-1,1\}^N$ via

$$H(\sigma) := -\frac{1}{2N}\sum_{jk=1}^{N}\sigma_j\sigma_k - h\sum_{j=1}^{N}\sigma_j.$$

The average of a function f is

$$\mathbb{E}[f] := \frac{\sum_{\sigma\in\Omega_N} e^{-\beta H(\sigma)} f(\sigma)}{\sum_{\sigma\in\Omega_N} e^{-\beta H(\sigma)}}$$

We investigate the behavior of the macroscopic random variable $X := \frac{1}{N}\sum_{j=1}^{N}\sigma_j$. The law of X is encoded in the Laplace transform $\mathbb{E}[e^{tX}]$.

[1: 11.04.2025]

[2: 16.04.2025]

Proposition 1.1. *For all $t \in \mathbb{R}$ it holds*

$$\mathbb{E}[e^{tX}] = \langle e^{\frac{t\varphi}{\beta}}\rangle \; e^{-\frac{t^2}{2\beta N}} e^{-ht}, \tag{1.3}$$

where

$$\langle g(\varphi)\rangle := \frac{\int_{\mathbb{R}} d\varphi\; e^{-NF(\varphi)} g(\varphi)}{\int_{\mathbb{R}} d\varphi\; e^{-NF(\varphi)}}$$

and

$$F(\varphi) := \frac{(\varphi-\beta h)^2}{2\beta} - \ln\cosh\varphi \tag{1.4}$$

where remember that $\cosh x = \frac{e^x+e^{-x}}{2}$.

Proof. Remember

$$\mathbb{E}[e^{tX}] = \frac{\sum_{\sigma\in\Omega_N} e^{-\beta H(\sigma)} e^{\frac{t}{N}\sum_{j=1}^{N}\sigma_j}}{\sum_{\sigma\in\Omega_N} e^{-\beta H(\sigma)}}. \tag{1.5}$$

We can reorganize $\beta H(\sigma)$ as follows

$$-\beta H(\sigma) = \frac{\beta}{2N}\left(\sum_{j=1}^{N}\sigma_j\right)^2 + h\left(\sum_{j=1}^{N}\sigma_j\right).$$

Using (1.2) we argue

$$e^{\frac{\beta}{2N}\left(\sum_{j=1}^{N}\sigma_j\right)^2} = \left(\tfrac{N}{2\pi\beta}\right)^{\frac{1}{2}}\int_{\mathbb{R}} d\varphi\; e^{-\frac{N}{2\beta}\varphi^2} e^{\varphi\sum_{j=1}^{N}\sigma_j}.$$

Inserting this in the denominator of (1.5) we obtain

$$\begin{aligned}
&\sum_{\sigma\in\Omega_N} e^{-\beta H(\sigma)} = \left(\tfrac{N}{2\pi\beta}\right)^{\frac{1}{2}}\int_{\mathbb{R}} d\varphi\; e^{-\frac{N}{2\beta}\varphi^2}\sum_{\sigma\in\Omega_N} e^{(\varphi+\beta h)\sum_{j=1}^{N}\sigma_j}\\
&= 2^N\left(\tfrac{N}{2\pi\beta}\right)^{\frac{1}{2}}\int_{\mathbb{R}} d\varphi\; e^{-\frac{N}{2\beta}\varphi^2}(\cosh(\varphi+\beta h))^N\\
&= K_N\int_{\mathbb{R}} d\varphi\; e^{-\frac{N}{2\beta}\varphi^2}(\cosh(\varphi+\beta h))^N = K_N\int_{\mathbb{R}} d\varphi\; e^{-\frac{N}{2\beta}(\varphi-\beta h)^2}(\cosh\varphi)^N,
\end{aligned}$$

where $K_n := 2^N \left(\frac{N}{2\pi\beta}\right)^{\frac{1}{2}}$, and we used

$$\sum_{\sigma\in\Omega_N} e^{(\varphi+\beta h)\sum_{j=1}^N \sigma_j} = \prod_{j=1}^N \sum_{\sigma_j\in\{-1,1\}} e^{(\varphi+\beta h)\sigma_j} = (2\cosh(\varphi+\beta h))^N,$$

followed by the coordinate change $\varphi \to \varphi - \beta h$. Repeating the same procedure for the numerator of (1.5) we obtain

$$\sum_{\sigma\in\Omega_N} e^{-\beta H(\sigma)} e^{\frac{t}{N}\sum_{j=1}^N \sigma_j} = K_N \int_{\mathbb{R}} d\varphi\; e^{-\frac{N}{2\beta}(\varphi-\beta h-\frac{t}{N})^2}(\cosh\varphi)^N.$$

The result now follows expanding the terms containing t in the exponent.

$\square$

To study the asymptotic behavior of $\langle g(\varphi)\rangle$ as $N\to\infty$ we use the following proposition.

Proposition 1.2 (Laplace's principle (scalar version))**.** *Let $f, g \in C^\infty(\mathbb{R})$ be two given functions. Assume*

(a) f *admits a unique global minimum in* x_0 *and* $f''(x_0) > 0$,

(b) $\inf_{\substack{x \text{ local min}\\ x\neq x_0}} [f(x) - f(x_0)] > 0$

(c) $\exists N_0 > 0$ *such that* $\int_{\mathbb{R}} dx\; e^{-N_0 f(x)} < \infty$ *and* $\int_{\mathbb{R}} dx\; e^{-N_0 f(x)}|g(x)| < \infty$.

Then for $N\to\infty$ we have

(i) $\int_{\mathbb{R}} dx\; e^{-Nf(x)} = e^{-Nf(x_0)}\frac{\sqrt{2\pi}}{\sqrt{Nf''(x_0)}}\left(1+O\left(\frac{1}{N}\right)\right)$.

(ii) $\langle g\rangle := \frac{\int_{\mathbb{R}} dx\; e^{-Nf(x)}g(x)}{\int_{\mathbb{R}} dx\; e^{-Nf(x)}} = g(x_0) + \frac{1}{2N}\left[\frac{g''(x_0)}{f''(x_0)} - \frac{g'(x_0)f^{(3)}(x_0)}{f''(x_0)^2} - \frac{g(x_0)f^{(4)}(x_0)}{4f''(x_0)^2}\right] + o\left(\frac{1}{N}\right)$

If we have k global minima $x_1,\dots,x_k$, under the same assumptions for each minimum, we obtain

(i)′ $\int_{\mathbb{R}} dx\; e^{-Nf(x)} = \sum_{j=1}^k e^{-Nf(x_j)}\frac{\sqrt{2\pi}}{\sqrt{Nf''(x_j)}}\left(1+O\left(\frac{1}{N}\right)\right)$,

(ii)′ $\langle g\rangle = \frac{1}{\sum_{j=1}^k \frac{1}{\sqrt{f''(x_j)}}}\sum_{j=1}^k \frac{1}{\sqrt{f''(x_j)}}\left(g(x_j) + \frac{1}{2N}\left[\frac{g''(x_j)}{f''(x_j)} - \frac{g'(x_j)f^{(3)}(x_j)}{f''(x_j)^2} - \frac{g(x_j)f^{(4)}(x_j)}{4f''(x_j)^2}\right] + o\left(\frac{1}{N}\right)\right)$

Informal proof of (i) For $N \gg 1$ the measure concentrates on a small region near the minimum point x_0

$$\int_{\mathbb{R}} dx\; e^{-Nf(x)} \simeq \int_{|x-x_0|<\varepsilon} dx\; e^{-Nf(x)}.$$

For small ε the function is well approximated by its Taylor expansion

$$f(x) \simeq f(x_0) + f''(x_0)\frac{(x-x_0)^2}{2}.$$

Hence

$$\int_{\mathbb{R}} dx\; e^{-Nf(x)} \simeq \int_{|x-x_0|<\varepsilon} dx\; e^{-Nf(x)} \simeq e^{-Nf(x_0)}\int_{|x-x_0|<\varepsilon} dx\; e^{-Nf''(x_0)\frac{(x-x_0)^2}{2}}.$$

We claim we can replace above the integral on $\mathbb{R}$

$$\int_{\mathbb{R}} dx\, e^{-Nf(x)} \simeq e^{-Nf(x_0)} \int_{|x-x_0|<\varepsilon} dx\, e^{-Nf''(x_0)\frac{(x-x_0)^2}{2}} \simeq e^{-Nf(x_0)} \int_{\mathbb{R}} dx\, e^{-Nf''(x_0)\frac{x^2}{2}} = e^{-Nf(x_0)} \frac{\sqrt{2\pi}}{\sqrt{Nf''(x_0)}}.$$

Proof. Here we only prove $\lim_{N\to\infty}\langle g\rangle = g(x_0)$. For the other statements see the lecture notes on *Functional integrals involving commuting and anticommuting variables* from Winter Semester 2024/2025.

- The integral in (i) is well defined $\forall N \geq N_0$ since

$$\begin{aligned} 0 < \int_{\mathbb{R}} dx\, e^{-Nf(x)} &= e^{-Nf(x_0)} \int_{\mathbb{R}} dx\, e^{-N(f(x)-f(x_0))} \leq e^{-Nf(x_0)} \int_{\mathbb{R}} dx\, e^{-N_0(f(x)-f(x_0))} \\ &= e^{-(N-N_0)f(x_0)} \int_{\mathbb{R}} dx\, e^{-N_0 f(x)} < \infty, \end{aligned}$$

where we used $f(x) - f(x_0) \geq 0$. The same argument shows that the integrals in (ii) are well defined $\forall N \geq N_0$. In the following we can assume $f(x_0) = 0$ and $x_0 = 0$. If this is not the case, we consider $\tilde{f}(x) := f(x_0 + x) - f(x_0)$.

- Since f is smooth and (a) and (b) hold, there exists $\varepsilon_0 > 0$ such that $\forall 0 < \varepsilon \leq \varepsilon_0$ we have

$$f(x) = f(x) - f(x_0) \geq \min\{f(\varepsilon), f(-\varepsilon)\} \qquad \forall |x| \geq \varepsilon. \tag{1.6}$$

For $\varepsilon \ll 1$ we also have, using $f(0) = f'(0) = 0$,

$$f(x) = \frac{f''(0)}{2}x^2 + R(x), \qquad \text{with} \quad |R(x)| \leq K_3\varepsilon^3 \qquad \forall |x| \leq \varepsilon, \tag{1.7}$$

for some constant $K_3 > 0$. Replacing $x = \pm\varepsilon$ we obtain, using $f''(0) > 0$,

$$f(\pm\varepsilon) = \frac{f''(0)}{2}\varepsilon^2 + O(\varepsilon^3) \geq \frac{f''(0)}{4}\varepsilon^2.$$

- We decompose the integral in the denominator as follows

$$\int_{\mathbb{R}} dx\, e^{-Nf(x)} = \int_{|x|<\varepsilon} dx\, e^{-Nf(x)} + \int_{|x|\geq\varepsilon} dx\, e^{-Nf(x)} = I_1(\varepsilon) + I_2(\varepsilon).$$

Using the bounds above, we argue, for $N \geq \frac{N_0}{2}$,

$$|I_2(\varepsilon)| = \int_{|x|\geq\varepsilon} dx\, e^{-Nf(x)} = \int_{|x|\geq\varepsilon} dx\, e^{-\frac{N}{2}f(x)} e^{-\frac{N}{2}f(x)} \leq \sup_{|x|\geq\varepsilon} e^{-\frac{N}{2}f(x)} \int_{|x|\geq\varepsilon} dx\, e^{-N_0 f(x)} \leq e^{-c_0 N\varepsilon^2} c_1$$

where we defined

$$c_0 := \frac{f''(0)}{8}, \qquad c_1 := \int_{\mathbb{R}} dx\, e^{-N_0 f(x)}.$$

We choose now $\varepsilon = \varepsilon_N$ such that

- $\lim_{N\to\infty} \varepsilon_N = 0$
- $\lim_{N\to\infty} N\varepsilon_N^2 = \infty$.

Hence we set

$$\varepsilon_N := \frac{N^\delta}{N^{\frac{1}{2}}}, \qquad 0 < \delta < \frac{1}{2}. \tag{1.8}$$

We will optimize δ later.

• For $|x| < \varepsilon$ we replace f by its Taylor expansion at order 2 (1.7). We obtain

$$\begin{aligned} I_1(\varepsilon) &= \int_{|x|<\varepsilon} dx\ e^{-\frac{1}{2}f''(0)Nx^2} + \int_{|x|<\varepsilon} dx\ e^{-\frac{1}{2}f''(0)Nx^2}(e^{-NR(x)} - 1) \\ &= \frac{1}{N^{\frac{1}{2}}} \int_{|x|<\varepsilon N = N^\delta} dx\ e^{-\frac{1}{2}f''(0)x^2} + \int_{|x|<\varepsilon} dx\ e^{-\frac{1}{2}f''(0)Nx^2}(e^{-NR(x)} - 1) \\ &= \frac{1}{N^{\frac{1}{2}}} \left(\int_{\mathbb{R}} dx\ e^{-\frac{1}{2}f''(0)x^2} + \tilde{I}_1(\varepsilon) \right) = \frac{1}{N^{\frac{1}{2}}} \left(\frac{\sqrt{2\pi}}{\sqrt{f''(0)}} + \tilde{I}_1(\varepsilon) \right) \end{aligned}$$

where

$$\tilde{I}_1(\varepsilon) = -\int_{|x|>N^\delta} dx\ e^{-\frac{1}{2}f''(0)x^2} + N^{\frac{1}{2}} \int_{|x|<\varepsilon} dx\ e^{-\frac{1}{2}f''(0)Nx^2}(e^{-NR(x)} - 1).$$

The first integral in this sum is bounded by

$$\left| \int_{|x|>N^\delta} dx\ e^{-\frac{1}{2}f''(0)x^2} \right| \le\ e^{-c_0 N^{2\delta}} \int_{\mathbb{R}} dx\ e^{-\frac{3}{8}f''(0)x^2} = e^{-c_0 N^{2\delta}} \frac{\sqrt{2\pi}}{\sqrt{\frac{3}{4}f''(0)}}.$$

To estimate the second integral we use (1.7) and

$$\left| e^{-NR(x)} - 1 \right| = \left| NR(x) \int_0^1 dt\ e^{-tNR(x)} \right| \le K_3(N\varepsilon^3) e^{K_3(N\varepsilon^3)}.$$

Inserting this bound we obtain

$$\begin{aligned} &N^{\frac{1}{2}} \left| \int_{|x|<\varepsilon} dx\ e^{-\frac{1}{2}f''(0)Nx^2}(e^{-NR(x)} - 1) \right| \le K_3(N\varepsilon^3) e^{K_3(N\varepsilon^3)} N^{\frac{1}{2}} \int_{\mathbb{R}} dx\ e^{-\frac{1}{2}f''(0)Nx^2} \\ &= K_3(N\varepsilon^3) e^{K_3(N\varepsilon^3)} \frac{\sqrt{2\pi}}{\sqrt{f''(0)}}. \end{aligned}$$

Choosing $\delta < \frac{1}{6}$ we obtain

$$N\varepsilon^3 = N^{3\delta - \frac{1}{2}} \to_{N\to\infty} 0,$$

hence

$$|\tilde{I}_1(\varepsilon)| \le\ c_2 N^{3\delta - \frac{1}{2}} \to_{N\to\infty} 0,$$

for some $c_2 > 0$.

• Putting all this together we have

$$\int_{\mathbb{R}} dx\ e^{-Nf(x)} = \frac{1}{N^{\frac{1}{2}}} \left(\frac{\sqrt{2\pi}}{\sqrt{f''(0)}} + \tilde{I}_1(\varepsilon) + N^{\frac{1}{2}} I_2(\varepsilon) \right)$$

where

$$|\tilde{I}_1(\varepsilon) + N^{\frac{1}{2}} I_2(\varepsilon)| \le c_2 N^{3\delta - \frac{1}{2}} + c_1 N^{\frac{1}{2}} e^{-c_0 N\varepsilon^2} \to_{N\to\infty} 0.$$

• Similarly, the integral in the numerator can be decomposed as

$$\int_{\mathbb{R}} dx\ e^{-Nf(x)} = \frac{1}{N^{\frac{1}{2}}}\left(g(0)\frac{\sqrt{2\pi}}{\sqrt{f''(0)}} + I_{rem}\right)$$

where

$$I_{rem} = -g(0)\int_{|x|>N^{\delta}} dx\ e^{-\frac{1}{2}f''(0)x^2} + N^{\frac{1}{2}}\int_{|x|<\varepsilon} dx\ e^{-\frac{1}{2}f''(0)Nx^2}(g(x)e^{-NR(x)} - g(0))$$
$$+ N^{\frac{1}{2}}\int_{|x|\geq\varepsilon} dx\ e^{-Nf(x)}g(x).$$

Using the same arguments as above we obtain

$$|I_{rem}| \leq c_4\varepsilon_N + c_5 N^{3\delta-\frac{1}{2}} \to_{N\to\infty} 0$$

□

[2: 16.04.2025]

[3: 23.04.2025]

Application to the mean field Ising model Remember cf (1.3) and (1.4),

$$\mathbb{E}[e^{tX}] = \langle e^{\frac{t\varphi}{\beta}}\rangle\ e^{-\frac{t^2}{2\beta N}} e^{-ht},$$

with

$$\langle g(\varphi)\rangle := \frac{\int_{\mathbb{R}} d\varphi\ e^{-NF(\varphi)}g(\varphi)}{\int_{\mathbb{R}} d\varphi\ e^{-NF(\varphi)}}$$

and

$$F(\varphi) := \frac{(\varphi-\beta h)^2}{2\beta} - \ln\cosh\varphi.$$

Proposition 1.3.

(i) The critical points of F are the solutions of

$$\frac{\varphi-\beta h}{\beta} = \tanh\varphi. \tag{1.9}$$

We denote by $\varphi_m = \varphi_m(h,\beta)$ the largest critical point.

(ii) Assume $\beta \leq 1$. Then F admits a unique critical point $\varphi_0 = \varphi_0(h,\beta) := \varphi_m$. This point is a global minimum and $F''(\varphi_0) > 0$. The function $h \mapsto \varphi(h,\beta)$ is continuous and satisfies

$$\varphi_0(-h,\beta) = -\varphi_0(h,\beta), \qquad \varphi_0(h,\beta) > 0\forall h > 0.$$

In particular $\varphi_0(0,\beta) = 0$.

(iii) Assume $\beta > 1$. We distinguish two cases.

(a) If $h = 0$ F has two (global) minimum points in $\pm\varphi_\beta$ and one local maximum in $\varphi = 0$, where $\varphi_\beta := \varphi_m$. Moreover

$$F''(\varphi_\beta) = F''(-\varphi_\beta) > 0.$$

(b) There exists a $h_\beta > 0$ such that, for $0 < h < h_\beta$ the function F has 3 critical points $\varphi_-(h,\beta) < \varphi_0(h,\beta) < 0 < \varphi_+(h,\beta)$, where

- *$\varphi_+ := \varphi_m$ is the unique global minimum point and $F''(\varphi_+) > 0$,*
- *φ_- is a local minimum point,*
- *φ_0 is a local maximum point.*

Moreover all these are continuous functions of h and

$$\lim_{h\downarrow 0} \varphi_+ = \varphi_\beta \quad \lim_{h\downarrow 0} \varphi_- = -\varphi_\beta, \quad \lim_{h\downarrow 0} \varphi_0 = 0.$$

Proof. See Section 3.2 in the lecture notes on *Functional integrals involving commuting and anticommuting variables* from Winter Semester 2024/2025 □

Proposition 1.4.

(i) Assume $h = 0$.

(a) If $\beta \leq 1$ then $\lim_{N\to\infty} \mathbb{E}[e^{tX}] = 1$

(b) If $\beta > 1$ then $\lim_{N\to\infty} \mathbb{E}[e^{tX}] = e^{\frac{\varphi_\beta t}{\beta}} + e^{-\frac{\varphi_\beta t}{\beta}}$.

(ii) Assume $h > 0$.

(a) If $\beta \leq 1$ then $\lim_{N\to\infty} \mathbb{E}[e^{tX}] = e^{-th} e^{\frac{\varphi_0(h,\beta)t}{\beta}}$. In particular

$$\lim_{N\to\infty} \lim_{h\downarrow 0} \mathbb{E}[e^{tX}] = 1 = \lim_{h\downarrow 0} \lim_{N\to\infty} \mathbb{E}[e^{tX}].$$

(b) If $\beta > 1$ then $\lim_{N\to\infty} \mathbb{E}[e^{tX}] = e^{\frac{\varphi_+(h,\beta)t}{\beta}}$. In particular

$$\lim_{N\to\infty} \lim_{h\downarrow 0} \mathbb{E}[e^{tX}] = e^{\frac{\varphi_\beta t}{\beta}} + e^{-\frac{\varphi_\beta t}{\beta}} \neq e^{\frac{\varphi_\beta t}{\beta}} = \lim_{h\downarrow 0} \lim_{N\to\infty} \mathbb{E}[e^{tX}]. \tag{1.10}$$

Proof. Apply Proposition 1.2 and 1.3. □

Note that (1.10) implies the limit measure does not recover the symmetry $\varphi \to -\varphi$ as $h \downarrow 0$. In this case one says the system exhibits spontaneous symmetry breaking.

1.3 Example 2: Ising with long range interactions

We replace $\{1, \dots, N\}$ with the finite volume $\Lambda_L := [-L, L]^d \cap \mathbb{Z}^d$. The set of possible spin configurations becomes $\Omega_{\Lambda_L} := \{-1, 1\}^{\Lambda_L}$. The energy of a configuration $\sigma \in \Omega_{\Lambda_L}$ is defined by

$$H(\sigma) := -\frac{1}{2} \sum_{jk \in \Lambda_L} J_{jk} \sigma_j \sigma_k - h \sum_{j \in \Lambda_L} \sigma_j, \tag{1.11}$$

where $J_{jk} = J_{kj} \geq 0 \ \forall j, k \in \Lambda_L$.
To define J we introduce the lattice Laplacian.

Definition 1.5 (lattice Laplacian)**.** *The lattice (or graph) Laplacian on $\mathbb{Z}^d$ is the linear operator $\Delta\colon \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ defined by*

$$(\Delta f)(j) := \sum_{k\in\mathbb{Z}^d, |k-j|=1} (f(j) - f(k)) = \sum_{k\in\mathbb{Z}^d, |k-j|=1} \Delta_{jk} f(k) \tag{1.12}$$

where $-\Delta_{jk} = 2d\mathbf{1}_{j=k} - \mathbf{1}_{|j-k|=1}$.

This operator is well defined and bounded (exercise) with $\|\Delta\| \leq 4d$. For all $f \in \ell^2(\mathbb{Z}^d)$ we have (exercise)

$$(f, -\Delta f)_{\ell^2(\mathbb{Z}^d)} = \sum_{|i-j|=1} (f(i) - f(j))^2 \geq 0, \tag{1.13}$$

hence $-\Delta \geq 0$ as a quadratic form. One can show that $-\Delta$ is self-adjoint with spectrum $\sigma(-\Delta) = \sigma_{ac}(-\Delta) = [0, 4d]$.

Definition 1.6 (finite volume lattice Laplacian)**.** *For any $\Lambda \subset\subset \mathbb{Z}^d$ the finite volume Laplacian Δ_Λ with Dirichlet boundary conditions is the matrix $\Delta_\Lambda \in \mathbb{R}^{\Lambda\times\Lambda}_{sym}$ defined by*

$$(\Delta_\Lambda)_{ij} := \Delta_{ij}.$$

For all $f \in \ell^2(\Lambda)$ we have (exercise)

$$(f, -\Delta f)_{\ell^2(\Lambda)} = \sum_{ij\in\Lambda, |i-j|=1} (f(i) - f(j))^2 + \sum_{i\in\Lambda} d_i f(i)^2 > 0, \tag{1.14}$$

where $d_i := \sum_{j\notin\Lambda, |i-j|=1} 1$. Hence $-\Delta_\Lambda > 0$ as a quadratic form.

Using these notations, we define

$$J_{ij} := (-\Delta_{\Lambda_L} + 1)^1, \tag{1.15}$$

where $-\Delta_{\Lambda_L} + 1 := -\Delta_{\Lambda_L} + \mathrm{Id} \in \mathbb{R}^{\Lambda_L\times\Lambda_L}_{sym}$.

Proposition 1.7. *J is well defined and satisfies $J > 0$ as a quadratic form and $J_{jk} > 0$ $\forall j, k \in \Lambda_L$.*

Note that $M > 0$ does not imply $M_{jk} > 0$ $\forall i, j$. Indeed $M := -\Delta_\Lambda$ satisfies $M > 0$ but $M_{ij} < 0$ for $|i - j| = 1$. Also, $M_{jk} > 0$ $\forall i, j$ does not imply $M > 0$. As an example take the matrix

$$M := \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}$$

with $\varepsilon > 1$.

Proof of Proposition 1.7. $-\Delta_{\Lambda_L} + 1 > 0$ as a quadratic form and hence it is invertible with $(-\Delta_{\Lambda_L} + 1)^{-1} > 0$.

We show now that $J_{jk} > 0$ holds. For this we reformulate J_{ij} as infinite sum of positive terms. Note that

$$-\Delta_{\Lambda_L} = 2d - P$$

where $P_{ij} = \mathbf{1}_{|i-j|=1}$ is the adjacency matrix of the graph Λ_L. Therefore we can write

$$J^{-1} = (2d + 1) - P = (2d + 1)\left(1 - \frac{1}{2d+1}P\right), \qquad J = \frac{1}{2d+1}\left(1 - \frac{1}{2d+1}P\right)^{-1}.$$

We have $\|P\|_{op} \leq 2d$ since

$$\sum_j (Pf)_j^2 = \sum_j (\sum_k P_{jk} f_k)^2 = \sum_j (\sum_k \sqrt{P_{jk}}\sqrt{P_{jk}} f_k)^2 \leq \sum_j (\sum_k P_{jk})(\sum_{k'} P_{jk} f_{k'}^2)$$
$$\leq 2d \sum_{k'} \sum_j P_{jk'} f_{k'}^2 \leq (2d)^2 \sum_{k'} f_{k'}^2,$$

where we used $P_{jk} \geq 0$, Cauchy-Schwartz and $0 < \sum_k P_{jk} \leq 2d$.
It follows

$$\|\frac{1}{2d+1} P\|_{op} \leq \frac{2d}{2d+1} < 1,$$

hence the Neumann series

$$J = \frac{1}{2d+1} \sum_{n \geq 0} \left(\frac{1}{2d+1} P \right)^n$$

converges in operator norm. We conclude

$$J_{jk} = \frac{1}{2d+1} \sum_{n \geq 0} \frac{1}{(2d+1)^n} \sum_{i_1,\dots,i_{n-1}} P_{ji_1} P_{i_1 i_2} \cdots P_{i_{n-1}k} > 0$$

since $P_{jk} \geq 0 \ \forall jk$ and at least some term in the sum is strictly positive. □

Set now

$$\mathbb{E}_{\Lambda_L}[f] := \frac{\sum_{\sigma \in \Omega_{\Lambda_L}} e^{-\beta H(\sigma)} f(\sigma)}{\sum_{\sigma \in \Omega_{\Lambda_L}} e^{-\beta H(\sigma)}}$$

with the energy function $H(\sigma)$ defined in (1.11). We study the Laplace transform of the random variable $X_{\Lambda_L} := \frac{1}{|\Lambda_L|} \sum_{j \in \Lambda_L} \sigma_j$.

Proposition 1.8. *It holds*

$$\mathbb{E}_{\Lambda_L} \left[e^{\frac{t}{|\Lambda_L|} \sum_{j \in \Lambda_L} \sigma_j} \right] = \langle e^{\frac{t}{\beta |\Lambda_L|} \sum_{j \in \Lambda_L} (1+d_j)\varphi_j} \rangle_{\Lambda_L} \ e^{-\frac{t^2}{2\beta} \frac{|\Lambda_L| + |\partial^{ext} \Lambda_L|}{|\Lambda_L|^2}},$$

where $\partial^{ext} \Lambda_L \{ j \notin \Lambda_l | \ \mathrm{dist}\,(j, \Lambda_L) = 1 \}$,

$$\langle g(\varphi) \rangle_{\Lambda_L} := \frac{\int_{\mathbb{R}^{\Lambda_L}} d\varphi^{\Lambda_L} \ e^{-F(\varphi)} g(\varphi)}{\int_{\mathbb{R}^{\Lambda_L}} d\varphi^{\Lambda_L} \ e^{-F(\varphi)}}$$

and

$$F(\varphi) := \frac{(\varphi, J^{-1}\varphi)}{2\beta} - \sum_{j \in \Lambda_L} \left[\ln \cosh(\varphi_j + \beta h) \right].$$

Proof. exercise □

[3: 23.04.2025]

[4: 25.04.2025]

2 Transfer operator approach

2.1 Ising model with nearest neighbor interaction

We consider the finite volume $\Lambda_L := [-L, L] \cap \mathbb{Z} = \{-L, -L+1, \ldots, -1, 0, 1, \ldots, L\}$. The set of possible spin configurations is $\Omega_{\Lambda_L} := \{-1, 1\}^{\Lambda_L}$. The energy of a configuration $\overline{\sigma} \in \Omega_{\Lambda_L}$ is defined by

$$H(\overline{\sigma}) := -\frac{1}{2} \sum_{jk \in \Lambda_L} \mathbf{1}_{|j-k|=1} \sigma_j \sigma_k - h \sum_{j \in \Lambda_L} \sigma_j = -\sum_{j=-L}^{L-1} \sigma_j \sigma_{j+1} - h \sum_{j=-L}^{L} \sigma_j. \tag{2.1}$$

If we do not introduce additional conditions on the boundary points $j = \pm L$ we say we have *simple boundary conditions*. The average is defined as usual by

$$\mathbb{E}\left[f(\overline{\sigma})\right] := \frac{\sum_{\overline{\sigma} \in \Omega_{\Lambda_L}} f(\overline{\sigma}) e^{-\beta H(\overline{\sigma})}}{\sum_{\overline{\sigma} \in \Omega_{\Lambda_L}} e^{-\beta H(\overline{\sigma})}}.$$

Our goal is to study the Laplace transform $\mathbb{E}\left[e^{\frac{t}{|\Lambda_L|} \sum_{j=-L}^{L} \sigma_j}\right]$, the mean $\mathbb{E}\left[\sigma_j\right]$ and the covariance $\mathbb{E}\left[\sigma_j \sigma_k\right] - \mathbb{E}\left[\sigma_j\right] \mathbb{E}\left[\sigma_k\right]$ in the thermodynamic limit $L \to \infty$.

Dual representation To formulate the dual representation we need some notation. Setting $\Omega_0 = \{-1, 1\}$ the one-spin configuration space we introduce the transfer operator

$$\begin{array}{llll} \mathcal{T}\colon & \mathbb{R}^{\Omega_0} \to & \mathbb{R}^{\Omega_0} \\ & f \mapsto & (\mathcal{T} f)(\sigma) := \sum_{\sigma' \in \Omega_0} T(\sigma, \sigma') f(\sigma') \end{array}$$

where we defined

$$T(\sigma, \sigma') := e^{\frac{V(\sigma)}{2}} K(\sigma, \sigma') e^{\frac{V(\sigma')}{2}}, \qquad K(\sigma, \sigma') = e^{\beta \sigma \sigma'}, \quad V(\sigma) = \beta h \sigma.$$

We this definition $\mathcal{T}^n f(\sigma) = \sum_{\sigma' \in \Omega_0} T^n(\sigma, \sigma') f(\sigma')$ where

$$T^n(\sigma, \sigma') = \sum_{\sigma_1, \ldots, \sigma_{n-1} \in \Omega_0} T(\sigma, \sigma_1) T(\sigma_1, \sigma_2) \cdots T(\sigma_{n-1}, \sigma').$$

We also introduce the (real) scalar product

$$\langle f, g \rangle := \sum_{\sigma \in \Omega_0} f(\sigma) g(\sigma),$$

and the left/right boundary functions $f_l(\sigma) = f_r(\sigma) := e^{\frac{1}{2}\beta h \sigma}$. For more general boundary conditions we have $f_l \neq f_r$.

Theorem 2.1. *With the notations above the following statements hold.*

(i) The partition function admits the dual representation

$$Z = \sum_{\overline{\sigma} \in \Omega^{\Lambda_L}} e^{-\beta H(\overline{\sigma})} = \langle f_l, \mathcal{T}^{2L} f_r \rangle.$$

(ii) *For all $j_0 \in \Lambda_L$ the mean in j_0 admits the dual representation*

$$\mathbb{E}\left[\sigma_j\right] = \frac{\langle f_l, \mathcal{T}^{L+j_0}\mathcal{O}\mathcal{T}^{L-j_0} f_r\rangle}{\langle f_l, \mathcal{T}^{2L} f_r\rangle},$$

where $\mathcal{O}f(\sigma) := \sigma f(\sigma)$ for all $\sigma \in \Omega_0$.

(iii) *For all $j_0 < k_0 \in \Lambda_L$ we have*

$$\mathbb{E}\left[\sigma_{j_0}\sigma_{k_0}\right] = \frac{\langle f_l, \mathcal{T}^{L+j_0}\mathcal{O}\mathcal{T}^{k_0-j_0}\mathcal{O}\mathcal{T}^{L-k_0} f_r\rangle}{\langle f_l, \mathcal{T}^{2L} f_r\rangle}.$$

(iv) *For $t \in \mathbb{R}$ we have*

$$\mathbb{E}\left[e^{\frac{t}{|\Lambda_L|}\sum_{j=-L}^{L}\sigma_j}\right] = \frac{\langle \tilde{f}_l, \tilde{\mathcal{T}}^{2L}\tilde{f}_r\rangle}{\langle f_l, \mathcal{T}^{2L} f_r\rangle},$$

where $\tilde{\mathcal{T}}f(\sigma) = \sum_{\sigma' \in \Omega_0}\tilde{T}(\sigma,\sigma')f(\sigma')$ is defined by

$$\tilde{T}(\sigma,\sigma') := e^{\frac{1}{2}\left(\beta h + \frac{t}{|\Lambda|}\right)\sigma} K(\sigma,\sigma') e^{\frac{1}{2}\left(\beta h + \frac{t}{|\Lambda|}\right)\sigma'}$$

and $\tilde{f}_l(\sigma) = \tilde{f}_r(\sigma) := e^{\frac{1}{2}\left(\beta h + \frac{t}{|\Lambda|}\right)\sigma}$.

Proof.
(i) We argue, using $\beta h \sigma_j = 2\frac{V(\sigma_j)}{2}$,

$$\begin{aligned} Z &= \left[\prod_{j=-L}^{L}\sum_{\sigma_j\in\Omega_0}\right]\prod_{j=-L}^{L-1} e^{\beta\sigma_j\sigma_{j+1}}\prod_{j=-L}^{L} e^{\beta h\sigma_j} \\ &= \left[\prod_{j=-L}^{L}\sum_{\sigma_j\in\Omega_0}\right]\ e^{\frac{1}{2}V(\sigma_{-L})}\left[\prod_{j=-L}^{L-1} T(\sigma_j,\sigma_{j+1})\right]\ {}^{\frac{1}{2}V(\sigma_L)} = \langle f_l, \mathcal{T}^{2L} f_r\rangle. \end{aligned}$$

$(ii)-(iii)$ similar arguments.

(iv) Same argument as in (i) but in the numerator we replace βh with $\beta h + \frac{t}{|\Lambda|}$ which modifies the definition of $\mathcal{T}$ in $\tilde{\mathcal{T}}$ and the definition of f_l into $\tilde{f}_l$.

□