Summary of the course

Nonlinear partial differential equations 1

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This is only a summary of the main results and arguments discussed in class and *not* a complete set of lecture notes. These notes can thus not replace the careful study of the literature. The following books are recommended:

- L.C. Evans, *Partial differential equations*, Amer. Math. Soc., 1998 (2nd edition 2010).
- D.Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer, 1998 (reprinted as Classics in Mathematics).
- M. Giaquinta, L. Martinazzi, *An Introduction to the Regularity Theory for Elliptic Systems, Harmonic Maps and Minimal Graphs*, Edizioni della Normale.
- L. Boccardo, G. Croce, *Elliptic Partial Differential Equations*, De Gruyter.

A large amount of material comes from the book by Evans. These notes are only for the use of the students in the class V4B1 at Bonn University, Fall term 2023-24.

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1 Preliminary definitions

1.1 Introduction

A partial differential equation of order k is an equation involving an unknown function $u\colon \Omega \to \mathbb{R}^m$, with $\Omega \subset \mathbb{R}^d$, and its partial derivatives up of order k :

$$F\big(\{\partial^\alpha u(x)\}_{0\le|\alpha|\le k}, x\big) = 0 \quad \forall x \in \Omega.$$

Here $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ is a multi-index and we used the notation $\partial^\alpha u = \prod_{j=1}^k \partial_{x_j}^{\alpha_j} u$. We always assume the function is regular enough so that partial derivatives commute.

A *linear* PDE of order k can be written as

$$\sum_{0\le|\alpha|\le k} a_\alpha(x)\partial^\alpha u(x) + f(x) = 0, \tag{1.1}$$

where the coefficients $a_\alpha\colon \Omega \to \mathbb{R}$ may depend on x, and $f\colon \Omega \to \mathbb{R}^m$ is the non homogeneous term.

There are several levels of nonlinearity, depending of the behavior of the highest order derivatives. A *semilinear* PDE of order k takes the form

$$\sum_{|\alpha|=k} a_\alpha(x)\partial^\alpha u(x) + F\big(\{\partial^\beta u(x)\}_{0\le|\beta|\le k-1}, x\big) = 0. \tag{1.2}$$

This equation is linear in the highest order derivatives, with coefficients a_α independent of u.

A *quasilinear* PDE of order k takes the form

$$\sum_{|\alpha|=k} a_\alpha(\big(\{\partial^\beta u(x)\}_{0\le|\beta|\le k-1}, x\big)\partial^\alpha u(x) + F\big(\{\partial^\beta u(x)\}_{0\le|\beta|\le k-1}, x\big) = 0. \tag{1.3}$$

This equation is linear in the highest order derivatives, with coefficients a_α depending on u and derivatives of order less than k.

A *fully nonlinear* PDE of order k is nonlinear in the $k-$th order derivatives.

The most famous examples of linear second order PDEs are:

- Laplace equation $\Delta u(x) = 0$, $u\colon \Omega \to \mathbb{R}$, with $\Omega \subset \mathbb{R}^d$,
- heat equation $\partial_t u(t,x) - \Delta u(t,x) = 0$, $u\colon \mathbb{R}\times\Omega \to \mathbb{R}$,
- wave equation $\partial_t^2 u(t,x) - \Delta u(t,x) = 0$, $u\colon \mathbb{R}\times\Omega \to \mathbb{R}$.

Examples of semilinear PDEs are:

- nonlinear heat equation: $\partial_t u(t,x) - \Delta u(t,x) = f(u(t,x))$, with $u\colon \mathbb{R}\times\Omega \to \mathbb{R}$;
- incompressible Navier-Stokes equation: $\partial_t u_j(t,x) - \Delta u_j(t,x) + u\cdot\nabla u_j(t,x) + \partial_j p(t,x) = 0$, with p a given function, $u\colon \mathbb{R}\times\Omega \to \mathbb{R}^d$, and $u\cdot\nabla := \sum_{j=1}^d u_j\partial_j$.

An example of quasilinear PDEs is the incompressible Euler equation

$$\partial_t u_j(t,x) + u \cdot \nabla u_j(t,x) + \partial_j p(t,x) = 0,$$

while the Hamiltion-Jacobi equation

$$\partial_t u(t,x) + H(\nabla u(t,x)) = 0$$

is fully nonlinear.

Some tools we have seen to study these equations are:

- linearization: approximate the solution of the nonlinear PDE by solving a linearized version;
- fixed point in some Banach space (used for example for nonlinear heat and Navier-Stokes equation)
- minimization (calculus of variations): reformulate the problem of solving the PDE into finding a minimizer for some functional $I(u)$.

Here we will develop systematically these and other tools.

1.2 Sobolev spaces: definition and some properties

Definition 1.1. *Let $\Omega \subset \mathbb{R}^d$ be open, $f \in L^1_{loc}(\Omega)$, i.e. $f \in L^1(K)$ $\forall K \subset \Omega$ compact.*

(i) f is weakly differentiable if there exist d functions $g_1, \dots, g_d$ in $L^1_{loc}(\Omega)$ such that

$$\int_\Omega f\, \partial_i \varphi \, dx = -\int_\Omega g_i \, \varphi \, dx \quad \forall \varphi \in C_c^\infty(\Omega). \tag{1.4}$$

In this case, g_i is called the weak derivative of f in the direction i.

(ii) f is k times weakly differentiable if, for all multiindices $\alpha \in \mathbb{N}^n$ with $|\alpha| \le k$, there exist $g^{(\alpha)} \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega f\, \partial^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega g^\alpha \, \varphi \, dx \quad \forall \varphi \in C_c^\infty(\Omega). \tag{1.5}$$

Notation The functions g_i and g^α are called weak derivatives and are still denoted by $\partial_i f$ and $\partial^\alpha f$, respectively.

Remarks

- The weak derivative is unique up to a null set, i.e. $\{g \in L^1_{loc}(\Omega) \mid g = \partial^\alpha f\}$ is a single equivalence class in $L^1_{loc}(\Omega)$.
- The weak derivative and the usual derivative agree if $g \in C^k(\Omega)$.
- In the following, we will usually make no notational distinction between functions and their equivalence classes.

Definition 1.2. *Let $\Omega \subset \mathbb{R}^d$ be open, $1 \le p \le \infty$ and $k \in \mathbb{N} \setminus \{0\}$.*
The Sobolev space $W^{k,p}(\Omega)$ consists of all $f \in L^p(\Omega)$ which are k times weakly differentiable with all weak derivatives in $L^p(\Omega)$.

$$W^{k,p}(\Omega) := \{f \in L^p(\Omega) \mid f \ k \textit{ times weakly differentiable with } \partial^\alpha f \in L^p(\Omega) \forall |\alpha| \le k\}$$

We define $\|\cdot\|_{W^{k,p}(\Omega)} \colon W^{k,p}(\Omega) \to [0,\infty)$ by

$$\|f\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{0 \le |\alpha| \le k} \|\partial^\alpha f\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}} & p < \infty \\ \sum_{0 \le |\alpha| \le k} \|\partial^\alpha f\|_{L^\infty(\Omega)} & p = \infty \end{cases} \tag{1.6}$$

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^d$ be open, $1 \le p \le \infty$ and $k \in \mathbb{N} \setminus \{0\}$.*

(i) *$(W^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)})$ is a Banach space.*

(ii) *$W^{k,2}(\Omega)$ is a Hilbert space with the scalar product*

$$(f,g)_{W^{k,2}(\Omega)} := \sum_{0 \le |\alpha| \le k} (\partial^\alpha f, \partial^\alpha g)_{L^2(\Omega)}, \tag{1.7}$$

where $(f,g)_{L^2(\Omega)} := \int_\Omega fg \, dx \ \forall f,g \in L^2(\Omega)$.

We will often use the notation $H^k(\Omega) := W^{k,2}(\Omega)$.

Proof. See lecture notes for FA or the book by Evans. □

Remark. Let $d = 1$ and $I = (a,b) \subset \mathbb{R}$ an open bounded interval. Then $f \in W^{1,1}(I) \Rightarrow \exists c \in \mathbb{R}$ such that the function

$$\tilde{f}(x) := c + \int_a^x f'(t)\,dt \quad \forall x \in [a,b] \tag{1.8}$$

satisfies $\tilde{f} \in [f]$. Note that $\tilde{f} \in C([a,b])$, and $\tilde{f}$ is differentiable a.e. with $\tilde{f}' = f'$ a.e..
If $f \in W^{1,p}(I)$ with $1 < p \le \infty$, then we have in addition $\tilde{f} \in C^{0,\alpha}(\bar{I})$ with $\alpha := 1 - \frac{1}{p}$ and

$$[\tilde{f}]_\alpha := \sup_{t \neq t' \in I} \frac{|f(t) - f(t')|}{|t - t'|^\alpha} \le \|f'\|_{L^p(\Omega)}.$$

Definition 1.4. *Let $\Omega \subset \mathbb{R}^d$ be open, $1 \le p < \infty$ and $k \in \mathbb{N} \setminus \{0\}$. We define the space*

$$W_0^{k,p}(\Omega) := \{f \in W^{k,p}(\Omega) \mid \exists j \mapsto f_j \in C_c^\infty(\Omega) \textit{ such that } f_j \to f \textit{ in } W^{k,p}(\Omega)\}. \tag{1.9}$$

This is therefore the completion of $C_c^\infty(\Omega)$ in the Sobolev norm. For $p = 2$ we will write $H_0^k(\Omega) := W_0^{k,p}(\Omega)$.

Remarks The space $W_0^{k,p}(\Omega)$ corresponds to the set of Sobolev functions "with zero boundary value". We will make this more precise later after we introduce the trace. $W_0^{k,p}(\Omega)$ is a closed linear subspace of $W^{k,p}(\Omega)$. Moreover, $W_0^{k,p}(\mathbb{R}^d) = W^{k,p}(\mathbb{R}^d)$, but $W_0^{k,p}(\Omega) \subsetneq W^{k,p}(\Omega)$ when $\Omega \subsetneq \mathbb{R}^d$. For $k = 0$ we have $W^{0,p}(\Omega) = L^p(\Omega) = W_0^{0,p}(\Omega)$ since $C_c^\infty(\Omega)$ is dense in $L^P(\Omega)$.

Theorem 1.5 (approximation by smooth functions)**.** *Let $\Omega \subset \mathbb{R}^d$ be open, $p \in [1, \infty)$, $k \in \mathbb{N}$. Then*

(i) $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ *is dense in* $W^{k,p}(\Omega)$,

(ii) $C_c^\infty(\mathbb{R}^d)$ *is dense in* $W^{k,p}(\mathbb{R}^d)$,

(iii) $C_c^\infty(\Omega)$ *is dense in* $W_0^{k,p}(\Omega)$.

Proof. $(i), (ii)$ See lecture notes on FA or the book by Evans.
(iii) Follows directly from the definition of $W_0^{k,p}(\Omega)$. □

Theorem 1.6 (Rellich)**.** *Let $\Omega \subset \mathbb{R}^d$ open and bounded, $1 \leq p < \infty$ and $k \geq 1$.*
Let $n \mapsto u_n \in W_0^{k,p}(\Omega)$ be a bounded sequence, and $u \in W_0^{k-1,p}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{k-1,p}(\Omega)$.

Then $u_n \to u$ strongly in $W_0^{k-1,p}(\Omega)$.

Proof. See lecture notes on FA. □

[1: 9.10.2023]

[2: 12.10.2023]

Theorem 1.7 (Sobolev inequalities in $\mathbb{R}^d$)**.**

(i) Let $1 \leq p < d$ and let

$$p^* := \frac{dp}{d-p} \quad \text{or, equivalently,} \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}. \tag{1.10}$$

Then it holds

$$\|u\|_{L^{p^*}(\mathbb{R}^d)} \leq \frac{p^*(d-1)}{d} \|\nabla u\|_{L^p(\mathbb{R}^d)} \quad \forall u \in W^{1,p}(\mathbb{R}^d). \tag{1.11}$$

(ii) Let $1 \leq d < p < \infty$ and let

$$\alpha := 1 - \frac{d}{p}. \tag{1.12}$$

Then every $u \in W^{1,p}(\mathbb{R}^d)$ has a Hölder continuous representative $\tilde{u} \in C^{0,\alpha}(\mathbb{R}^d)$, and there exist a constant $C_{p,d} > 0$ such that

$$[\tilde{u}]_{C^{0,\alpha}} := \sup_{x \neq y} \frac{|\tilde{u}(x) - \tilde{u}(y)|}{|x-y|^\alpha} \leq C_{pd} \|\nabla u\|_{L^p} \quad \forall u \in W_0^{1,p}(\mathbb{R}^d). \tag{1.13}$$

Proof. See lecture notes on FA or Evans. □

Remark. If $u \in L^p(\mathbb{R}^d) \cap C^{0,\alpha}(\mathbb{R}^d)$ then u is bounded and

$$\sup_{x\in\mathbb{R}^d} |u(x)| \leq C([u]_{C^{0,\alpha}(\mathbb{R}^d)} + \|u\|_{L^p(\mathbb{R}^d)})$$

for some constant $C > 0$ independent of u. Indeed we have, for each $x \in \mathbb{R}^d$ and $r > 0$,

$$u(x) = \frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)dy + \frac{1}{|B_r(x)|}\int_{B_r(x)} (u(x) - u(y))dy.$$

The result now follows from

$$|u(x)| \leq \frac{1}{|B_r(x)|}\int_{B_r(x)} |u(y)|dy + \frac{r^\alpha}{|B_r(x)|}\int_{B_r(x)} \frac{|u(x)-u(y)|}{|x-y|^\gamma})dy \leq \frac{1}{|B_r(0)|^{\frac{1}{p}}}\|u\|_{L^p(\mathbb{R}^d)} + r^\alpha [u]_{C^{0,\alpha}}$$

and optimizing in $r > 0$.

Theorem 1.8 (Sobolev embedding)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. The following statements hold.*

(i) Assume $1 \leq p < d$. Then $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ $\forall 1 \leq q \leq p^*$, where $p^* := \frac{pd}{d-p}$.

The embedding

$$\begin{aligned} I\colon \quad & W_0^{1,p}(\Omega) \to L^q(\Omega) \\ & u \mapsto I(u) := u \end{aligned}$$

is continuous, and

$$\|u\|_{L^q(\Omega)} \leq C\ \|\nabla u\|_{L^p(\Omega)} \quad \forall u \in W_0^{1,p}(\Omega),$$

where the constant C depends on p, q and Ω.

Moreover, if $1 \leq q < p^*$, the injection is also compact.

(ii) Assume $1 \leq d < p < \infty$. Then $W_0^{1,p}(\Omega) \subset C^{0,\beta}(\Omega)$ $\forall 0 < \beta \leq \alpha := 1 - \frac{d}{p}$.

This means, that each $u \in W_0^{1,p}(\Omega)$ has a representative $\tilde{u} \in C^{0,\beta}(\Omega)$.

The embedding

$$\begin{aligned} I_\beta\colon \quad & W_0^{1,p}(\Omega) \to C^{0,\beta}(\Omega) \\ & [u] \mapsto I([u]) := \tilde{u} \end{aligned}$$

is continuous, and

$$[\tilde{u}]_{C^{0,\beta}} \leq C\|\nabla u\|_{L^p} \quad \forall u \in W_0^{1,p}(\Omega), \tag{1.14}$$

where the constant C depends on p, β and Ω.

Moreover, if $\beta < \alpha$, the embedding is also compact.

Proof. See lecture notes on FA or Evans. □

Remark. If $u \in L^p(\Omega) \cap C^{0,\alpha}(\Omega)$ then u is bounded and

$$\sup_{x\in\Omega} |u(x)| \leq C([u]_{C^{0,\alpha}(\Omega)} + \|u\|_{L^p(\Omega)})$$

for some constant $C > 0$ independent of u. Indeed we argue

$$u(x) = \frac{1}{|\Omega|}\int_\Omega u(y)dy + \frac{1}{|\Omega|}\int_\Omega (u(x) - u(y))dy.$$

The result now follows from

$$|u(x)| \leq \frac{1}{|\Omega|}\int_\Omega |u(y)|dy + \frac{\mathrm{diam}(\Omega)^\alpha}{|\Omega|}\int_\Omega \frac{|u(x)-u(y)|}{|x-y|^\gamma})dy \leq \frac{1}{|\Omega|^{\frac{1}{p}}}\|u\|_{L^p(\Omega)} + \mathrm{diam}(\Omega)^\alpha [u]_{C^{0,\alpha}}.$$

Two key tools to prove the above results are: convolution with mollifiers and partition of unity.

Lemma 1.9 (convolution)**.**
Let $\eta \in C_c^\infty(B_1(0))$ such that $\eta \geq 0$ and $\int_{\mathbb{R}^d} \eta dx = \int_{B_1(0)} \eta dx = \|\eta\|_{L^1} = 1$.
We define $\varepsilon \mapsto \eta_\varepsilon$ via $\eta_\varepsilon(x) := \varepsilon^{-d}\eta\left(\frac{1}{\varepsilon}x\right)$, for $\varepsilon > 0$. The following holds.

(i) $\forall \varepsilon > 0$ $\eta_\varepsilon \in C_c^\infty(B_\varepsilon(0); [0,\infty))$, and $\int_{\mathbb{R}^d} \eta_\varepsilon dx = \int_{B_\varepsilon(0)} \eta_\varepsilon dx = 1$.

(ii) Suppose $u \in L^p(\mathbb{R}^d)$, $p < \infty$. Then

  (a) $\eta_\varepsilon * u(x) = \int_{\mathbb{R}^d} \eta_\varepsilon(x-y)u(y)dy$ for a.e. $x \in \mathbb{R}^d$.
  (b) $\eta_\varepsilon * f \in L^p(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ and $\|\eta_\varepsilon * u\|_{L^p} \leq \|\eta_\varepsilon\|_{L^1}\|u\|_{L^p} = \|u\|_{L^p}$.
  (c) $\lim_{\varepsilon\to 0} \|\eta_\varepsilon * u - u\|_{L^p} = 0$.
  (d) If in addition $u \in W^{k,p}(\mathbb{R}^d)$ it holds

$$\partial^\alpha(\eta_\varepsilon * u) = \eta_\varepsilon * (\partial^\alpha u)$$

  for all multiindices α with $|\alpha| \leq k$.
  Hence we also have $\lim_{\varepsilon\to 0} \|\eta_\varepsilon * u - u\|_{W^{k,p}(\mathbb{R}^d)} = 0$.

Notation The functions η_ε are sometimes called mollifiers. For any sequence $\varepsilon_j \to 0$, the sequence $j \mapsto \eta_{\varepsilon_j}$ is called a standard Dirac sequence (or a sequence of mollifiers).

Proof. See lecture notes on FA. □

Definition 1.10 (Partition of unity)**.** *Let $A \subset \mathbb{R}^d$ be non empty, I be a finite or countable index set.*

(i) A family of sets $\{V_i\}_{i\in I}$ is an open cover of A if $V_i \subset \mathbb{R}^d$ is open and nonempty $\forall i$, and $A \subset \bigcup_{i\in I} V_i$.

(ii) An open cover $\{V_i\}_{i\in I}$ is locally finite if $\forall x \in \bigcup_{i\in I} V_i$ $\exists \varepsilon > 0$ such that

$$\#\{i \in I \mid B_\varepsilon(x) \cap V_i \neq \varnothing\} < \infty.$$

(iii) Let $\{V_i\}_{i\in I}$ be a locally finite open cover of A. A family of functions $\{\chi_i\}_{i\in I}$ is a partition of unity for A with respect to the cover $\{V_i\}_{i\in I}$ if

$$\chi_i \in C_c^\infty(V_i, [0,\infty)) \ \forall i \in I \quad \textit{and} \quad \sum_{i\in I} \chi_i(x) = 1 \ \forall x \in A.$$

Remarks.

- The sum $\sum_{i\in I}\chi_i(x)$ is finite $\forall x$ because the open cover is locally finite.
- In particular $0 \le \chi_i(x) \le 1 \ \forall x \in A$.

Lemma 1.11 (Existence of a partition of unity)**.** *Let $\Omega \subset \mathbb{R}^d$ be open.*
Suppose $\exists K_j \subset V_j \subset \bar{V}_j \subset \Omega$ for all $j \in \mathbb{N}$ such that

$$K_j, \bar{V}_j \text{ compact } \forall j, \qquad K_j \cap K_{j'} = \varnothing \ \forall j \neq j', \qquad \{V_j\}_{j\in\mathbb{N}} \text{ locally finite open cover of } \Omega.$$

Then there exists a partition of unity $\{\chi_j\}_{j\in\mathbb{N}}$ for Ω with respect to $\{V_j\}_{j\in\mathbb{N}}$.
The partition satisfies, in addition, $\chi_j(x) = 1 \ \forall x \in K_j$.

Remark. The sets K_j may be empty! In that case $\chi_j(x) < 1 \ \forall x \in V_j$.

Proof. See lecture notes on FA. □

1.3 Boundary regularity and its applications

All results in the previous section need no information on $\partial\Omega$. Indeed most of the results there are stated for functions in $W_0^{k,p}(\Omega)$ that 'take zero value' on the boundary. To allow for nonzero values at the boundary we need to require some regularity.

1.3.1 Boundary regularity

Let $\Omega \subset \mathbb{R}^d$ be an open set. Informally $\partial\Omega$ is 'regular' (Lipschitz, C^k, smooth...) if locally it can be represented by a function $\gamma\colon \mathbb{R}^{d-1} \to \mathbb{R}$ which is 'regular' (Lipschitz, C^k, smooth...).

Definition 1.12. *Let $\Omega \subset \mathbb{R}^d$ be an open set. We say that $\partial\Omega$ is C^k, $k \ge 1$, (resp. Lipschitz) if for all $x_0 \in \Omega$ $\exists r > 0$ and a C^k (resp. Lipschitz) function $\gamma\colon \mathbb{R}^{d-1} \to \mathbb{R}$ such that, after relabelling the variables and reorienting the axes, it holds*

$$\Omega \cap B_r(x_0) = \{x = (x_1, \dots, x_d) \in B_r(x_0) \,|\, x_d > \gamma(x_1, \dots, x_{d-1})\}.$$

Remark 1 If $\partial\Omega$ is C^1, then we can define the outward normal unit vector field $\nu\colon \partial\Omega \to \mathcal{S}^{d-1}$. Moreover, if $u \in C^1(\overline{\Omega})$ it holds $\partial_\nu u(x) = \nu_x \cdot \nabla u(x) \ \forall x \in \partial\Omega$. If $\partial\Omega$ the outward normal $\nu(x)$ is still well defined for a.e. $x \in \partial\Omega$

Remark 2: flattening the boundary. Assume $\partial\Omega$ is C^1. Then we can 'locally' flatten the boundary in the following way.
Let $x_0 \in \partial\Omega$, $r > 0$ and $\gamma\colon \mathbb{R}^{d-1} \to \mathbb{R}$ as above. We introduce the change of coordinates $\Phi\colon \mathbb{R}^d \to \mathbb{R}^d$ as follows

$$y = \Phi(x) := \begin{cases} y_i := x_i & \forall i = 1, \dots, d-1, \\ y_d := x_d - \gamma(x_1, \dots, x_{d-1}). \end{cases}$$

Φ is invertible with inverse

$$x = \Phi^{-1}(y) := \begin{cases} x_i := y_i & \forall i = 1, \dots, d-1, \\ x_d := y_d + \gamma(y_1, \dots, y_{d-1}). \end{cases}$$

Moreover, since γ is C^1, the Jacobian is well defined and equals 1. In the new coordinates the boundary is locally flat:

$$\Omega \cap B_r(x_0) = \{y \in \Phi(B_r(x_0)) \,|\, y_d = x_d - \gamma(x_1, \dots, x_{d-1}) > 0\}.$$

1.3.2 Approximation by smooth functions up to the boundary

Theorem 1.13. *Let $p \in [1, \infty)$, $k \in \mathbb{N}$.*

(i) Let $\Omega = \{x \in \mathbb{R}^d | \, x_d > 0\}$ be the upper half-plane. Then $C^\infty(\overline{\Omega}) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.

(ii) Assume $\Omega \subset \mathbb{R}^d$ is open and bounded with Lipschitz boundary. Then $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$.

Proof.
Let $\Omega \subset \mathbb{R}^d$ be open and $u \in W^{k,p}(\Omega)$ given. Our goal is to find a sequence of functions in $C^\infty(\overline{\Omega}) \cap W^{k,p}(\Omega)$ converging to u in the Sobolev norm.
To construct the sequence let $\varepsilon \mapsto \eta_\varepsilon$ be a Dirac sequence (cf. Lemma 1.9). For each $\varepsilon > 0$ we consider

$$(\eta_\varepsilon * u)(x) := (\eta_\varepsilon * \mathbf{1}_\Omega u)(x) = \int_{\mathbb{R}^d} \eta_\varepsilon(x-y)(\mathbf{1}_\Omega u)(y) \, dy = \int_\Omega \eta_\varepsilon(x-y) u(y) \, dy.$$

It holds $\eta_\varepsilon * u \in C^\infty(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, hence in particular $\eta_\varepsilon * u \in C^\infty(\overline{\Omega}) \cap L^p(\Omega)$. Moreover

$$\|\eta_\varepsilon * \mathbf{1}_\Omega u - u\|_{L^p(\Omega)} \leq \|\eta_\varepsilon * \mathbf{1}_\Omega u - \mathbf{1}_\Omega u\|_{L^p(\mathbb{R}^d)} \to_{\varepsilon \to 0} 0.$$

What about the derivatives? In general $u \in W^{k,p}(\Omega) \not\Rightarrow \mathbf{1}_\Omega u \in W^{k,p}(\mathbb{R}^d)$, hence we cannot even say if $\partial^\alpha(\eta_\varepsilon * u)$ is in $L^p(\mathbb{R}^d)$.
Note that we have $\partial^\alpha(\eta_\varepsilon * \mathbf{1}_\Omega u)(x) = \eta_\varepsilon * \mathbf{1}_\Omega \partial^\alpha u(x)$ for all x such that $B_\varepsilon(x) \subset \Omega$, i.e.

$$\partial^\alpha(\eta_\varepsilon * \mathbf{1}_\Omega u)(x) = \eta_\varepsilon * \mathbf{1}_\Omega \partial^\alpha u(x) \quad \forall x \in \Omega_\varepsilon := \{x \in \Omega | \, \mathrm{dist}\,(x, \partial\Omega) > \varepsilon\}.$$

To go further we need to use our information on $\partial\Omega$.

(i) In this case $\Omega = \{x \in \mathbb{R}^d | \, x_d > 0\}$, hence

$$\Omega_\varepsilon = \Omega + \varepsilon \hat{e}_d,$$

where $\hat{e}_d$ is the normal vector in the direction d. This fact, together with $\Omega_\delta \subset \Omega_\varepsilon \ \forall \delta > \varepsilon$, suggests to consider, for each $\varepsilon, \delta > 0$, the function $u \colon \mathbb{R}^d \to \mathbb{R}$ defined by

$$u_{\varepsilon,\delta}(x) := \eta_\varepsilon * \mathbf{1}_\Omega u(x + \delta \hat{e}_d) = [\eta_\varepsilon * \mathbf{1}_\Omega u] \circ \tau_\delta(x)$$

where $\tau_\delta \colon \mathbb{R}^d \to \mathbb{R}^d$ is the translation $\tau_\delta(x) := x + \delta \hat{e}_d$. This function satisfies $u_{\varepsilon,\delta} \in C^\infty(\mathbb{R}^d)$ in particular $u_{\varepsilon,\delta} \in C^\infty(\overline{\Omega}) \ \forall \varepsilon, \delta > 0$. Moreover

$$\partial^\alpha u_{\varepsilon,\delta}(x) = [\eta_\varepsilon * \mathbf{1}_\Omega \partial^\alpha u] \circ \tau_\delta(x) = [\eta_\varepsilon * \mathbf{1}_\Omega \partial^\alpha u](x + \delta \hat{e}_d) \qquad \forall x \in \Omega - (\delta - \varepsilon)\hat{e}_d.$$

Since $\overline{\Omega} \subset \Omega - (\delta - \varepsilon)\hat{e}_d \ \forall \delta > \varepsilon$ we set $\delta_\varepsilon := \lambda \varepsilon$, with $\lambda > 1$ fixed and consider the family of functions

$$u_\varepsilon := u_{\varepsilon, \delta_\varepsilon}.$$

We claim that $u_\varepsilon \in W^{k,p}(\Omega)$ for all $\varepsilon > 0$ and $\|u_\varepsilon - u\|_{W^{k,p}(\Omega)} \to_{\varepsilon \to 0} 0$. Indeed, for all $0 \leq |\alpha| \leq k$ it holds

$$\begin{aligned} \|\partial^\alpha u_\varepsilon\|_{L^p(\Omega)} &= \|(\eta_\varepsilon * \mathbf{1}_\Omega \partial^\alpha u) \circ \tau_{\lambda\varepsilon}\|_{L^p(\Omega)} \leq \|(\eta_\varepsilon * \mathbf{1}_\Omega \partial^\alpha u) \circ \tau_{\lambda\varepsilon}\|_{L^p(\mathbb{R}^d)} \\ &= \|\eta_\varepsilon * \mathbf{1}_\Omega \partial^\alpha u\|_{L^p(\mathbb{R}^d)} \leq \|\partial^\alpha u\|_{L^p(\Omega)} < \infty, \end{aligned}$$

hence $u_\varepsilon \in W^{k,p}(\Omega)$. Moreover

$$\|u_\varepsilon - u\|_{L^p(\Omega)} \leq \|u_\varepsilon - \mathbf{1}_\Omega u\|_{L^p(\mathbb{R}^d)} \leq \|u_\varepsilon - (\mathbf{1}_\Omega u)\circ\tau_{\lambda\varepsilon}\|_{L^p(\mathbb{R}^d)} + \|(\mathbf{1}_\Omega u)\circ\tau_{\lambda\varepsilon} - (\mathbf{1}_\Omega u)\|_{L^p(\mathbb{R}^d)}$$

Since $\lim_{h\to 0}\|f(x+h)-f(x)\|_{L^p} = 0$, we have

$$\lim_{\varepsilon\to 0}\|(\mathbf{1}_\Omega u)\circ\tau_{\lambda\varepsilon} - (\mathbf{1}_\Omega u)\|_{L^p(\mathbb{R}^d)} = 0.$$

Finally

$$\begin{aligned}&\|u_\varepsilon - (\mathbf{1}_\Omega u)\circ\tau_{\lambda\varepsilon}\|_{L^p(\mathbb{R}^d)} = \|(\eta_\varepsilon * \mathbf{1}_\Omega u)\circ\tau_{\lambda\varepsilon} - (\mathbf{1}_\Omega u)\circ\tau_{\lambda\varepsilon}\|_{L^p(\mathbb{R}^d)}\\ &= \|\eta_\varepsilon * \mathbf{1}_\Omega u - \mathbf{1}_\Omega u\|_{L^p(\mathbb{R}^d)} \to_{\varepsilon\to 0} 0\end{aligned}$$

The same argument holds for derivatives. This concludes the proof in case (i).

(ii) Assume $\partial\Omega$ is C^1. In this case the boundary is locally flat and we can adapt the argument from (i). To make this rigorous note that $\partial\Omega$ is compact and therefore we can find $x_1,\ldots,x_N \in \partial\Omega$ and $r_1,\ldots,r_N > 0$ such that

$$\partial\Omega \subset \bigcup_{j=1}^N B_{r_j}(x_j)$$

and $B_{r_j}(x_j)\cap\partial\Omega$ is flat after an appropriate change of coordinates. Set $V_0 \subset \bar{V}_0 \subset \Omega$ open such that

$$\overline{\Omega} \subset V_0 \cup \bigcup_{j=1}^N B_{r_j}(x_j).$$

Then there exists a partition of unity $\{\chi_j\}_{j=0,\ldots,N}$ for Ω wrt the open cover. In particular $u = u_0 + \sum_{j=1}^N u_j$, with $u_j = \chi_j u$.
Since u_0 has compact support in V_0 it holds $\eta_\varepsilon * u_0 \to u_0$ in $W^{k,p}(V_0)$. For $j \geq 1$ we adapt the construction from (i) (see Evans for details).
For the case of Lipschitz boundary see Lecture notes in FA. □

1.3.3 Trace

Theorem 1.14. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded with Lipschitz boundary, $p \in [1,\infty)$.*

(i) There exists a linear bounded operator $T\colon W^{1,p}(\Omega) \to L^p(\partial\Omega;\mathcal{H}^{d-1})$ such that

(a) $Tu = u_{|\partial\Omega}$ for all $u \in W^{1,p}(\Omega)\cap C(\overline{\Omega})$ and

(b) for all $u \in W^{1,p}(\Omega), \varphi \in C^1(\overline{\Omega}), j = 1,\ldots d$ it holds

$$\int_\Omega u\partial_j\varphi\; dx = -\int_\Omega \partial_j u\varphi\; dx + \int_{\partial\Omega}(Tu)\varphi\nu_j\; dx$$

(ii) $u \in W_0^{1,p}(\Omega) \quad \Leftrightarrow \quad Tu = 0.$

We call Tu the trace of u on $\partial\Omega$.

[2: 12.10.2023]

[3: 16.10.2023]

Proof of Theorem 1.14.
(i) To construct T we proceed as follows.

• For $u \in C^1(\overline{\Omega})$ the boundary value $u_{|\partial\Omega}$ is well defined, hence we *define*

$$T\colon C^1(\overline{\Omega}) \to \{\text{functions on } \partial\Omega\}$$

via $Tu := u_{|\partial\Omega}$. This map is linear. Moreover for all $u \in C^1(\overline{\Omega})$ it holds (proof later)

$$Tu \in L^p(\partial\Omega; \mathcal{H}^{d-1}), \qquad \text{and} \qquad \|Tu\|_{L^p(\partial\Omega)} \leq C\ \|u\|_{W^{1,p}(\Omega)} \tag{1.15}$$

where the constant $C = C(\Omega, p) > 0$ is independent of u.

• For each $u \in W^{1,p}(\Omega)$ there is a sequence $n \mapsto u_n \in C^1(\overline{\Omega})$ such that $\|u - u_n\|_{W^{1,p}(\Omega)} \to 0$. We have

$$\|Tu_n - Tu_m\|_{L^p(\partial\Omega)} \leq C\|u_n - u_m\|_{W^{1,p}(\Omega)}$$

hence $n \mapsto Tu_n$ is a Cauchy sequence in $L^p(\partial\Omega; \mathcal{H}^{d-1})$. Therefore the limit exists and we *define*

$$Tu := \lim_{n\to\infty} Tu_n = \lim_{n\to\infty} u_{n|\partial\Omega}.$$

The limit is independent of the approximating sequence (exercise). The map $T\colon W^{1,p}(\Omega) \to L^p(\partial\Omega)$ we have defined is bounded since

$$\|Tu\|_{L^p(\partial\Omega)} = \lim_{n\to\infty} \|Tu_n\|_{L^p(\partial\Omega)} \leq C \lim_{n\to\infty} \|u_n\|_{W^{1,p}(\Omega)} = C\|u\|_{W^{1,p}(\Omega)}.$$

The operator T we have constructed satisfies (a) and (b) (exercise, see also FA) Here we sketch the proof of (1.15). Let $u \in C^1(\overline{\Omega})$. our goal is to show $\|Tu\|_{L^p(\partial\Omega)} \leq C\ \|u\|_{W^{1,p}(\Omega)}$, and hence in particular $Tu \in L^p(\partial\Omega; \mathcal{H}^{d-1})$. By introducing a finite partition of unity we reduce the problem to study a function localized on a small piece of the boundary. We distisguish two cases.

Case 1. $\partial\Omega$ is C^1. In this case the boundary is locally flat, after an appropriate coordinate change. Assume $x_0 \in \partial\Omega$ and $r > 0$ are such that $\partial\Omega \cap B_r(x_0)$ is flat (without need of coordinate change), i.e.

$$\Omega \cap B_r(x_0) = \{x = (x', x_d) \in B_r(x_0) |\, x_d > 0\},$$

where $x' \in \mathbb{R}^{d-1}$. In particular $x_0 = (x'_0, 0)$, with $x'_0 \in \mathbb{R}^{d-1}$. We define

$$\Gamma := \partial\Omega \cap B_r(x_0) = \{(x', 0) \in B_r(x_0)\}.$$

Assume now $u \in C^1(\overline{\Omega})$ has support inside $B_r(x_0)$ (use the partition of unity). Then $\|u\|^p_{L^p(\Gamma)} = \int_{|x'-x'_0|<r} |u(x', 0)|^p dx'$. The idea now is to add one dimension by paying one derivative. Precisely

$$u(x', l) - u(x', 0) = \int_0^l \partial_{x_d} u(x', t)\, dt \qquad \forall l > 0 \text{ s.t. } [(x', 0), (x', l)] \subset \Omega.$$

Since $\operatorname{supp} u \subset B_r(x_0)$, then for each x' there exists a $l_{x'} > 0$ such that $[(x', 0), (x', l_{x'})] \subset \Omega$ and $u(x', l_{x'}) = 0$. Therefore

$$\begin{aligned} |u(x', 0)|^p &= \left| \int_0^{l_{x'}} \partial_{x_d} u(x', t) dt \right|^p \le \left(\int_0^{l_{x'}} |Du(x', t)| dt \right)^p = l^p_{x'} \left(\frac{1}{l_{x'}} \int_0^{l_{x'}} |Du(x', t)| dt \right)^p \\ &\le l^p_{x'} \frac{1}{l_{x'}} \int_0^{l_{x'}} |Du(x', t)|^p dt \le C \int_0^{l_{x'}} |Du(x', t)|^p dt \end{aligned} \tag{1.16}$$

where in the last steps we used Jensen's inequality and $l^{p-1}_{x'} \le C$ and $C = C_{\Omega,p} > 0$. It follows

$$\begin{aligned} \|u\|^p_{L^p(\Gamma)} &= \int_{|x'-x'_0|<r} |u(x', 0)|^p dx' \le C \int_{|x'-x'_0|<r} \int_0^{l_{x'}} |Du(x', t)|^p \, dt \, dx' \\ &\le C \int_\Omega |Du(x)|^p dx = C \|Du\|^p_{L^p(\Omega)}. \end{aligned}$$

This concludes the proof of Case 1.

Case 2 When $\partial\Omega$ is Lipschitz, we work directly on the integral. Let $x_0 \in \Omega$ $r > 0$ and $\gamma \colon \mathbb{R}^{d-1} \to \mathbb{R}$ a Lipschitz continuous function such that

$$\Omega \cap B_r(x_0) = \{x = (x', x_d) \in B_r(x_0) \,|\, x_d > \gamma(x')\}.$$

It holds $\partial\Omega \cap B_r(x_0) = \Phi(B^{d-1}_r(x'_0))$, where $\Phi \colon \mathbb{R}^{d-1} \to \mathbb{R}^d$ is defined via $\Phi(x') := (x', \gamma(x'))$ and is Lipschitz continuous. Assume $u \in C^1(\overline{\Omega})$ has support inside $B_r(x_0)$ (use the partition of unity). We argue, using the area formula,

$$\begin{aligned} &\int_{\partial\Omega \cap B_r(x_0)} |u(x)|^p d\mathcal{H}^{d-1}(x) = \int_{\Phi(B^{d-1}_r(x'_0))} |u(x)|^p d\mathcal{H}^{d-1}(x) \\ &= \int_{B^{d-1}_r(x'_0)} |u(x', \gamma(x'))|^p \sqrt{1 + |D\gamma(x')|^2} dx' \le (1 + \|D\gamma\|_{L^\infty}) \int_{B^{d-1}_r(x'_0)} |u(x', \gamma(x'))|^p dx' \\ &\le C' \int_{B^{d-1}_r(x'_0)} \int_0^{l_{x'}} |Du(x', t)|^p dt dx' \le C' \int_\Omega |Du(x)|^p dx = C' \|Du\|^p_{L^p(\Omega)} \end{aligned}$$

where in the last line we used (1.16) and we defined $C' := C(1 + \|D\gamma\|_{L^\infty})$. This concludes th proof of part (i).

(ii) ($\Rightarrow$) Assume $u \in W_0^{1,p}(\Omega)$.
Then there exists a sequence $n \mapsto u_n \in C_c^\infty(\Omega)$, such that $\|u - u_n\|_{W^{1,p}(\Omega)} \to 0$.
Since $u_n \in C_c^\infty(\Omega) \subset C^1(\overline{\Omega})$ it holds $Tu_n = u_{n|\partial\Omega} = 0\ \forall n$, and hence $Tu = \lim_{n\to\infty} Tu_n = 0$.

($\Leftarrow$) (sketch) Assume $Tu = 0$. Since $u \in W^{1,p}(\Omega)$ and $\partial\Omega$ is Lipschitz there is a sequence $n \mapsto u_n \in C^\infty(\overline{\Omega})$ such that $\lim_{n\to\infty} \|u - u_n\|_{W^{1,p}} = 0$.
Intuitively, $Tu = 0$ forces u to be "very small" near the boundary so we can replace u_n with $\tilde{u}_n \in C_c^\infty(\Omega)$ without changing the limit. Precisely, assuming $\Omega \cap B_r(x_0) = \{x = (x', x_d) \in B_r(x_0) |\, x_d > 0\}$, we have

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^p} \int_0^\varepsilon \int_{|x'-x_0'|<r} |u(x', x_d)| dt dx' = 0.$$

A similar result holds for Lipschitz boundary. For more details see Evans and FA lecture notes. $\square$

1.3.4 Extensions

Let $u \in W^{1,p}(\Omega)$ be a given function. Is it possible to define an extension $\tilde{u}$ of u such that $\tilde{u}_{|\Omega} = u$ and $\tilde{u} \in W^{1,p}(\mathbb{R}^d)$? If $u \in W_0^{1,p}(\Omega)$, it is enough to extend by zero, i.e. $\tilde{u} := \mathbf{1}_\Omega u \in W^{1,p}(\mathbb{R}^d)$ and satisfies $\tilde{u}_{|\Omega} = u$ and $D\tilde{u} = \mathbf{1}_\Omega Du$.
For a general function $u \in W^{1,p}(\Omega)$, $\mathbf{1}_\Omega u \in L^p(\mathbb{R}^d)$, but is not weakly differentiable. The solution is to let the function $\tilde{u}$ take non zero values on a set V a bit larger than Ω.

Theorem 1.15. *Let $1 \le p < \infty$, $\Omega \subset \mathbb{R}^d$ open and bounded with Lipschitz boundary. Then for each $V \subset \mathbb{R}^d$ open and bounded set with $\overline{\Omega} \subset V$, there exists a linear operator*

$$E\colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$$

such that, $\forall u \in W^{1,p}(\Omega)$ it holds

(i) $Eu(x) = u(x)$ *for a.e.* $x \in \Omega$,

(ii) $\operatorname{supp} E(u) \subset V$,

(iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^d)} \le C\|u\|_{W^{1,p}(\Omega)}$, *where $C = C_{p,\Omega,V} > 0$ is a constant. In particular this means the operator E is bounded.*

Eu is called an extension of u to $\mathbb{R}^d$.

Proof. Here we consider the proof in the case Ω has C^1 boundary (for the general case see lecture notes FA).
Since Ω has C^1 boundary, the problem can be reduced (after introducing a partition of unity, and eventually a coordinate change) to study a flat piece of boundary.
Assume $x_0 \in \partial\Omega$, $r > 0$ is such that $B_r(x_0) \cap \partial\Omega$ is flat i.e.

$$B^+ := B_r(x_0) \cap \Omega = \{x \in B_r(x_0) |\, x_d > 0\}.$$

We also define $B_- := \{x \in B_r(x_0) |\, x_d < 0\}$.

Let $u \in W^{1,p}(\Omega)$. We assume that $\operatorname{supp} u \subset B_r(x_0) \cap \overline{\Omega}$. The strategy is to extend u first to a function $\tilde{u} \in W^{1,p}(\mathbb{R}^d)$, with support in the whole ball $B_r(x_0)$. From $\tilde{u}$ we then construct an extension Eu with support in V.

Step 1: extension to the ball. We define

$$\tilde{u}(x) = \begin{cases} u_+(x) & x \in \Omega \\ u_-(x) & x \in \overline{\Omega}^c, \end{cases}$$

with

$$u_+(x) := u(x) = u(x', x_d), \qquad u_-(x) = u_-(x', x_d) := u(x', -x_d) = u \circ R(x', x_d),$$

where $x' \in \mathbb{R}^{d-1}$ and R is the reflection operator $R(x', x_d) := (x', -x_d)$.
It holds $u_+ \in W^{1,p}(\Omega)$ and $u_- \in W^{1,p}(\overline{\Omega}^c)$. Moreover $\operatorname{trace}(u_+) = \operatorname{trace}(u_-)$
(Idea: $\exists n \mapsto u_n \in C^1(\overline{\Omega})$ converging to u in $W^{1,p}(\Omega)$. Then $n \mapsto u_n \circ R \in C^1(\Omega^c)$ and converges to u_- in $W^{1,p}(\overline{\Omega}^c)$. The result follows from $u_{n|\partial\Omega} = (u_n \circ R)_{n|\partial\Omega}$).
It follows (Exercise 1.2) that $\tilde{u} \in W^{1,p}(\mathbb{R}^d)$ with

$$D\tilde{u}(x) = \begin{cases} Du_+(x) & x \in \Omega \\ Du_-(x) & x \in \overline{\Omega}^c. \end{cases}$$

Moreover $\|\tilde{u}\|^p_{W^{1,p}(\mathbb{R}^d)} = \|u_+\|^p_{W^{1,p}(\Omega)} + \|u_-\|^p_{W^{1,p}(\overline{\Omega}^c)} = 2\|u\|_{W^{1,p}(\Omega)}$.

Step 2: extension to V. There exists a function $\zeta \in C_c^\infty(V)$ such that $\zeta \geq 0$ and $\zeta_{|\Omega} = 1$. We define then

$$Eu := \zeta\tilde{u}.$$

With this definition $Eu \in W^{1,p}(\mathbb{R}^d)$, $Eu_{|\Omega} = u$, and $\operatorname{supp} Eu \subset V$. Finally

$$\begin{aligned}
\|Eu\|_{L^p(\mathbb{R}^d)} &\leq \|u\|_{L^p(B^+)} + \|\zeta u_-\|_{L^p(B^-)} \leq (1 + \|\zeta\|_{L^\infty(\mathbb{R}^d)})\|u\|_{L^p(\Omega)}, \\
\|D(Eu)\|_{L^p(\mathbb{R}^d)} &\leq \|Du\|_{L^p(B^+)} + \|D(\zeta u_-)\|_{L^p(B^-)} \\
&\leq (1 + \|\zeta\|_{L^\infty(\mathbb{R}^d)})\|Du\|_{L^p(B^+)} + \|D\zeta\|_{L^\infty(\mathbb{R}^d)}\|u\|_{L^p(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}.
\end{aligned}$$

□

1.3.5 Sobolev embeddings

Theorem 1.16 (Rellich II)**.** *Let $\Omega \subset \mathbb{R}^d$ open and bounded with Lipschitz boundary, $1 \leq p < \infty$ and $k \geq 1$.*
Let $n \mapsto u_n \in W^{k,p}(\Omega)$ be a bounded sequence, and $u \in W^{k-1,p}(\Omega)$ such that $u_n \rightharpoonup u$ in $W^{k-1,p}(\Omega)$.

Then $u_n \to u$ strongly in $W^{k-1,p}(\Omega)$.

Proof. See lecture notes on FA. □

Theorem 1.17 (Sobolev embedding)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and bounded with Lipschitz boundary. The following statements hold.*

(i) Assume $1 \leq p < d$. Then $W^{1,p}(\Omega) \subset L^q(\Omega)$ $\forall 1 \leq q \leq p^*$, where $p^* := \frac{pd}{d-p}$.

The embedding

$$\begin{aligned} I: \quad & W^{1,p}(\Omega) \to L^q(\Omega) \\ & u \mapsto I(u) := u \end{aligned}$$

is continuous, i.e.

$$\|u\|_{L^q(\Omega)} \leq C\ \|u\|_{W^{1,p}(\Omega)} \quad \forall u \in W^{1,p}(\Omega),$$

where the constant C depends on p, q and Ω.

Moreover, if $1 \leq q < p^*$, the injection is also compact.

(ii) Assume $1 \leq d < p < \infty$. Then $W^{1,p}(\Omega) \subset C^{0,\beta}(\Omega)$ $\forall 0 < \beta \leq \alpha := 1 - \frac{d}{p}$.

This means, that each $u \in W^{1,p}(\Omega)$ has a representative $\tilde{u} \in C^{0,\beta}(\Omega)$.

The embedding

$$I_\beta\colon \quad \begin{aligned} &W^{1,p}(\Omega) \to C^{0,\beta}(\Omega)\\ &[u] \mapsto I([u]) := \tilde{u}\end{aligned}$$

is continuous, and

$$[\tilde{u}]_{C^{0,\beta}} \leq C\|u\|_{W^{1,p}(\Omega)} \quad \forall u \in W^{1,p}(\Omega), \tag{1.17}$$

where the constant C depends on p, β and Ω.

Moreover, if $\beta < \alpha$, the embedding is also compact.

Remark. In the Sobolev inequality above, the norm of the gradient $\|\nabla u\|_{L^p(\Omega)}$ (appearing in the case of $W_0^{1,p}$) is now replaced by the Sobolev norm $\|u\|_{W^{1,p}(\Omega)}$. Indeed the inequality $\|u\|_{L^q(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}$ does hold for constant functions.

Proof. Sketch in the case $p < d$.
Fix $V \subset \mathbb{R}^d$ open and bounded with $\overline{\Omega} \subset V$. Since $\partial\Omega$ is Lipschitz there exists an extension operator $E\colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$, such that for all $u \in W^{1,p}(\Omega)$,

$$Eu_{|\Omega} = u, \qquad \|Eu\|_{W^{1,p}(\mathbb{R}^d)} \leq C\|u\|_{W^{1,p}(\Omega)},$$

and $E(u)$ has support in V. Then, by Theorem 1.7, and the extension theorem,

$$\|u\|_{L^{p^*}(\Omega)} = \|Eu\|_{L^{p^*}(\Omega)} \leq \|Eu\|_{L^{p^*}(\mathbb{R}^d)} \leq C\,\|D(Eu)\|_{L^p(\mathbb{R}^d)} \leq C\|Eu\|_{W^{1,p}(\mathbb{R}^d)} \leq C'\,\|u\|_{W^{1,p}(\Omega)}.$$

□

[3: 16.10.2023]

[4: 19.10.2023]

2 Elliptic partial differential equations of order 2

2.1 Weak formulation

In this section we always assume $\Omega \subset \mathbb{R}^d$ open and bounded with Lipschitz boundary. We will look for solutions $u\colon \overline{\Omega} \to \mathbb{R}$ of the system

$$\begin{cases} Lu = f & \text{in } \Omega\\ u_{|\partial\Omega} = g \end{cases} \tag{2.1}$$

where L is a linear partial partial differential operator of order 2

$$Lu(x) = -\sum_{ij=1}^{d} a_{ij}(x)\partial_i\partial_j u(x) + \sum_{j=1}^{d} b_j(x)\partial_j u(x) + c(x)u(x),$$

$f\colon \Omega \to \mathbb{R}$ is the non-homogeneous term, the coefficients $a\colon \Omega \to \mathbb{R}^{d\times d}_{sym}$, $b\colon \Omega \to \mathbb{R}^d$ and $c\colon \Omega \to \mathbb{R}$ are matrix-valued, vector-valued and scalar-valued functions respectively. Finally $g\colon \partial\Omega \to \mathbb{R}$ gives the boundary value of u. The system (2.1) is called a *Dirichlet boundary value problem.*

Regularity. In order for the PDE above to make sense we need at least $u \in C^2(\Omega)$.

Divergence and non-divergence form Assume $a \in C^1(\Omega)$. We can reorganize the second order derivatives as follows

$$\begin{aligned} -\sum_{ij=1}^{d} a_{ij}(x)\partial_i\partial_j u(x) &= -\sum_i \partial_i\Big[\sum_j a_{ij}\partial_j u\Big](x) + \sum_j \Big[\sum_i \partial_i a_{ij}(x)\Big]\partial_j u(x) \\ &= -\mathrm{div}\,(aDu)(x) + \sum_j \Big[\sum_i \partial_i a_{ij}(x))\Big]\partial_j u(x) \end{aligned}$$

Hence we can write Lu in two ways

$$L(u) = \begin{cases} -\mathrm{Tr}\,[a\partial\otimes\partial]u + b\cdot Du + cu & \text{non-divergence form} \\ -\mathrm{div}\,(aDu) + \tilde{b}\cdot Du + cu & \text{divergence form,} \end{cases}$$

where $\tilde{b}_j = b_j + \sum_i \partial_i a_{ij}$.

Definition 2.1.

(i) L is called elliptic if $a(x) > 0$ for a.e. $x \in \Omega$.
[i.e. $(\xi, a(x)\xi) > 0 \forall \xi \in \mathbb{R}^d$ $\xi \neq 0$]

(ii) L is called uniformly elliptic if there exists a constant $\theta > 0$ such that $a(x) \geq \theta\,\mathrm{Id}$ for a.e. $x \in \Omega$.

Remark 1. $a(x) \in \mathbb{R}^{d\times d}_{sym}$, hence $a(x)$ is diagonalizable with real eigenvalues and an o.n. basis of eigenvectors. In the eigenvector basis the matrix is diagonal $a(x) = \mathrm{diag}\,(\lambda_1(x), \ldots, \lambda_d(x))$. In this basis we have

$$(\xi, a(x)\xi) = \sum_j \lambda_j(x)\xi_j^2.$$

Therefore $a(x) > 0$ iff $\lambda_j(x) > 0$ $\forall j$. In particular $a(x) > 0$ iff $a(x) \geq \theta_x \mathrm{Id}$, where $\theta_x = \min_j \lambda_j(x) > 0$.
a is uniformly elliptic iff the eigevalues of $a(x)$ are bounded away from zero *uniformly* in x.

Remark 2. If $a = Id, b = c = 0$ then $Lu = -\Delta u$. The operator $-\Delta$ is then uniformly elliptic with $\theta = 1$.

Weak formulation: version 1. Assume L is in divergence form $Lu = -\mathrm{div}\,(aDu)+b\cdot Du+cu$ and $a_{ij}, b_j, c, f, u \in C^\infty(\overline{\Omega})$.
If $Lu = f$ we have

$$\int_\Omega \xi(x)Lu(x)\,dx = \int_\Omega \xi(x)f(x)\,dx \qquad \forall \xi \in C_c^\infty(\Omega).$$

Integrating by parts we obtain

$$-\int_\Omega \xi \mathrm{div}\,(au)\,dx = -\int_\Omega \mathrm{div}\,(\xi aDu)\,dx + \int_\Omega D\xi \cdot aDu\,dx = \int_\Omega D\xi \cdot aDu\,dx,$$

where the boundary contribution disappears since $\xi \in C_c^\infty(\Omega)$. Hence

$$\int_\Omega [D\xi \cdot aDu\,dx + \xi(b\cdot Du + cu)]\,dx = \int_\Omega \xi f dx \qquad \forall \xi \in C_c^\infty(\Omega).$$

The integrals above remain well defined also when $a_{ij}, b_j, c \in L^\infty$, $u, \xi \in W^{1,2}(\Omega)$ $f \in L^2(\Omega)$.
The boundary of Ω is Lipschitz, hence $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$. Therefore we can replace $u \in C^\infty(\overline{\Omega})$ by $u \in H^1(\Omega)$. Since $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, we can replace $\xi \in C_c^\infty(\Omega)$ by $\xi \in H_0^1(\Omega)$.
To properly encode the boundary value, note that, since $\partial\Omega$ is Lipschitz, $\mathrm{Tr} : H^1(\Omega) \to L^2(\partial\Omega)$ is well defined. Hence, if $g \in L^2(\partial\Omega, \mathcal{H}^{d-1})$ we replace $u_{|\partial\Omega} = g$ by $\mathrm{Tr}\,u = g$.

Remark. If we use $\xi \in C^\infty(\overline{\Omega})$, we obtain an additional term

$$\int_\Omega \mathrm{div}\,(\xi aDu)\,dx = \int_{\partial\Omega} \xi(x)(aDu)(x)\cdot \nu_x\,d\mathcal{H}^{d-1}.$$

But $Du_{|\partial\Omega}$ is not well defined for $u \in H^1(\Omega)$. This term disappears if instead of Dirichlet we require homogeneous Neuman boundary conditions $(aDu)(x)\cdot\nu_x = 0$ for a.e. $x \in \partial\Omega$.

Definition 2.2. *Assume* $a_{ij}, b_j, c \in L^\infty$, $\forall i,j = 1,\dots,d$, $f \in L^2(\Omega)$, *and* $g \in L^2(\partial\Omega, \mathcal{H}^{d-1})$.

(i) The bilinear form B_L *associated to the formal differential operator* $Lu = -\mathrm{div}\,(aDu) + b\cdot Du + cu$ *is defined by*

$$\begin{aligned} B_L\colon \quad & H^1(\Omega)\times H^1(\Omega) \to \mathbb{R} \\ & (u,v) \mapsto B[u,v] := \textstyle\int_\Omega [aDu\cdot Dv + (b\cdot Du + cu)v]\,dx. \end{aligned} \tag{2.2}$$

(ii) A function $u \in H^1(\Omega)$ *is called a weak solution of* (2.1) *if*

$$\begin{cases} B[u,v] = (f,v)_{L^2(\Omega)} & \forall v \in H^1_0(\Omega) \\ \mathrm{Tr}\,u = g. \end{cases} \tag{2.3}$$

Remark 1: zero boundary value. Assume we have zero boundary condition $g = 0$. Then $u \in H^1(\Omega)$ is a weak solution of $Lu = f$ in Ω with $u_{|\partial\Omega} = 0$ iff $B_L[u,v] = (f,v)_{L^2(\Omega)}$ $\forall v \in H^1_0(\Omega)$ and $\mathrm{Tr}\,u = 0$.
By Theorem (1.14), $\mathrm{Tr}\,u = 0$ iff $u \in H^1_0(\Omega)$.
Hence $u \in H^1(\Omega)$ is a weak solution of $Lu = f$ in Ω with $u_{|\partial\Omega} = 0$ iff $u \in H^1_0(\Omega)$ and $B_L[u,v] = (f,v)_{L^2(\Omega)}$ $\forall v \in H^1_0(\Omega)$.

Remark 2: reducing to zero boundary value. Can we always reduce to the case $g = 0$? Consider $u \in H^1(\Omega)$ a weak solution of $Lu = f$ in Ω with $u_{|\partial\Omega} = g$. Assume $u_0 \in H^1(\Omega)$ is a function satisfying $\mathrm{Tr}\,u_0 = g$. Then the function $w := u - u_0 \in H^1_0(\Omega)$ and solves the (formal) equation $Lw = f - Lu_0$ in Ω with $w_{|\partial\Omega} = 0$.
The non homogeneous term f has to be replaced by

$$\tilde{f} = f - Lu_0 = f + \mathrm{div}\,(aDu_0) - b\cdot Du_0 - cu_0.$$

Note that, while $b\cdot Du_0 + cu_0 \in L^2(\Omega)$ for all $u_0 \in H^1(\Omega)$, the term $\mathrm{div}\,(aDu_0)$ is not well defined. The corresponding integral formulation has to be rearranged as follows

$$\int_\Omega fv\,dx \to \int_\Omega \tilde{f}v\,dx = \int_\Omega [v(f - b\cdot Du_0 - cu_0) - Dv\cdot aDu_0]\,dx = \int_\Omega [f_0 v + \sum_j f_j\partial_j v]\,dx$$

where

$$f_0 := f - b\cdot Du_0 - cu_0, \qquad f_j := -(aDu_0)_j.$$

Note that $f_0, f_j \in L^2(\Omega)$ for all $f \in L^2(\Omega)$, $a,b,c \in L^\infty(\Omega)$ and $u_0 \in H^1(\Omega)$. Therefore we need to replace

$$(f,v)_{L^2(\Omega)} \to (f_0,v)_{L^2(\Omega)} + \sum_{j=1}^d (f_j,\partial_j v)_{L^2(\Omega)}.$$

This motivates the following more general definition of weak solution.

Definition 2.3. *(Weak solution version 2) For* $f = (f_0, f_1, \dots, f_d) \in L^2(\Omega)^{d+1}$ *we define*

$$\begin{aligned} \langle f,\cdot\rangle\colon \quad & H^1_0(\Omega) \to \mathbb{R} \\ & v \mapsto \langle f,v\rangle := (f_0,v)_{L^2(\Omega)} + \textstyle\sum_{j=1}^d (f_j,\partial_j v)_{L^2(\Omega)}. \end{aligned} \tag{2.4}$$

We say that $u \in H^1_0(\Omega)$ is a weak solution of

$$\begin{cases} Lu = f_0 - \sum_j \partial_j f_j & \text{in } \Omega \\ u_{|\partial\Omega} = 0 \end{cases}$$

if

$$B_L[u,v] = \langle f, v\rangle \qquad \forall v \in H^1_0(\Omega). \tag{2.5}$$

Note that $f_0 - \sum_j \partial_j f_j$ is a formal expression only. In the special case $f = (f_0, 0, \dots, 0)$ we obtain $\langle f, v\rangle = (f_0, v)_{L^2}$, hence we are back to the first version of weak formulation.

Lemma 2.4. *Let $f \in L^2(\Omega), g \in L^2(\partial\Omega, \mathcal{H}^{d-1})$, with $g \in Range(\mathrm{Tr}\,)$, i.e. $\exists u_0 \in H^1(\Omega)$ such that $\mathrm{Tr}\, u_0 = g$. Consider the two systems*

$$(a)\begin{cases} Lu = f & \text{in } \Omega \\ u_{|\partial\Omega} = g \end{cases} \qquad (b)\begin{cases} Lu = \tilde f := f_0 - \sum_j \partial_j f_j & \text{in } \Omega \\ u_{|\partial\Omega} = 0 \end{cases},$$

with $f_0 := f - b\cdot Du_0 - cu_0, f_j := -(aDu_0)_j$. The following holds:

$u \in H^1(\Omega)$ is a weak solution of (a) iff $\tilde u := u - u_0 \in H^1_0(\Omega)$ is a weak solution of (b), i.e.

$$\mathrm{Tr}\, u = g \quad \text{and} \quad B_L[u,v] = (f,v)_{L^2(\Omega)}\ \forall v \in H^1_0(\Omega) \qquad \Leftrightarrow \qquad B_L[\tilde u, v] = \langle \tilde f, v\rangle\ \forall v \in H^1_0(\Omega).$$

Proof. The proof follows from $\mathrm{Tr}\,\tilde u = \mathrm{Tr}\, u - \mathrm{Tr}\, u_0 = g - g = 0$ iff $\tilde u \in H^1_0(\Omega)$, and

$$B_L[\tilde u, v] = B_L[u,v] - B_L[u_0, v] = (f,v)_{L^2(\Omega)} - \int_\Omega [Dv\cdot aDu_0 + v(b\cdot Du_0 + cu_0)]\,dx = \langle \tilde f, v\rangle.$$

□

Remark. Let $f = (f_0, f_1, \dots, f_d) \in L^2(\Omega)^{d+1}$. Then $\langle f, \cdot\rangle \in H^1_0(\Omega)^* = H^1_0(\Omega)'$ (dual space) and

$$\|\langle f,\cdot\rangle\|_{H^1_0(\Omega)^*} \le \|f\|_{L^2(\Omega)^{d+1}} = \left(\sum_{j=0}^d \|f\|^2_{L^2(\Omega)}\right)^{\frac12}.$$

Notation: we often write $H^{-1}(\Omega) := H^1_0(\Omega)^*$. The following theorem gives the precise relation between $H^{-1}(\Omega)$ and $L^2(\Omega)^{d+1}$.

Theorem 2.5 (dual space of $H^1_0(\Omega)$)**.**
Let $\Omega \subset \mathbb{R}^d$, be an open set, not necessarily bounded.
Let $\sim$ be the equivalence relation on $L^2(\Omega)^{d+1}$ defined by

$$f \sim g \qquad \Leftrightarrow \qquad \langle f, v\rangle = \langle g, v\rangle \quad \forall v \in H^1_0(\Omega).$$

Then $H^1_0(\Omega)^* =: H^{-1}(\Omega) = L^2(\Omega)^{d+1}/\sim$, i.e.

$$\forall T \in H^{-1}(\Omega) \quad \exists!\ [f] \in L^2(\Omega)^{d+1}/\sim \quad s.t. \quad T = \langle g, \cdot\rangle \quad \forall g \in [f].$$

Moreover

$$\|T\|_{H^{-1}(\Omega)} = \inf\{\|f\|_{L^2(\Omega)^{d+1}} \mid f \in L^2(\Omega)^{d+1}, T = \langle f, \cdot\rangle\}$$

Proof.
We have already proved above that $\langle f, \cdot\rangle \in H^{-1}(\Omega)\ \forall f \in L^2(\Omega)^{d+1}$.
Let now $T \in H^{-1}(\Omega)$. Since $H_0^1 \simeq (H_0^1)^*$ there exists a unique function $f_T \in H_0^1(\Omega)$ such that

$$T(v) = (f_T, v)_{H^1(\Omega)} \qquad \forall v \in H_0^1(\Omega).$$

We have

$$(f_T, v)_{H^1(\Omega)} = (f_T, v)_{L^2(\Omega)} + \sum_{j=1}^{d} (\partial_j f_T, \partial_j v)_{L^2(\Omega)} = \langle \tilde{f}_T, v\rangle,$$

where $\tilde{f}_T = (f_T, \partial_1 f_T, \ldots, \partial_d f_T)$. Therefore $T(\cdot) = \langle f, \cdot\rangle\ \forall f \in [\tilde{f}_T]$.

Finally we have

$$\|T\|_{H^{-1}(\Omega)} = \|\langle f, \cdot\rangle\|_{H^{-1}(\Omega)} \le \|f\|_{L^2(\Omega)^{d+1}} \qquad \forall f \in [\tilde{f}_T],$$

hence

$$\|T\|_{H^{-1}(\Omega)} \le \inf\{\|f\|_{L^2(\Omega)^{d+1}} \mid f \in L^2(\Omega)^{d+1}, T = \langle f, \cdot\rangle\}.$$

Equality is obtained noting that

$$\|T\|_{H^{-1}(\Omega)} = \|f_T\|_{H_0^1(\Omega)} = \|\tilde{f}_T\|_{L^2(\Omega)^{d+1}}$$

where $\|T\|_{H^{-1}(\Omega)} = \|f_T\|_{H_0^1(\Omega)}$ holds since the map $\Phi\colon H^{-1}(\Omega) \to H_0^1(\Omega)$ is an isometry. □

[4: 19.10.2023]

[5: 23.10.2023]

Remark 1. For all $f = (f_0, 0, \ldots, 0)$, with $f_0 \in L^2(\Omega)$ we have $\langle f, \cdot\rangle = (f_0, \cdot)_{L^2(\Omega)} \in H^{-1}(\Omega)$. Hence $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$.

Remark 2: coupling. The map

$$\begin{aligned} \langle \cdot, \cdot\rangle_{H^{-1}H_0^1}\colon \quad & L^2(\Omega)^{d+1}/\sim\ \times H_0^1(\Omega) \to \mathbb{R} \\ & (f, v) \mapsto \langle f, v\rangle := (f_0, v)_{L^2(\Omega)} + \textstyle\sum_{j=1}^d (f_j, \partial_j v)_{L^2(\Omega)} \end{aligned} \tag{2.6}$$

defines a *coupling* between $\langle f, \cdot\rangle \in H^{-1}(\Omega)$ and $v \in H_0^1(\Omega)$. This coupling is bounded

$$|\langle f, v\rangle| \le \ \|\langle f, \cdot\rangle\|_{H^{-1}(\Omega)} \|v\|_{H_0^1(\Omega)}.$$

In the following we will use often the notation $\langle \cdot, \cdot\rangle_{H^{-1}H_0^1}$ instead of $\langle \cdot, \cdot\rangle$ to stress the coupling structure.

2.2 Existence of weak solutions

2.2.1 Energy estimates and first existence theorem

Remember that, given $f \in L^2(\Omega)^{d+1}$, the function $u \in H_0^1(\Omega)$ is a weak solution of

$$\begin{cases} Lu = f_0 - \sum_j \partial_j f_j & \text{in } \Omega \\ u_{|\partial\Omega} = 0 \end{cases} \tag{2.7}$$

if

$$B_L[u, v] = \langle f, v\rangle_{H^{-1}H_0^1} \qquad \forall v \in H_0^1(\Omega).$$

Our basic tool to investigate existence and uniqueness of solutions is Lax-Milgram.

Theorem 2.6 (Lax-Milgram)**.** *Let H be a $\mathbb{K}-$Hilbert space (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and $B\colon H \times H \to \mathbb{K}$ is a sesquilinear form (i.e. linear in the second variable and antilinear in the first). Suppose in addition that there exist constants $\alpha, \beta > 0$ such that*

(i) (continuity) $\quad |B[u,v]| \leq \alpha \|u\|_H \,\|v\|_H \quad \forall u, v \in H$;

(ii) (coercivity, positivity) $\quad \operatorname{Re} B[u,u] \geq \beta \|u\|_H^2 \quad \forall u \in H$.

Then $\forall T \in H^* \ \exists!\ u_T \in H$ such that

$$B[u_T, v] = T(v) \quad \forall v \in H. \tag{2.8}$$

Proof. See lecture notes on FA or Evans. □

In our case we have a real Hilbert space $H = H_0^1(\Omega)$ and a bilinear form $B = B_L$.

Remark 1. B_L always satisfies (i). Indeed

$$\begin{aligned}|B_L[u,v]| &= \left| \int_\Omega [aDu \cdot Dv + (b \cdot Du + cu)v]\, dx \right| \\ &\leq \|a\|_{L^\infty(\Omega)} \|Du\|_{L^2(\Omega)} \|Dv\|_{L^2(\Omega)} + \|b\|_{L^\infty(\Omega)} \|Du\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|c\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\leq C \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}\end{aligned}$$

where $\|a\|_{L^\infty(\Omega)} := \sum_{ij} \|a_{ij}\|_{L^\infty(\Omega)}$, $\|b\|_{L^\infty(\Omega)} := \max_j \|b_j\|_{L^\infty(\Omega)}$, and $C > 0$ is some constant depending on a, b, c.

Remark 2. B_L does not satisfy (ii) in general! As an example consider $a(x) := \mathrm{Id}$, $b_j(x) := b_0 x_j$ and $c(x) := c_0$, with $b_0 > 0$ and $c_0 := b_0 d/4$. For $u \in C_c^1(\Omega) \subset H_0^1(\Omega)$ we argue

$$\int_\Omega u(x)(b \cdot Du)(x) dx = b_0 \int_\Omega x \cdot D(\frac{u^2}{2}) dx = -b_0 d \int_\Omega \frac{u^2}{2} dx = -\frac{b_0 d}{2} \|u\|_{L^2(\Omega)}^2,$$

hence we obtain

$$B_L[u,u] = \|Du\|_{L^2(\Omega)}^2 - \frac{b_0 d}{4} \|u\|_{L^2(\Omega)}^2 \qquad \forall u \in C_c^1(\Omega).$$

Given a $u_0 \in C_c^1(\Omega)$ we can always find $b_0 > 0$ large enough such that $B_L[u_0, u_0] < 0$.

Remark 3. Assume

- $a(x) \geq \theta \mathrm{Id}$, for a.e. $x \in \Omega$ with $\theta > 0$ (i.e. L is uniformly elliptic),
- $c(x) \geq 0$ for a.e. $x \in \Omega$ and
- $b = 0$.

Then

$$B_L[u,u] = \int_\Omega aDu \cdot Du\, dx + \int_\Omega cu^2\, dx \geq \theta \|Du\|_{L^2(\Omega)}^2 + \int_\Omega cu^2 dx \geq \theta \|Du\|_{L^2(\Omega)}^2.$$

By Poincaré inequality it follows $B_L[u,u] \geq C' \|u\|_{H_0^1(\Omega)}^2$, for some $C' > 0$.
Hence, Lax-Milgram ensures that $\forall f \in L^2(\Omega)^{d+1}/\sim$ there exists a unique weak solution of (2.7).

In the general case, we only have "almost coercivity". This is the content of the next theorem.

Theorem 2.7 (energy estimates).
Consider the formal differential operator $Lu = -\mathrm{div}\,(aDu) + b\cdot Du + cu$ *with* $a_{ij}, b_j, c \in L^\infty(\Omega)$, *and assume* a *is uniformly elliptic, i.e.* $a(x) \geq \theta \mathrm{Id}$, *for a.e.* $x \in \Omega$ *with* $\theta > 0$.

Then $\exists \alpha, \beta > 0$ *and* $\gamma \geq 0$ *such that* $\forall u, v \in H^1_0(\Omega)$ *the following inequalities hold:*

(i) $|B_L[u,v]| \leq \alpha \|u\|_{H^1_0(\Omega)} \|v\|_{H^1_0(\Omega)}$,

(ii) $B_L[u,u] \geq \beta \|u\|^2_{H^1_0(\Omega)} - \gamma \|u\|^2_{L^2(\Omega)}$.

Proof.
(i) see Remark 1 above.

(ii) We compute

$$\begin{aligned}
B_L[u,u] &= \int_\Omega aDu\cdot Du dx + \int_\Omega [ub\cdot Du + cu^2]\,dx \\
&\geq \theta \|Du\|^2_{L^2(\Omega)} - \|b\|_{L^\infty(\Omega)} \|Du\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} - \|c\|_{L^\infty(\Omega)} \|u\|^2_{L^2(\Omega)} \\
&\geq \left(\theta - \tfrac{\varepsilon \|b\|_{L^\infty(\Omega)}}{2}\right) \|Du\|_{L^2(\Omega)} - \left(\tfrac{\|b\|_{L^\infty(\Omega)}}{2\varepsilon} + \|c\|_{L^\infty(\Omega)}\right) \|u\|^2_{L^2(\Omega)}
\end{aligned}$$

where in the last line we used Young's inequality $\|Du\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \leq \frac{\varepsilon}{2}\|Du\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon}\|u\|^2_{L^2(\Omega)}$. We choose now $\varepsilon > 0$ small enough such that

$$\theta - \tfrac{\varepsilon \|b\|_{L^\infty(\Omega)}}{2} \geq \frac{\theta}{2}.$$

Setting $\gamma := \frac{\|b\|_{L^\infty(\Omega)}}{2\varepsilon} + \|c\|_{L^\infty(\Omega)} \geq 0$ we obtain

$$B_L[u,u] \geq \frac{\theta}{2}\|Du\|^2_{L^2(\Omega)} - \gamma \|u\|^2_{L^2(\Omega)}.$$

By Poincaré inequality it follows

$$B_L[u,u] \geq \beta \|u\|^2_{H^1_0(\Omega)} - \gamma \|u\|^2_{L^2(\Omega)},$$

for some $\beta > 0$. □

Theorem 2.8 (first existence theorem for weak solutions).
Consider the formal differential operator $Lu = -\mathrm{div}\,(aDu) + b\cdot Du + cu$ *with* $a_{ij}, b_j, c \in L^\infty(\Omega)$, *and assume* a *is uniformly elliptic, i.e.* $a(x) \geq \theta \mathrm{Id}$, *for a.e.* $x \in \Omega$ *with* $\theta > 0$.

Then there exists a constant $\gamma \geq 0$ *such that* $\forall \mu \geq \gamma, f \in L^2(\Omega)^{d+1}/\sim$ *there exists a unique weak solution* $u \in H^1_0(\Omega)$ *of*

$$\begin{cases} Lu + \mu u = f_0 - \sum_j \partial_j f_j & \text{in } \Omega \\ u_{|\partial\Omega} = 0 \end{cases} \tag{2.9}$$

Proof. By Theorem 2.7, $\exists \alpha, \beta > 0$ and $\gamma \geq 0$ such that
$|B_L[u,v]| \leq \alpha \|u\|_{H^1_0(\Omega)} \|v\|_{H^1_0(\Omega)}$, and
$B_L[u,u] + \gamma \|u\|^2_{L^2(\Omega)} \geq \beta \|u\|^2_{H^1_0(\Omega)}$ $\forall u, v \in H^1_0(\Omega)$.

$L + \mu\mathrm{Id}$ is the operator obtained from L by replacing the coefficient $c(x)$ by $c(x) + \mu$. A function $u \in H^1_0(\Omega)$ is a weak solution of (2.9) if $B_{L+\mu\mathrm{Id}}[u,\cdot] = \langle f, \cdot\rangle_{H^{-1}H^1_0}$, where

$$B_{L+\mu\mathrm{Id}}[u,v] = B_L[u,v] + \mu(u,v)_{L^2(\Omega)}.$$

Then $\forall u, v \in H_0^1(\Omega)$ $\mu \geq \gamma$, we have

$$|B_{L+\mu \mathrm{Id}}[u,v]| \leq |B_L[u,v]| + \mu |(u,v)_{L^2(\Omega)}| \leq (\alpha+\mu)\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)},$$
$$B_{L+\mu \mathrm{Id}}[u,u] = B_L[u,u] + \mu\|u\|^2_{L^2(\Omega)} \geq \beta\|u\|^2_{H_0^1(\Omega)} + (\mu-\gamma)\|u\|^2_{L^2(\Omega)} \geq \beta\|u\|^2_{H_0^1(\Omega)}.$$

The result now follows by Lax-Milgram. □

2.2.2 Fredholm dychotomy and second existence theorem

Inverse of L: rigorous formulation. Intuitively the equation $Lu = f$ has a solution if L is "invertible". Lax-Milgram says that, if $L > 0$ then L is invertible. The energy estimate says that, for L uniformly elliptic, there is a $\gamma \geq 0$ such that $L + \gamma \mathrm{Id} > 0$, and hence $L + \mu \mathrm{Id}$ is invertible $\forall \mu \geq \gamma$. To make the notion of L^{-1} precise we need some definitions.

We can associate to the formal differential operator $Lu = -\mathrm{div}\,(aDu) + b \cdot Du + cu$ at least two linear operators $T_L\colon H_0^1(\Omega) \to H^{-1}(\Omega)$ and $\hat{T}_L\colon H_0^1(\Omega) \to H_0^1(\Omega)$. We mostly work with T_L. The second operator $\hat{T}_L$ is more convenient to define the adjoint of L.

The operator T_L. We define

$$\begin{aligned} T_L\colon \quad & H_0^1(\Omega) \to H^{-1}(\Omega) \\ & u \mapsto T_L(u) := B_L[u,\cdot], \end{aligned} \tag{2.10}$$

By continuity of B_L we have

$$\|T_L(u)\|_{H^{-1}(\Omega)} = \sup_{\|v\|_{H_0^1(\Omega)}=1} |B_L[u,v]| \leq \alpha\|u\|_{H_0^1(\Omega)},$$

hence T_L is linear and bounded with $\|T_L(u)\|_{op} \leq \alpha$.

With these definitions, $u \in H_0^1(\Omega)$ is a weak solution of $Lu = f_0 - \sum_j \partial_j f_j$ iff $T_L(u) = B_L[u,\cdot] = \langle f, \cdot \rangle_{H^{-1}H_0^1}$. Existence of weak solutions can be now formulated as follows: *for all $f \in L^2(\Omega)^{d+1}/\sim$ there exists a unique weak solution iff the operator T_L is invertible.*

Assume T_L is invertible. The inverse satisfies the following properties.

(i) $T_L^{-1}\colon H^{-1}(\Omega) \to H_0^1(\Omega)$ is linear and bounded (from the inverse operator theorem, FA)

(ii) The operator $T^{-1}_{L|L^2}\colon L^2(\Omega) \to L^2(\Omega)$ defined via

$$T^{-1}_{L|L^2}(f) := T_L^{-1}((f,\cdot)_{L^2(\Omega)}) \in H_0^1(\Omega) \subset L^2(\Omega), \tag{2.11}$$

is <u>compact</u>. This follows from the fact that the operator $L^2(\Omega) \to H_0^1(\Omega)$ defined via $f \mapsto T_L^{-1}((f,\cdot)_{L^2(\Omega)}) \in H_0^1(\Omega)$ is bounded and the injection $I\colon H_0^1(\Omega) \to L^2(\Omega)$ is compact (use weak compactness and Rellich, see FA). The compactness of $T^{-1}_{L|L^2}$ will be crucial to prove existence results.

The operator $\hat{T}_L$. We define

$$\begin{aligned} \hat{T}_L\colon \quad & H_0^1(\Omega) \to H_0^1(\Omega) \\ & u \mapsto \hat{T}_L(u) := \Phi^{-1}(T_L(u)), \end{aligned} \tag{2.12}$$

where $\Phi\colon H_0^1(\Omega) \to H^{-1}(\Omega)$ is the standard bijective isometry $\Phi(u) := (u, \cdot)_{H_0^1(\Omega)}$. Then $\hat{T}_L(u)$ is the unique vector $u_L \in H_0^1(\Omega)$ such that

$$\Phi(u_L) = (u_L, \cdot)_{H_0^1(\Omega)} = T_L(u) = B_L[u, \cdot].$$

Since Φ is an isometry $\hat{T}_L$ is linear and bounded with $\|\hat{T}_L(u)\|_{op} = \|T_L(u)\|_{op} \leq \alpha$. Moreover, T_L is invertible iff $\hat{T}_L$ is invertible. In this case $\hat{T}_L^{-1}$ is linear and bounded with $\|\hat{T}_L^{-1}\|_{op} = \|T_L^{-1}\|_{op}$.

Adjoint operator. The adjoint of $\hat{T}_L\colon H_0^1(\Omega) \to H_0^1(\Omega)$ is the unique linear bounded operator $\hat{T}_L^\dagger\colon H_0^1(\Omega) \to H_0^1(\Omega)$ satisfying

$$(\hat{T}_L u, v)_{H_0^1(\Omega)} = (u, \hat{T}_L^\dagger v)_{H_0^1(\Omega)} \qquad \forall u, v \in H_0^1(\Omega).$$

If T_L is invertible, then the adjoint of $T_{L|L^2}^{-1}\colon L^2(\Omega) \to L^2(\Omega)$ is the unique linear bounded operator $T_{L|L^2}^{-1\dagger}\colon L^2(\Omega) \to L^2(\Omega)$ satisfying

$$(T_{L|L^2}^{-1} f, g)_{L^2(\Omega)} = (f, T_{L|L^2}^{-1\dagger}, g)_{L^2(\Omega)} \qquad \forall f, g \in L^2(\Omega).$$

The following lemma summarizes important properties of these adjoints.

Lemma 2.9.
Consider the formal differential operator $Lu = -\operatorname{div}(aDu) + b \cdot Du + cu$ with $a_{ij}, b_j, c \in L^\infty(\Omega)$. The following holds.

(i) $\hat{T}_L^\dagger = \hat{T}_{L^*}$ *with*
$$L^* v := -\operatorname{div}(aDv) - b \cdot Dv + (c - \operatorname{div} b).$$

(ii) $B_L[u, v] = B_{L^*}[v, u]$ $\forall u, v \in H_0^1(\Omega)$.

(iii) Assume L is uniformly elliptic. Then L^* is uniformly elliptic. Moreover the parameters α, β, γ from the energy estimate are the same for L and L^*.

(iv) Assume $T_L\colon H_0^1(\Omega) \to H^{-1}(\Omega)$ is invertible and consider $T_L^{-1}{}_{|L^2}\colon L^2(\Omega) \to L^2(\Omega)$. Then the adjoint operator on $L^2(\Omega)$ satisfies
$$T_{L|L^2}^{-1\dagger} = T_{L^*|L^2}^{-1}.$$

Remark 1. For $b \in L^\infty(\Omega)$, $\operatorname{div} b$ is not well defined. Hence $v \operatorname{div} b$ is a formal expression, that makes sense only after integrating by parts, just as in the case of $\operatorname{div}(aDu)$.

Remark 2. While a function $u \in H_0^1(\Omega)$ is a weak solution of $Lu = f_0 - \sum_j \partial_j f_j$ with $u_{|\Omega} = 0$ if

$$B_L[u, v] = \langle f, v \rangle_{H^{-1} H_0^1} \qquad \forall v \in H_0^1(\Omega), \tag{2.13}$$

a function $v \in H_0^1(\Omega)$ is a weak solution of $L^* v = f_0 - \sum_j \partial_j f_j$ with $v_{|\Omega} = 0$ if

$$B_{L^*}[v, u] = B_L[u, v] = \langle f, u \rangle_{H^{-1} H_0^1} \qquad \forall u \in H_0^1(\Omega). \tag{2.14}$$

[5: 23.10.2023]

[6: 26.10.2023]

Proof.
$(i) + (ii)$ By construction $B_L[u,v] = (\hat{T}_L u, v)_{H_0^1(\Omega)} = (u, \hat{T}_L^\dagger v)_{H_0^1(\Omega)} = (\hat{T}_L^\dagger v, u)_{H_0^1(\Omega)}$. It is then enough to find L^* such that

$$B_{L^*}[v,u] = (\hat{T}_{L^*} v, u)_{H_0^1(\Omega)} = (\hat{T}_L^\dagger v, u)_{H_0^1(\Omega)} = B_L[u,v] \qquad \forall u, v \in H_0^1(\Omega).$$

Performing integration by parts and using $a^T = a$, we get

$$\begin{aligned} B_L[u,v] &= \int_\Omega [aDu \cdot Dv + (b \cdot Du + cu)v]\, dx \\ &= \int_\Omega [(aDv) \cdot Du + (-\mathrm{div}\,(bv) + cv)u]\, dx \\ &= \int_\Omega u\,[-\mathrm{div}\,(aDv) - \mathrm{div}\,(bv) + cv]\, dx = \int_\Omega u\, L^* v\, dx, \end{aligned}$$

where the last two integrals make sense only when $aDv, bv \in H^1(\Omega)$.

(iii) We have

$$\begin{aligned} &|B_{L^*}[v,u]| = |B_L[u,v]| \leq \alpha \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} \\ &B_{L^*}[u,u] + \gamma \|u\|^2_{L^2(\Omega)} = B_L[u,u] + \gamma \|u\|^2_{L^2(\Omega)} \geq \beta \|u\|^2_{H_0^1(\Omega)}. \end{aligned}$$

(iv) Our goal is to show that

$$(T^{-1}_{L|L^2} f, g)_{L^2(\Omega)} = (f, T^{-1}_{L^*|L^2} g)_{L^2(\Omega)} \qquad \forall f, g \in L^2(\Omega).$$

Note that

$$\begin{aligned} &(a)\ u_f := T^{-1}_{L|L^2} f \quad \text{satisfies} \quad B_L[u_f, \cdot] = (f, \cdot)_{L^2(\Omega)} \\ &(b)\ u_g^* := T^{-1}_{L^*|L^2} g \quad \text{satisfies} \quad B_{L^*}[u_g^*, \cdot] = B_L[\cdot, u_g^*] = (g, \cdot)_{L^2(\Omega)}. \end{aligned}$$

We argue

$$(T^{-1}_{L|L^2} f, g)_{L^2(\Omega)} = (u_f, g)_{L^2(\Omega)} = (g, u_f)_{L^2(\Omega)} \overset{(b)}{=} B_L[u_f, u_g^*] \overset{(a)}{=} (f, u_g^*)_{L^2(\Omega)} = (f, T^{-1}_{L^*|L^2} g).$$

This concludes the proof. □

Theorem 2.10 (second existence theorem for weak solutions)**.**
Consider the formal differential operator $Lu = -\mathrm{div}\,(aDu) + b \cdot Du + cu$ with $a_{ij}, b_j, c \in L^\infty(\Omega)$, and assume a is uniformly elliptic, i.e. $a(x) \geq \theta \mathrm{Id}$, for a.e. $x \in \Omega$ with $\theta > 0$.

Then exactly one of the following holds.

(α) T_L is invertible i.e. $\forall f \in L^2(\Omega)^{d+1}/\sim\ \exists!\ u \in H_0^1(\Omega)$ weak solution of the non-homogeneous problem

$$(*)_f := \begin{cases} Lu = f_0 - \sum_j \partial_j f_j & \text{in } \Omega \\ u_{|\partial\Omega} = 0 \end{cases}$$

(β) $\ker T_L \neq \{0\}$ i.e. $\exists$ at least one $u \in H_0^1(\Omega)$, $u \neq 0$, weak solution of the homogeneous problem

$$(**) := \begin{cases} Lu = 0 & \text{in } \Omega \\ u_{|\partial\Omega} = 0 \end{cases}$$

This is called the <u>Fredholm alternative</u> or Fredholm dychotomy.

Moreover, remember that

$$\ker T_L := \{u \in H_0^1(\Omega) |\, u \text{ weak solution of } Lu = 0 \text{ in } \Omega, u_{|\partial\Omega} = 0\},$$
$$\ker T_{L^*} := \{u \in H_0^1(\Omega) |\, u \text{ weak solution of } L^*u = 0 \text{ in } \Omega, u_{|\partial\Omega} = 0\}.$$

Then, if (β) *holds, we have* $\dim \ker T_L = \dim \ker T_{L^*} < \infty$ *and* $(*)_f$ *has a weak solution iff*

$$\langle f, v\rangle_{H^{-1}H_0^1} = 0 \qquad \forall v \in \ker T_{L^*}.$$

Proof.
Case 1: $\gamma = 0$. In this case we can apply Lax-Milgram to show that T_L is invertible, i.e. we are in case (α).

Case 2: $\gamma > 0$.
$u \in H_0^1(\Omega)$ is a weak solution of $(*)_f$ iff $B_L[u, \cdot] = \langle f, \cdot\rangle_{H^{-1}H_0^1}$, iff

$$T_{L+\gamma \mathrm{Id}}\,(u) = B_{L+\gamma \mathrm{Id}}\,[u, \cdot] = \langle f, \cdot\rangle_{H^{-1}H_0^1} + (\gamma u, \cdot)_{L^2(\Omega)} = \langle g_u, \cdot\rangle_{H^{-1}H_0^1},$$

where $g_u := (f_0 + \gamma u, f_1, \ldots, f_d) \in L^2(\Omega)^{d+1}$. By Theorem 2.8, $T_{L+\gamma \mathrm{Id}}$ is invertible, hence $u \in H_0^1(\Omega)$ is a weak solution of $(*)_f$ iff u satisfies the fixed-point equation

$$u := T^{-1}_{L+\gamma \mathrm{Id}}\,(\langle g_u, \cdot\rangle) = T^{-1}_{L+\gamma \mathrm{Id}}\,(\langle f, \cdot\rangle) + \gamma T^{-1}_{L+\gamma \mathrm{Id}}\,((u, \cdot)_{L^2(\Omega)}) = G_f + Ku$$

where we defined $G_f := T^{-1}_{L+\gamma \mathrm{Id}}\,(\langle f, \cdot\rangle)$, and $K = K_{L,\gamma} \colon L^2(\Omega) \to L^2(\Omega)$ is the operator defined by

$$K(u) = K_{L,\gamma}(u) := \gamma T^{-1}_{L+\gamma \mathrm{Id}}\,((u, \cdot)_{L^2(\Omega)}) = \gamma T^{-1}_{L+\gamma \mathrm{Id}\,|L^2}(u).$$

Note that, since $K(L^2(\Omega)) \subset H_0^1(\Omega)$ and $G_f \in H_0^1(\Omega)$ we have

$$u \in L^2(\Omega) \ \text{ solution of } u = G_g + Ku \Rightarrow u \in H_0^1(\Omega).$$

Then

$$\begin{aligned} u \text{ weak solution of } (*)_f \quad &\Leftrightarrow \quad (\mathrm{Id} - K)u = G_f \\ u \text{ weak solution of } (**) \quad &\Leftrightarrow \quad (\mathrm{Id} - K)u = 0. \end{aligned}$$

The second line implies that

$$\ker T_L = \ker(\mathrm{Id} - K) = \ker(\mathrm{Id} - K_{L,\gamma}), \qquad \ker T_{L^*} = \ker(\mathrm{Id} - K_{L^*,\gamma}). \tag{2.15}$$

Moreover, using Lemma 2.9 (iv), we have

$$K_{L*,\gamma} = \gamma T^{-1}_{L^*}{}_{|L^2(\Omega)} = \gamma (T^{-1}_{L}{}_{|L^2(\Omega)})^{\dagger} = K^{\dagger}_{L,\gamma} = K^{\dagger}.$$

Hence

$$\ker T_{L^*} = \ker(\mathrm{Id} - K_{L^*,\gamma}) = \ker(\mathrm{Id} - K^{\dagger}). \tag{2.16}$$

The operator K is compact, hence $\mathrm{Id} - K$ is a Fredholm operator of index zero (see FA or Appendix in Evans) and therefore exactly one of the following hold

(a) $\mathrm{Id} - K$ is invertible (hence in particular $(\mathrm{Id} - K)^{-1}$ is linear and bounded)

(b) $\ker(\mathrm{Id} - K) \neq \{0\}$ and $\mathrm{Ran}\,(\mathrm{Id} - K) = \overline{\mathrm{Ran}\,(\mathrm{Id} - K)} \subsetneq L^2(\Omega)$.

If (a) holds, then $\forall f \in L^2(\Omega)^{d+1}/\sim$ there is a unique weak solution $u := (\mathrm{Id} - K)^{-1} G_f$ of $(*)_f$, hence we are in case (α).
If (b) holds, then there is at least one function $u \in \ker(\mathrm{Id} - K)$, $u \neq 0$. Since u is also a weak solution of $(**)$ we are in case (β).

Assume now (β) holds. Since $\mathrm{Id} - K$ is a Fredholm operator of index zero we have (see Appendix in Evans)

(i) $\dim\ker(\mathrm{Id} - K) < \infty$,

(ii) $\big(\ker(\mathrm{Id} - K^\dagger)\big)^\perp = \mathrm{Ran}\,(\mathrm{Id} - K)$,

(iii) $\dim\ker(\mathrm{Id} - K^\dagger) = \dim\ker(\mathrm{Id} - K)$.

$(i) + (iii)$ together with (2.15) and (2.16) imply $\dim\ker T_L = \dim\ker T_{L^*} < \infty$.

Finally, let $f \in L^2(\Omega)^{d+1}$, $f \neq 0$. Our goal is to show that u is a weak solution of $(*)_f$ iff $\langle f, v\rangle_{H^{-1}H_0^1} = 0\ \forall v \in \ker T_{L^*}$.

Indeed, for each $v \in \ker T_{L^*}$, we have

$$\begin{aligned}\langle f, v\rangle_{H^{-1}H_0^1} &= B_{L+\gamma\mathrm{Id}}[G_f, v] = B_L[G_f, v] + \gamma(G_f, v)_{L^2(\Omega)} \\ &= B_{L^*}[v, G_f] + \gamma(G_f, v)_{L^2(\Omega)},\end{aligned}$$

where in the first equality we used the definition of G_f. In particular, for $v \in \ker T_{L^*}$ we have $T_{L^*}(v) = B_{L^*}[v, \cdot] = 0$, hence

$$\langle f, v\rangle_{H^{-1}H_0^1} = \gamma(G_f, v)_{L^2(\Omega)} \qquad \forall v \in \ker T_{L^*}. \tag{2.17}$$

We will use this identity to prove the two implications.
($\Rightarrow$) Assume u is a weak solution of $(*)_f$.
Then $(\mathrm{Id} - K)u = G_f$ and hence $G_f \in \mathrm{Ran}\,(\mathrm{Id} - K) = \big(\ker(\mathrm{Id} - K^\dagger)\big)^\perp = (N^*)^\perp$. Hence

$$(G_f, v)_{L^2(\Omega)} = 0 \qquad \forall v \in \ker T_{L^*},$$

abd the result follows from (2.17).
($\Leftarrow$) Assume $\langle f, v\rangle_{H^{-1}H_0^1} = 0\ \forall v \in \ker T_{L^*}$. Then, using (2.17) and $\gamma > 0$

$$0 = (G_f, v)_{L^2(\Omega)} \qquad \forall v \in \ker T_{L^*},$$

and hence $G_f \in (\ker T_{L^*})^\perp = \mathrm{Ran}\,(\mathrm{Id} - K)$. Therefore, there exists at least one $u \in H_0^1(\Omega)$ such that $(\mathrm{Id} - K)u = G_f$, i.e. u is a weak solution of $(*)_f$.

□

Remark 1 Assume (β) holds, i.e. $N \neq \{0\}$.

- Let $v \in N$, $v \neq 0$. If u is a weak solution of $(*)_f$ then $u + \lambda v$ is also a weak solution, for all $\lambda \in \mathbb{R}$. Hence the problem $(*)_f$ has either no weak solution or infinitely many of them.
- There exists at least one $f \in L^2(\Omega)^{d+1}$ such that $(*)_f$ has no weak solution. Indeed suppose a solution exists for all $f \in L^2(\Omega)^{d+1}$, and let $v \in N^*$. It holds
$$\langle f, v\rangle_{H^{-1}H_0^1} = 0 \qquad \forall f \in L^2(\Omega)^{d+1}$$
i.e. $T(v) = 0\ \forall T \in H^{-1}(\Omega)$. This implies $(u, v)_{H^1(\Omega)} = 0\ \forall u \in H_0^1(\Omega)$ and hence $v = 0$. It follows $N^* = \{0\}$ and hence, since $\dim N^* = \dim N$, $N = \{0\}$ which contradicts the assumption we are in case (β).

Remark 2 If $u \neq 0$ is a weak solution of $Lu = 0$ in Ω, $u_{|\partial\Omega} = 0$, then u can be seen as eigenvector of L for the eigenvalue zero. We will make this precise in the next subsection.

[6: 26.10.2023]

[7: 30.10.2023]

2.2.3 Spectrum of L and third existence theorem

Remember (FA): let X be a complex Banach space and $T \in \mathcal{L}(X)$. The spectrum of T is the set

$$\sigma(T) := \{\lambda \in \mathbb{C} | \, T - \lambda \mathrm{Id} \text{ not invertible } \}$$

We distinguish three types of spectrum:

$$\begin{aligned}
\sigma_p(T) &:= \{\lambda \in \mathbb{C} | \ker(T - \lambda \mathrm{Id}\,) \neq \{0\}\} \\
\sigma_c(T) &:= \{\lambda \in \mathbb{C} | \ker(T - \lambda \mathrm{Id}\,) = \{0\}, \ \mathrm{Ran}\,(T - \lambda \mathrm{Id}\,) \subsetneq \overline{\mathrm{Ran}\,(T - \lambda \mathrm{Id}\,)} = X\} \\
\sigma_r(T) &:= \{\lambda \in \mathbb{C} | \ker(T - \lambda \mathrm{Id}\,) = \{0\}, \ \overline{\mathrm{Ran}\,(T - \lambda \mathrm{Id}\,)} \subsetneq X\}
\end{aligned}$$

In our case we are only interested in real valued weak solutions of $Lu = f_0 - \sum_j \partial_j f_j$ where $u = -\mathrm{div}\,(aDu) + b \cdot Du + cu$.
Set $\lambda \in \mathbb{R}$. A function $u \in H_0^1(\Omega)$ is a weak solution of the formal PDE $Lu = \lambda u$, with b.c. $u_{|\partial\Omega} = 0$, iff $B_L[u, \cdot] = \lambda(u, \cdot)_{L^2}$ iff $\ker T_{L-\lambda \mathrm{Id}} \neq \{0\}$.
Assume L is uniformly elliptic. Then $L - \lambda \mathrm{Id}$ is also uniformly elliptic and, by Fredholm alternative, we have: $\ker T_{L-\lambda \mathrm{Id}} \neq \{0\} \Leftrightarrow T_{L-\lambda \mathrm{Id}}$ is not invertible.
This motivates the following definition.

Definition 2.11 (real spectrum of L)**.** *Consider the formal differential operator $Lu = -\mathrm{div}\,(aDu) + b \cdot Du + cu$ with $a_{ij}, b_j, c \in L^\infty(\Omega)$, and L uniformly elliptic.*

The real spectrum of L is the set

$$\begin{aligned}
\Sigma(L) &:= \{\lambda \in \mathbb{R} | \, T_{L-\lambda \mathrm{Id}} \textit{ not invertible } \} \\
&= \{\lambda \in \mathbb{R} | \ker T_{L-\lambda \mathrm{Id}} \neq \{0\}\} = \{\lambda \in \mathbb{R} | \, \lambda \textit{ is an eigenvalue of } L\}.
\end{aligned}$$

Remark Note that $\hat{T}_L \in \mathcal{L}(H_0^1(\Omega))$ but $\Sigma(L) \neq \{\lambda \in \mathbb{R} | \ \hat{T}_L - \lambda \mathrm{Id} \text{ not invertible } \}$. Indeed $\lambda \in \Sigma(L)$ iff $T_{L-\lambda \mathrm{Id}}$ is not invertible iff $\hat{T}_{L-\lambda \mathrm{Id}}$ is not invertible, but $\hat{T}_{L-\lambda \mathrm{Id}} \neq \hat{T}_L - \lambda \mathrm{Id}$.
To see this note that

$$\Phi(\hat{T}_{L-\lambda \mathrm{Id}}\,(u)) = T_{L-\lambda \mathrm{Id}}\,(u) = B_L[u, \cdot] + \lambda(u, \cdot)_{L^2} \neq B_L[u, \cdot] + \lambda(u, \cdot)_{H_0^1} = \Phi(\hat{T}_L(u) - \lambda u).$$

Theorem 2.12 (third existence theorem for weak solutions)**.** *Consider the formal differential operator $Lu = -\mathrm{div}\,(aDu) + b \cdot Du + cu$ with $a_{ij}, b_j, c \in L^\infty(\Omega)$, and L uniformly elliptic. Let α, β, γ be the constants from the energy bounds.*

Let $\Sigma(L) \subset \mathbb{R}$ be the real spectrum of L. The following hold.

(i) $\Sigma(L) \subset (-\gamma, \infty)$

(ii) $\Sigma(L)$ *is finite or countable.*

(iii) If Σ is infinite, then $\Sigma = \{\lambda_k\}_{k=1}^\infty$, with $-\gamma < \lambda_k \leq \lambda_{k+1}$ $\forall k \geq 1$ and $\lim_{k\to\infty} \lambda_k = \infty$.

Proof.
(i) By the first existence theorem 2.8, the operator $T_{L+\mu \mathrm{Id}}$ is invertible $\forall \mu \geq \gamma$, hence for all $\lambda \leq -\gamma$. It follows $\Sigma \subset (-\gamma, \infty)$.

$(ii) + (iii)$ We assume now $\lambda + \gamma > 0$. $\lambda \in \Sigma \Leftrightarrow \lambda$ is a real eigenvalue $\Leftrightarrow \exists u \in H_0^1(\Omega)$, $u \neq 0$, weak solution of $Lu = \lambda u$, $u_{|\partial\Omega} = 0 \Leftrightarrow \exists u \in H_0^1(\Omega)$, $u \neq 0$, such that $B_L[u,v] = \lambda(u,v)_{L^2(\Omega)}$ $\forall v \in H_0^1(\Omega) \Leftrightarrow \exists u \in H_0^1(\Omega)$, $u \neq 0$, such that

$$B_{L+\gamma \mathrm{Id}}[u,v] = B_L[u,v] + \gamma(u,v)_{L^2(\Omega)} = (\lambda+\gamma)(u,v)_{L^2(\Omega)} \qquad \forall v \in H_0^1(\Omega).$$

Since $T_{L+\gamma \mathrm{Id}}$ is invertible (true by Theorem 2.8) this holds iff $\exists u \neq 0$ solution of the fixed point equation

$$u = (\lambda+\gamma) T^{-1}_{L+\gamma \mathrm{Id}}\Big((u,\cdot)_{L^2}\Big).$$

We define $K\colon L^2(\Omega) \to L^2(\Omega)$ via

$$K(f) := T^{-1}_{L+\gamma \mathrm{Id}}\Big((f,\cdot)_{L^2}\Big). \tag{2.18}$$

Note that K is compact and $K(L^2(\Omega)) \subset H_0^1(\Omega)$. Therefore, $\lambda \in (-\gamma, \infty)$ is a real eiganvalue iff $\exists u \in L^2(\Omega)$ $u \neq 0$ solution of

$$Ku = \frac{1}{\lambda+\gamma} u,$$

i.e. $(\lambda+\gamma)^{-1}$ is an eigenvalue for the operator K.
We extend K to the operator $K^e \in \mathcal{L}(L^2(\Omega;\mathbb{C}))$ defined via $K^e(u+iv) := K(u) + iK(v)$, $\forall u, v \in L^2(\Omega)$.
Using $\overline{K^e(u+iv)} = K^e(\overline{u+iv})$ we have

$$\lambda \in \sigma_p(K^e) \cap \mathbb{R} \Leftrightarrow \exists u \in L^2(\Omega), u \neq 0 \text{ st } Ku = \lambda u,$$

hence

$$\lambda \in \Sigma(L) \quad \Leftrightarrow \quad \frac{1}{\lambda+\gamma} \in \sigma_p(K^e) \cap \mathbb{R} \setminus \{0\}. \tag{2.19}$$

Since K is compact, K^e is compact and therefore (see lecture notes in FA): $\sigma(K^e) \setminus \{0\} = \sigma_p(K^e) \setminus \{0\}$, $\sigma(K^e)$ is finite or countable and the only accumulation point is zero. Moreover, for all $\lambda \in \sigma_p(K^e) \setminus \{0\}$ we have $\dim\ker(K^e - \lambda \mathrm{Id}) < \infty$. (i) and (iii) now follow from (2.19). $\square$

Remark 1. Both T_L and $\hat{T}_L$ are bounded operators, but the spectrum of L is unbounded. This is not a contradiction. Indeed

$$\|T_L\|_{op} = \sup_{u \in H_0^1(\Omega), u \neq 0} \frac{\|T_L(u)\|_{H^{-1}}}{\|u\|_{H_0^1}} = \sup_{u,v \in H_0^1(\Omega), u,v \neq 0} \frac{|B_L[u,v]|}{\|u\|_{H_0^1}\|v\|_{H_0^1}} \leq \alpha,$$

where in the last inequality we used the energy estimate. Assume $\Sigma(L)$ is infinite, so that $\Sigma(L) = \{\lambda_k\}_{k=1}^\infty$, with $\lambda_k \leq \lambda_{k+1}$ $\forall k \geq 1$ and $\lim_{k\to\infty} \lambda_k = \infty$. Let $u_k \in H_0^1(\Omega)$ be an eigenvector associated to λ_k. This means $B_L[u_k, v] = \lambda_k(u_k, v)_{L^2(\Omega)}$ $\forall v \in H_0^1(\Omega)$. Inserting $u = v = u_k$, in the estimate for $\|T_L\|_{op}$ we obtain

$$\alpha \geq \|T_L\|_{op} \geq \frac{|B_L[u_k,u_k]|}{\|u_k\|^2_{H_0^1}} = |\lambda_k| \frac{\|u_k\|^2_{L^2}}{\|u_k\|^2_{H_0^1}}.$$

Note that the norm in the numerator differs from the norm in the denominator. If they where the same, then we would get $|\lambda_k| \leq \alpha \ \forall k$ which in turn would imply the real spectrum is bounded. Now, we can for example normalize u_k such that $\|u_k\|_{H^1_0} = 1$. It follows, using $\lim_{k\to\infty} \lambda_k = \infty$,

$$\lim_{k\to\infty} \|u_k\|_{L^2} = 0.$$

This means $\|u_k\|^2_{L^2} \to 0$ and $\|Du_k\|^2_{L^2} \to 1$, which is possible if u_k is strongly oscillating.

Remark 2. For each $\lambda \in \mathbb{R} \setminus \Sigma$ the operator $T_{L-\lambda \mathrm{Id}} : H^1_0(\Omega) \to H^{-1}(\Omega)$ is invertible. Since $T_{L-\lambda \mathrm{Id}}$ is linear and bounded, the inverse $T^{-1}_{L-\lambda \mathrm{Id}} : H^{-1}(\Omega) \to H^1_0(\Omega)$ is linear and bounded. Moreover

$$\lim_{\mathrm{dist}\,(\lambda,\Sigma(L))\to 0} \|T^{-1}_{L-\lambda \mathrm{Id}}\|_{op} = \infty.$$

(exercise)

Symmetric elliptic operators. Set $b = 0$ i.e $Lu = -\mathrm{div}\,(aDu) + cu$. Then $L^* = L$ and we say the operator L is symmetric. The following hold:

• For $L = L^*$, the bilinear form associated to L is symmetric

$$B_L[u, v] = B_L[v, u] \qquad \forall u, v \in H^1_0(\Omega).$$

As a consequence, if L is uniformly elliptic $B_{L+\gamma \mathrm{Id}}[\cdot,\cdot]$ defines an inner product on $H^1_0(\Omega)$.

• If L is uniformly elliptic we also have:

$$K^\dagger = (T^{-1}_{L+\gamma \mathrm{Id}\,|L^2})^\dagger = T^{-1}_{(L+\gamma \mathrm{Id}\,)^*|L^2} = T^{-1}_{L+\gamma \mathrm{Id}\,|L^2} = K,$$

where we used $(L + \gamma)^* = L^* + \gamma = L + \gamma$. It follows $K^{e\dagger} = K^e$ and hence $\sigma(K^e) \subset \mathbb{R}$.

Theorem 2.13.
Consider the formal differential operator $Lu = -\mathrm{div}\,(aDu) + cu$ where $a_{ij}, c \in L^\infty(\Omega)$.
Assume L is uniformly elliptic and $\gamma = 0$ i.e. $B_L[u, u] \geq \beta \|u\|^2_{H^1_0} \ \forall u \in H^1_0(\Omega)$.
The following hold.

(i) $B_L[\cdot,\cdot]$ *defines an inner product on* $H^1_0(\Omega)$.

(ii) $\Sigma(L) = \{\lambda_n\}_{n=1}^\infty$, *with* $0 < \lambda_n \leq \lambda_{n+1} \ \forall n$ *and* $\lim_{n\to\infty} \lambda_n = \infty$.

(iii) $\exists$ *and o.n. basis* $\{e_n\}_{n=1}^\infty$ *of* $(H^1_0(\Omega), B_L)$ *such that* e_n *is a weak solution of* $Le_n = \lambda_n e_n$, *with* $e_{n|\partial\Omega} = 0$, $\forall n \geq 1$.

(iv) $\lambda_1 > 0$ *is called principal eigenvalue and can be computed via the following variational formula*

$$\lambda_1 = \min_{u\in H^1_0(\Omega), \|u\|_{L^2(\Omega)}=1} B_L[u, u] = \min_{u\in H^1_0(\Omega), u\neq 0} \frac{B_L[u, u]}{\|u\|^2_{L^2(\Omega)}}.$$

Proof.
(i) follows from the symmetry of B_L and the energy estimates.

$(ii)+(iii)$ Since $\gamma = 0$ we have $K = T^{-1}_{L|L^2(\Omega)}$ and $\Sigma(L)$ is finite of countable with $\Sigma(L) \subset (0,\infty)$. Since $K^{e\dagger} = K^e$ we have, using the 3rd existence theorem,

$$\sigma(K^e) = \sigma(K^e) \cap \mathbb{R} \subset [-\gamma, \infty) = [0,\infty) = \{0\} \cup \frac{1}{\Sigma(L)}.$$

Moreover $\ker K^e = \ker K = \{0\}$ and hence, using the spectral theorem for compact self-adjoint operators, there exists an o.n. basis $\{e_n\}_{n=1}^\infty$ of $L^2(\Omega)$ such that $Ke_n = \frac{1}{\lambda_n} e_n$ where $\lambda_n \in \Sigma(L)$ for all $n \geq 1$. From $\dim \ker(K^e - \lambda_n^{-1}\mathrm{Id}\,) < \infty \ \forall n$ it follows that $\Sigma(L)$ is infinite and hence (ii) holds.

It remains to show that $\{e_n\}_{n=1}^\infty$ is also an o.n. basis for $(H_0^1(\Omega), B_L)$. From $Ke_n = \frac{1}{\lambda_n} e_n$ it follows that $e_n \in H_0^1(\Omega)$ satisfies $B_L[e_n, \cdot] = \lambda_n (e_n, \cdot)_{L^2(\Omega)}$ and hence

$$B_L[e_n, e_m] = \lambda_n (e_n, e_m)_{L^2(\Omega)} = \lambda_n \delta_{nm}.$$

Therefore $\{\lambda_n^{-\frac{1}{2}} e_n\}_{n=1}^\infty$ is an o.n. family in $(H_0^1(\Omega), B_L)$. To see it is also a basis note that

$$B_L[u, e_n] = \sum_{k=1}^\infty (u, e_k)_{L^2} B_L[e_k, e_n] = \lambda_n (u, e_n)_{L^2},$$

hence

$$B_L[u, e_n] = 0 \ \forall n \quad \Rightarrow \quad (u, e_n)_{L^2} = 0 \ \forall n \quad \Rightarrow \quad u = 0,$$

where in the last step we used that $\{e_n\}_{n=1}^\infty$ is basis for $L^2(\Omega)$. $\square$

2.3 Weak solutions in unbounded domains

In this section we inquire if the above results remains valid in unbounded domain $\Omega = \mathbb{R}^d$. The bilinear form associated to L becomes

$$\begin{aligned} B_L\colon \quad & H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d) \to \mathbb{R} \\ & (u,v) \mapsto B[u,v] := \textstyle\int_{\mathbb{R}^d} [aDu \cdot Dv + (b \cdot Du + cu)v]\, dx, \end{aligned} \tag{2.20}$$

where we used $H^1(\mathbb{R}^d) = H_0^1(\mathbb{R}^d)$. The mapping of $H^1(\mathbb{R}^d)^*$ into $L^2(\mathbb{R}^d)^{d+1}$ given by Thm 2.5 works also in infinite domain. Given $f \in L^2(\mathbb{R}^d)^{d+1}$, a function $u \in H^1(\mathbb{R}^d)$ is a weak solution of $Lu = f_0 - \sum_j \partial_j f_j$ if

$$B_L[u,v] = \langle f, \cdot \rangle_{H^{-1}H^1(\mathbb{R}^d)} \quad \forall v \in H^1(\mathbb{R}^d).$$

The energy etimates 2.7 work also for $\Omega = \mathbb{R}^d$ (exercise), but the constant γ is generally worse. Indeed while for $L = -\Delta$ on bounded domain we have $\gamma = 0$ using Poincaré inequality, this does not hold on $\mathbb{R}^d$. In this last case we argue

$$B_L[u,u] = \|Du\|^2_{L^2(\mathbb{R}^d)} = \|u\|^2_{H^1(\mathbb{R}^d)} - \|u\|^2_{L^2(\mathbb{R}^d)},$$

hence we need $\gamma = 1$. Since the energy estimates work, also the first existence theorem holds. On the contrary, the second existence theorem does not hold in general, while the injection $I\colon H^1(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is <u>not compact</u>.

To recover compactness, we can modify the regularity of the coefficients in L. An important example is the *Schrödinger operator*, constructed as follows.

Consider $V\colon \mathbb{R}^d \to \mathbb{R}$ a measurable function and assume V is bounded below i.e. $V(x) \geq C$ $\forall x \in \mathbb{R}^d$ for some constant $C \in \mathbb{R}$. The formal differential operator $Hu := -\Delta u + Vu$ is the Schrödinger operator with multiplicative potential V. The bilinear map

$$(u,v)_{H^1_V} := (u,v)_{H^1(\mathbb{R}^d)} + \int_{\mathbb{R}^d} (V-C)\, uv\, dx$$

is well defined on

$$H^1_V := \{u \in H^1(\mathbb{R}^d) |\ \sqrt{V-C}u \in L^2(\mathbb{R}^d)\}.$$

The pair $(H^1_V, (\,,)_{H^1_V})$ is a real Hilbert space (exercise). A function $u \in H^1_V$ is a weak solution of $Hu = f_0 - \sum_j \partial_j f_j$ iff

$$(Du, Dv)_{L^2(\mathbb{R}^d)} + \int_{\mathbb{R}^d} V\, uv\, dx = B_H[u,v] = \langle f, v\rangle \qquad \forall v \in H^1_V.$$

Again, the energy estimates and the first existence theorem hold. Assume now $\lim_{|x|\to\infty} V(x) = \infty$. Then the injection $I\colon H^1_V \to L^2(\mathbb{R}^d)$ is compact and the second and third existence theorem hold (see exercise sheet).

[7: 30.10.2023]

[8: 2.11.2023]

2.4 Regularity theory

2.4.1 Preliminary definitions and estimates

Assume $\Omega \subset \mathbb{R}^d$, open and bounded. We consider the formal differential operator $Lu = -\mathrm{div}\,(aDu) + b \cdot Du + cu$ with $a_{ij}, b_j, c \in L^\infty(\Omega)$, and a uniformly elliptic, i.e. $a(x) \geq \theta \mathrm{Id}$, for a.e. $x \in \Omega$ with $\theta > 0$.

Assume $u \in H^1_0(\Omega)$ is a weak solution of $Lu = f$ in Ω with $u_{|\partial\Omega} = 0$ and $f \in L^2(\Omega)$. Depending on the regularity of a, b, c and f we will show that u may be more regular than just H^1. The key idea is to bound norms for higher order derivatives by norms of lower order ones.

Example 1. Consider $L = -\Delta$, $f \in L^2(\Omega)$, and assume $u \in H^1_0(\Omega)$ is a weak solution of $Lu = f$ in Ω with $u_{|\partial\Omega} = 0$. The following statements hold.

(i) If in addition $u \in C^3_c(\Omega)$, then $-\Delta u = f$ holds pointwise a.e. in Ω and

$$\|D^2 u\|_{L^2(\Omega)} = \|\Delta u\|_{L^2(\Omega)} = \|f\|_{L^2(\Omega)}.$$

(ii) If in addition $u \in C^4_c(\Omega)$ and $f \in H^1(\Omega)$, then $-\Delta \partial_j u = \partial_j f$ holds pointwise a.e. in Ω $\forall j = 1, \ldots, d$ and

$$\|D^2 \partial_j u\|_{L^2(\Omega)} = \|\Delta \partial_j u\|_{L^2(\Omega)} = \|\partial_j f\|_{L^2(\Omega)}.$$

Proof. (i) Since u is a weak solution it holds

$$\int_\Omega Du \cdot Dv\, dx = \int_\Omega f v\, dx \qquad \forall v \in C^\infty_c(\Omega).$$

Integrating by parts (possible since $u \in C^2(\Omega)$) we obtain $\int_\Omega Du \cdot Dv\, dx = \int_\Omega (-\Delta u) v\, dx$, (there are no boundary contributions since v and u have compact support). Hence

$$\int_\Omega (-\Delta u - f) v\, dx = 0 \qquad \forall v \in C_c^\infty(\Omega),$$

which implies $-\Delta u - f = 0$ pointwise a.e. in Ω. Moreover

$$\begin{aligned}
\|D^2 u\|^2_{L^2(\Omega)} &= \sum_{ij} \|\partial_i \partial_j u\|^2_{L^2(\Omega)} = \sum_{ij} \int_\Omega \partial_i \partial_j u \ \partial_i \partial_j u\, dx \\
&= \sum_{ij} \int_\Omega \partial_i [\partial_j u \ \partial_i \partial_j u]\, dx - \sum_j \int_\Omega \partial_j u \ \partial_j \Delta u\, dx = -\sum_j \int_\Omega \partial_j u \ \partial_j \Delta u\, dx \\
&= \int_\Omega \Delta u \ \Delta u\, dx = \|\Delta u\|^2_{L^2(\Omega)} = \|f\|^2_{L^2(\Omega)},
\end{aligned}$$

where in the second line we used that $u \in C^3(\Omega)$ and has compact support, and in the last identity we used $-\Delta u = f$ pointwise a.e. in Ω.

(ii) The argument is the same as above. This time we need $u \in C_c^4(\Omega)$ to perform integration by parts and get the identity for $\|D^2 \partial_j u\|_{L^2(\Omega)}$. □

Remark. Note that, for each $u \in H_0^1(\Omega)$ there is a sequence $n \to u_n \in C_c^\infty(\Omega)$ such that $\|u - u_n\|_{H^1(\Omega)} \to 0$. But this does not imply that the sequence $n \to D^2 u_n$ converges in $L^2(\Omega)$, unless $u \in H_0^2(\Omega)$. In the following we will show that $u \in H_0^1(\Omega)$ weak solution of $Lu = f$ implies (under certain conditions) $u \in H^2(\Omega)$, and not $u \in H_0^2(\Omega)$. That means we need to prove $Du \in H^1(\Omega)$. Therefore in the following we will consider both $u \in H_0^1(\Omega)$ and $u \in H^1(\Omega)$.

Lemma 2.14 (preliminary estimates). *Let $\Omega \subset \mathbb{R}^d$ open and bounded. We consider the formal differential operator $Lu = -\mathrm{div}\,(aDu) + b \cdot Du + cu$ with $a_{ij}, b_j, c \in L^\infty(\Omega)$, and L uniformly elliptic. Assume $f \in L^2(\Omega)$.*

(i) There exists a constant $C = C(a, b, c) > 0$ such that

$$\|Du\|_{L^2(\Omega)} \leq C\,[\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}] \tag{2.21}$$

holds $\forall u \in H_0^1(\Omega)$ weak solution of $Lu = f$ in Ω with $u_{|\partial\Omega} = 0$, i.e. $B_L[u, v] = (f, v)_{L^2(\Omega)}$ $\forall v \in H_0^1(\Omega)$.

(ii) For all W open with $W \subset\subset \Omega$ (i.e. $\overline{W}$ is compact and $\overline{W} \subset \Omega$) There exists a constant $C = C(a, b, c, W) > 0$ such that

$$\|Du\|_{L^2(W)} \leq C\,[\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}] \tag{2.22}$$

holds $\forall u \in H^1(\Omega)$ weak solution of $Lu = f$ in Ω (no boundary condition) i.e $B_L[u, v] = (f, v)_{L^2(\Omega)}$ $\forall v \in H_0^1(\Omega)$.

Proof.
(i) Assume $u \in H_0^1(\Omega)$ weak solution of $Lu = f$ in Ω with $u_{|\partial\Omega} = 0$. Then $B_L[u, v] = (f, v)_{L^2(\Omega)}$ $\forall v \in H_0^1(\Omega)$. Replacing $v = u$ and using the energy bound we get

$$\begin{aligned}
\beta \|Du\|^2_{L^2(\Omega)} \leq \beta \|u\|^2_{H_0^1(\Omega)} &\leq B_L[u, u] + \gamma \|u\|^2_{L^2(\Omega)} = (f, u)_{L^2(\Omega)} + \gamma \|u\|^2_{L^2(\Omega)} \\
&\leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} + \gamma \|u\|^2_{L^2(\Omega)} \\
&\leq C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)})^2.
\end{aligned}$$

(ii) Assume $u \in H^1(\Omega)$ weak solution of $Lu = f$ in Ω. This means $B_L[u,v] = (f,v)_{L^2(\Omega)}$ $\forall v \in H_0^1(\Omega)$. Note that now we cannot replace $v = u$ since $u \notin H_0^1(\Omega)$. The solution is to add a cut-off function as follows.

Let W be open with $W \subset\subset \Omega$. Then there exists a function $\zeta \in C_c^\infty(\Omega)$ such that

$$0 \le \zeta \le 1, \quad \zeta_{|W} = 1.$$

This means in particular that $\overline{W} \subset \operatorname{supp}\zeta$. We define $v := \zeta^2 u$. Then $v \in H_0^1(\Omega)$ and $v_{|W} = u$, hence

$$\|Du\|_{L^2(W)} = \|\zeta Du\|_{L^2(W)} \le \|\zeta Du\|_{L^2(\Omega)}, \tag{2.23}$$

so we need to find a bound for $\|\zeta Du\|_{L^2(\Omega)}$. Since u is a weak solution and $v \in H_0^1(\Omega)$ it holds

$$(f, \zeta^2 u)_{L^2(\Omega)} = B_L[u, \zeta^2 u] = \int_\Omega (aDu)\cdot D(\zeta^2 u)\,dx + \int_\Omega (b\cdot Du + cu)\zeta^2 u\,dx,$$

where

$$\int_\Omega (aDu)\cdot D(\zeta^2 u)\,dx = \int_\Omega (a\zeta Du)\cdot(\zeta Du)\,dx + 2\int_\Omega (a\zeta Du)\cdot D\zeta u\,dx$$

By uniform ellipticity we argue, using also $0 \le \zeta \le 1$,

$$\begin{aligned}
\theta\|\zeta Du\|^2_{L^2(\Omega)} &\le \int_\Omega (a\zeta Du)\cdot(\zeta Du)\,dx \\
&= B_L[u,\zeta^2 u] - \int_\Omega (b\cdot Du + cu)\zeta^2 u\,dx - 2\int_\Omega u(D\zeta\cdot a\zeta Du)\,dx \\
&= (f,\zeta^2 u)_{L^2(\Omega)} - \int_\Omega (b\cdot\zeta Du + c\zeta u)\zeta u\,dx - 2\int_\Omega u(D\zeta\cdot a\zeta Du)\,dx \\
&\le \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \|b\|_{L^\infty(\Omega)}\|\zeta Du\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \|c\|_{L^\infty(\Omega)}\|u\|^2_{L^2(\Omega)} \\
&\qquad + 2\|D\zeta a\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)}\|\zeta Du\|_{L^2(\Omega)} \\
&= C_1\|\zeta Du\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \|c\|_{L^\infty}\|u\|^2_{L^2(\Omega)}
\end{aligned}$$

where we defined $\|b\|_{L^\infty(\Omega)} := \sum_j \|b_j\|_{L^\infty(\Omega)}$ and $\|D\zeta a\|_{L^\infty(\Omega)} := \sum_j \|(D\zeta a)_j\|_{L^\infty(\Omega)}$, and

$$C_1 := 2\|D\zeta a\|_{L^\infty(\Omega)} + \|b\|_{L^\infty(\Omega)}.$$

By Young's inequality $\|\zeta Du\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \le \frac{\varepsilon}{2}\|\zeta Du\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon}\|u\|^2_{L^2(\Omega)}$, for any $\varepsilon > 0$. Choosing now $\varepsilon < \theta/C_1$ we obtain

$$0 \le \frac{1}{2}\|\zeta Du\|^2_{L^2(\Omega)} \le C_1\frac{1}{2\varepsilon}\|u\|^2_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \|c\|_{L^\infty}\|u\|^2_{L^2(\Omega)} \le C_2(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)})^2$$

Inserting this in (2.23), we conclude the proof. □

Our goal now is to extend the arguments in Example 1 to $u \in H_0^1$. Since now $D^2 u$ is not defined we replace it by finite difference quotients.

Definition 2.15. *Let $u\colon \Omega \to \mathbb{R}$ be a function on $\Omega \subset \mathbb{R}^d$ open. For $\varepsilon > 0$, consider the set*

$$\Omega_\varepsilon := \{x \in \Omega \,|\, \operatorname{dist}(x, \partial\Omega) > \varepsilon\}.$$

Note that $\Omega_\varepsilon = \varnothing$ if $\varepsilon > \operatorname{diam}(\Omega)$.

(i) The $i-$th difference quotient of size $h \in \mathbb{R}$, $h \neq 0$ is the map

$$D_i^h u\colon \quad \begin{array}{ll} \Omega_{|h|} & \to \mathbb{R} \\ x & \mapsto D_i^h u(x) := \frac{u(x+he_i)-u(x)}{h} \end{array}, i = 1, \dots d.$$

(ii) The <u>difference quotient of size h</u> is the vector $D^h u := (D_1^h u, \dots D_1^h u)$.

Note that $D_i^h u$ is well defined on $\Omega_{|h|}$. Indeed for all $x \in \Omega_{|h|}$ it holds $x + he_i \in \Omega$ and hence $u(x + he_i)$ is well defined.

Lemma 2.16 (elementary properties of the difference quotient)**.**

(i) $u \in C^1(\Omega) \Rightarrow \lim_{h\to 0} D_i^h u(x) = \partial_i u(x)\ \forall x \in \Omega$.

(ii) $u \in L^p(\Omega) \Rightarrow D_i^h u \in L^p(\Omega_{|h|})$ *and*

$$\|D_i^h u\|_{L^p(\Omega_{|h|})} \leq \frac{2}{h}\|u\|_{L^p(\Omega)}.$$

(iii) $D_i^h(u_1 u_2)(x) = D_i^h u_1(x)\, u_2(x) + u_1(x + he_i)\, D_i^h u_2(x)$.

(iv) Assume $u \in L^p(\Omega)$ *and* $v \in L^q(\Omega)$, *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $\operatorname{supp} v$ *is compact. Then*

$$\int_\Omega v(x)\ D_i^h u(x)\, dx = -\int_\Omega D_i^{-h} v(x)\ u(x)\, dx.$$

(v) The discrete Laplace operator $\Delta_h u(x) := \sum_{i=1}^d D_i^{-h} D_i^h u$ *satisfies*

$$\Delta_h u(x) := \sum_{i=1}^d \frac{u(x + he_i) + u(x - he_i) - 2u(x)}{h^2}$$

and is well defined on $\Omega_{|h|}$.

Proof. Exercise

$\square$

Theorem 2.17 (connection between discrete and weak derivative)**.** *Let* $\Omega \subset \mathbb{R}^d$ *open.*

(i) Set $1 \leq p < \infty$, $u \in W^{1,p}(\Omega)$, *and* V *open with* $V \subset\subset \Omega$. *It holds*

$$\|D^h u\|_{L^p(V)} \leq \|Du\|_{L^p(\Omega)} \qquad \forall 0 < |h| \leq \frac{1}{2}\operatorname{dist}(V, \partial\Omega).$$

(ii) Set $1 < p < \infty$, $u \in L^p(\Omega)$, *and* V *open with* $V \subset\subset \Omega$. *Assume*

$$\sup_{0<|h|\leq\frac{1}{2}\operatorname{dist}(V,\partial\Omega)} \|D^h u\|_{L^p(V)} = C < \infty.$$

Then $u \in W^{1,p}(V)$ *and* $\|Du\|_{L^p(V)} \leq C$.

Proof. Exercise sheet. $\square$

Note that for $p = 1$ *(ii)* does not hold (see Exercise sheet).

2.4.2 Interior regularity

Theorem 2.18 (H^2 interior regularity)**.** *Assume $\Omega \subset \mathbb{R}^d$ is open and bounded. We consider the formal differential operator $Lu = -\text{div}\,(aDu) + b \cdot Du + cu$ with $a_{ij}, b_j, c \in L^\infty(\Omega)$, and L uniformly elliptic.*

Assume in addition $a \in C^1(\Omega)$, $f \in L^2(\Omega)$ and $u \in H^1(\Omega)$ is a weak solution of $Lu = f$ in Ω (no boundary condition), i.e. $B_L[u, v] = (f, v)_{L^2(\Omega)}$ $\forall v \in H_0^1(\Omega)$.

Then $u \in H^2_{loc}(\Omega)$ and $\forall V$ open with $V \subset\subset \Omega$, there is a constant $C = C(\Omega, V, a, b, c) > 0$ such that

$$\|u\|_{H^2(V)} \leq C \ \left[\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right] \tag{2.24}$$

Remark 1. Since $u \in H^2(\Omega)$ and $a \in C^1(\Omega)$, the equation $Lu = f$ holds pointwise a.e. in Ω.

Remark 2. The result above does not improve if we use $u \in H_0^1(\Omega)$ instead of $u \in H^1(\Omega)$, unless we require some boundary regularity (later).

Proof of Theorem 2.18. We replace Δu (which is not well-defined) with $\Delta_h u = \sum_{i=1}^d D_i^{-h} D_i^h u$. Since $u \in H^1(\Omega)$ it holds $\Delta_h u \in H^1(V)$ $\forall V$ open with $V \subset\subset \Omega$, and $\text{dist}\,(\overline{V}, \partial\Omega) > |h|$.

Let us now fix V open with $V \subset\subset \Omega$. We upgrade $\Delta_h u$ to a function in $H_0^1(\Omega)$ by adding a cut-off function as in Lemma 2.14. Precisely, there exists a W open with $V \subset\subset W \subset\subset \Omega$ and a function $\zeta \in C_c^\infty(W)$ such that $0 \leq \zeta \leq 1$ and $\zeta_{|V} = 1$. We define

$$v := -\sum_{i=1}^d D_i^{-h} \zeta^2 D_i^h u.$$

With this definition $v \in H_0^1(\Omega)$ $\forall 0 < |h| \leq h_0 := \frac{1}{4}\text{dist}\,(\overline{W}, \partial\Omega)$.

Claim. If u is a weak solution of $Lu = f$, then $\exists C > 0$ such that

$$\sum_{i,k=1}^d \|\zeta D_i^h \partial_k u\|^2_{L^2(\Omega)} \leq C \ [\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}]^2 \qquad \forall 0 < |h| \leq h_0.$$

We will prove this Claim below.

Consequence. Using $\zeta_{|V} = 1$ we get

$$\begin{aligned}\sup_{0<|h|\leq h_0} \|D_i^h \partial_k u\|_{L^2(V)} = \sup_{0<|h|\leq h_0} \|\zeta D_i^h \partial_k u\|_{L^2(V)} &\leq \sup_{0<|h|\leq h_0} \|\zeta D_i^h \partial_k u\|_{L^2(\Omega)} \\ &\leq C \ [\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}] \qquad \forall k, i.\end{aligned}$$

Therefore, by Theorem 2.17, $\partial_k u \in H^1(V)$ $\forall k$ and

$$\|\partial_i \partial_k u\|_{L^2(V)} \leq C \ [\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}].$$

It follows that $u \in H^2(V)$ and

$$\|D^2 u\|_{L^2(V)} \leq C_1 \ [\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}],$$

for some constant $C_1 > 0$. By Lemma 2.14 (ii) we also know that

$$\|Du\|_{L^2(V)} \leq C_2 \ [\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}],$$

for some constant $C_1 > 0$. We conclude

$$\|u\|_{H^2(V)} \leq C_3 \; [\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}],$$

for some constant $C_3 > 0$. It remains to prove the Claim.

$\square$

[8: 2.11.2023]

[9: 06.11.2023]

Proof of the Claim. Since u is a weak solution of $Lu = f$ in Ω and $v \in H_0^1(\Omega)$ we have $B_L[u, v] = (f, v)_{L^2(\Omega)}$. This can be reformulated as

$$\int_\Omega (aDu) \cdot Dv \, dx = (f_u, v)_{L^2(\Omega)}, \qquad \text{with} \qquad f_u := f - (b \cdot Du + cu).$$

Inserting the explicit for of v we get

$$\begin{aligned}\int_\Omega (aDu) \cdot Dv \, dx &= -\sum_{ikl} \int_\Omega \partial_i u \; a_{ij} \; \partial_j (D_k^{-h}(\zeta^2 D_k^h u)) \, dx = -\sum_{ikl} \int_\Omega \partial_i u \; a_{ij} \; D_k^{-h}(\partial_j(\zeta^2 D_k^h u)) \, dx \\ &= \sum_{ikl} \int_\Omega D_k^h(\partial_i u \; a_{ij}) \; \partial_j(\zeta^2 D_k^h u) \, dx,\end{aligned}$$

where in the second line we used $\operatorname{supp} \zeta \subset W$, $|h| \leq h_0 = \frac{1}{4} \operatorname{dist}(\overline{W}, \partial\Omega)$ and Lemma 2.16 (*iv*). Set now

$$g_1 := D_k^h(\partial_i u \; a_{ij}), \qquad g_2 := \partial_j(\zeta^2 D_k^h u).$$

By Lemma 2.16 (*iii*), we have

$$g_1(x) = D_k^h a_{ij}(x) \; (\partial_i u(x)) \; + \; a_{ij}(x + he_k) \; (D_k^h \partial_i u(x)),$$

where the first summand contains only a first order derivative in u, while the second summand has a second order "derivative" in u (actually one weak and one finite derivative). In the same way

$$g_2 = 2\zeta \partial_j \zeta \, (D_k^h u) \; + \; \zeta^2 (D_k^h \partial_j u),$$

where again the first summand contains only a first order "derivative" in u (actually a finite difference) and the second summand has a second order "derivative" in u (one weak and one finite derivative). Therefore

$$\begin{aligned} g_1 g_2 = \; & (\zeta D_k^h \partial_i u) \; a_{ij}(\cdot + he_k) \; (\zeta D_k^h \partial_j u) \\ & + 2\zeta \partial_i \zeta \, (D_k^h u) \; a_{ij}(\cdot + he_k) \; (D_k^h \partial_j u) + (\zeta \partial_i u) \; D_k^h a_{ij} \; (\zeta D_k^h \partial_j u) \\ & + 2\zeta \partial_i \zeta \, (D_k^h u) \; D_k^h a_{ij} \; (\partial_j u), \end{aligned}$$

where in the first line we have two second order derivatives (weak or discrete), in the second line one first and one second order derivative, and in the third line only first order derivatives in u. We reorganize the integrals above as follows

$$\int_\Omega (aDu) \cdot Dv \, dx = A_2 + A_1 + A_0,$$

where A_2 (resp. A_1, A_0) is the sum of all terms with two (resp. one, zero) second order derivatives in u. Precisely

$$
\begin{aligned}
A_2 &:= \sum_k \int_\Omega \big(a(\cdot + he_k)(\zeta D_k^h \partial u)\big) \cdot (\zeta D_k^h \partial u)\; dx \\
A_1 &:= \sum_k \int_\Omega \Big[2D_k^h u\; \Big((a(\cdot + he_k)(\zeta D_k^h \partial u)) \cdot \partial\zeta\Big) \;\;+\;\; \big((D_k^h a)\zeta \partial u\big) \cdot (\zeta D_k^h \partial u)\Big]\; dx \\
A_0 &:= \sum_k \int_\Omega 2(\zeta D_k^h u)\Big((D_k^h a)(\zeta \partial u) \cdot \partial\zeta,\Big)\; dx.
\end{aligned}
$$

Putting all this together we get

$$A_2 = (f_u, v)_{L^2(\Omega)} - A_0 - A_1,$$

where now the left hand side contains two second order derivatives, and the right hand-side at most one. We are now ready to prove the Claim.

Set $X := \sum_{j,k=1}^d \|\zeta D_k^h \partial_j u\|^2_{L^2(\Omega)}$. Our goal is to prove $X \le C[\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}]^2\; \forall 0 < |h| \le h_0$. By uniform ellipticity we have

$$A_2 \;\ge\; \theta \sum_{jk} \int_\Omega (\zeta D_k^h \partial_j u)^2 dx = \theta X,$$

hence

$$\theta X \le A_2 = (f_u, v)_{L^2(\Omega)} - A_0 - A_1 \le |(f_u, v)_{L^2(\Omega)}| + |A_0| + |A_1|. \tag{2.25}$$

We bound the three terms separately.

Bound on $|A_0|$. There is a constant $C_1 > 0$ independent of u such that

$$|A_0| \le C_1\big(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big)^2. \tag{2.26}$$

To prove this we argue, using $\operatorname{supp}\zeta \subset W$ and $0 \le \zeta \le 1$,

$$|A_0| \le \sum_{kij} 2\|D_k^h a_{ij}\|_{L^\infty(W)}\; \|\partial_j \zeta\|_{L^\infty(W)}\; \|D_i^h u\|_{L^2(W)} \|\partial_i u\|_{L^2(W)} \tag{2.27}$$

• Since $u \in H^1(\Omega)$ is a weak solution of $Lu = f$ and $W \subset\subset \Omega$, by Lemma 2.14 (ii) we have

$$\|Du\|_{L^2(W)} \le C_{W\Omega}\; \big(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big) \tag{2.28}$$

for some constant $C_W > 0$.

• To obtain the same bound on $\|D_k^h u\|_{L^2(W)}$ we argue in two steps. Set $W' := B_{2h_0}(W)$. Since $|h| \le h_0$

$$B_{|h|}(W) \subset\subset W' \subset\subset \Omega, \qquad \text{and} \qquad |h| \le h_0 = \frac{1}{2}\operatorname{dist}(W, \partial W').$$

Hence, by Theorem 2.17 (ii),

$$\|D_i^h u\|_{L^2(W)} \le \|Du\|_{L^2(W')} \le C_{WW'}\big(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big). \tag{2.29}$$

where in the second inequality we applied again Lemma 2.14 (ii).

• Since $a \in C^1(\overline{W'}) \cap L^\infty(\Omega)$ we have

$$\sup_{0 \le |h| \le h_0} \|D_i^h a\|_{L^\infty(W)} \le \sup_{x \in W'} |Da(x)| < \infty, \tag{2.30}$$

and

$$\|a(\cdot + he_i)\|_{L^\infty(W)} \le \|a\|_{L^\infty(\Omega)}. \tag{2.31}$$

Inserting all these estimates in (2.27) we obtain (2.26).

Bound on $|A_1|$. There is a constant $C_2 > 0$ independent of u such that, using also Young's inequality,

$$|A_1| \le C_2 \sum_{j,k=1}^{d} \|\zeta D_k^h \partial_j u\|_{L^2(\Omega)} \big(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big) \le \frac{C_2 \varepsilon}{2} X + \frac{C_2 d^2}{2\varepsilon} \big(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big)^2, \tag{2.32}$$

for all $\varepsilon > 0$. To prove this we argue, using again $\operatorname{supp} \zeta \subset W$ and $0 \le \zeta \le 1$,

$$\begin{aligned} |A_1| \le& \sum_{ijk} 2\|a_{ij}(\cdot + he_k)\|_{L^\infty(W)} \|\partial_i \zeta\|_{L^\infty(W)} \ \|D_k^h u\|_{L^2(W)} \| \|\zeta D_k^h \partial_j u\|_{L^2(W)} \\ &+ \sum_{kij} \|D_k^h a_{ij}\|_{L^\infty(W)} \ \|\partial_i u\|_{L^2(W)} \|\zeta D_k^h \partial_j u\|_{L^2(W)} \end{aligned}$$

The result follows applying (2.28), (2.29),(2.30) and (2.31).

Bound on $|(f_u, v)_{L^2(\Omega)}|$. There are constants $C_3, C_4 > 0$ independent of u such that, using also Young's inequality,

$$\begin{aligned} |(f_u, v)_{L^2(\Omega)}| &\le C_3 \big(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big)^2 + C_4 \sum_{j,k=1}^{d} \|\zeta D_k^h \partial_j u\|_{L^2(\Omega)} \ \big(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big) \\ &\le \frac{C_4 \varepsilon}{2} X + (C_3 + \frac{C_4 d^2}{2\varepsilon}) \big(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big)^2, \end{aligned} \tag{2.33}$$

for all $\varepsilon > 0$. To prove this we argue, using again $\operatorname{supp} \zeta \subset W$,

$$|(f_u, v)_{L^2(\Omega)}| = |(f_u, v)_{L^2(W)}| \le \|f_u\|_{L^2(W)} \|v\|_{L^2(W)} \tag{2.34}$$

We bound the last two terms separately.

• We have, using (2.28),

$$\|f_u\|_{L^2(W)} \le \|f\|_{L^2(\Omega)} + \|b\|_{L^\infty(\Omega)} \|Du\|_{L^2(W)} + \|c\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \le C\Big(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\Big),$$

for some constant $C > 0$.

• Finally, setting $w_k := \zeta^2 D_k^h u$

$$\|v\|_{L^2(W)}^2 = \sum_k \|D_k^{-h} \zeta^2 D_k^h u\|_{L^2(W)}^2 = \sum_k \|D_k^{-h} w_k\|_{L^2(W)}^2.$$

Since $w_k \in H^1(\Omega)$, $\operatorname{supp} w_k \subset W$ and $|h| \le \frac{1}{2} \operatorname{dist}(W, \partial\Omega)$, we have

$$\|D_k^{-h} w_i\|_{L^2(W)} \le \|\partial_k w_k\|_{L^2(\Omega)} = \|\partial_k w_k\|_{L^2(W)} = \|\partial_k (\zeta^2 D_k^h u)\|_{L^2(W)}.$$

Hence

$$\begin{aligned}\|v\|_{L^2(W)} &\le \sum_k \|\partial_k(\zeta^2 D_k^h u)\|_{L^2(W)} \le \sum_k \Big[\|2\zeta\partial_k\zeta D_k^h u\|_{L^2(W)} + \|\zeta^2 D_k^h \partial_k u)\|_{L^2(W)}\Big] \\ &\le \sum_k \Big[\|2\partial_k\zeta\|_{L^\infty(\Omega)}\|D_k^h u\|_{L^2(W)} + \|\zeta D_k^h\partial_k u)\|_{L^2(\Omega)}\Big] \\ &\le \sum_k \|\zeta D_k^h \partial_k u)\|_{L^2(\Omega)} + C'\Big(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\Big),\end{aligned}$$

where in the last line we used again (2.29) and $C' > 0$ is some constant. The result now follows inserting these bounds in (2.34).

Final bound on X. Putting all the above estimates together we obtain

$$\theta X \le \Big(\frac{C_2}{2} + \frac{C_4}{2}\Big)\varepsilon X \; + \; \Big(C_1 + \frac{C_2 d^2}{2\varepsilon} + C_3 + \frac{C_4 d^2}{2\varepsilon}\Big)\Big(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\Big)^2$$

The result follows choosing $\varepsilon > 0$ small enough. This completes the proof of the Claim. □

[9: 06.11.2023]

[10: 09.11.2023]

Higher regularity Let us go back to Example 1, i.e. $L = -\Delta$, $u \in C_c^\infty(\Omega)$ is a weak solution of $-\Delta u = f$ in Ω. We have seen that, if $f \in L^2(\Omega)$, then $-\Delta u = f$ holds pointwise a.e. in Ω and $\|D^2 u\|_{L^2(\Omega)} = \|f\|_{L^2(\Omega)}$. If in addition $f \in C^1(\Omega)$, then $-\Delta\partial_j u = \partial_j f$ holds pointwise a.e. in Ω and $\|D^2\partial_j u\|_{L^2(\Omega)} = \|\partial_j f\|_{L^2(\Omega)}$.

If $f \in C^m(\Omega)$, and $u \in C_c^\infty(\Omega)$ is a strong solution of $-\Delta u = f$ we can derive both terms and obtain $-\Delta\partial_j^\alpha u = \partial_j^\alpha f$, $\forall|\alpha| \le m$. Then $w = \partial_j^\alpha u$ is a weak solution of $-\Delta w = \partial_j^\alpha f$ in Ω and hence

$$\|D^2\partial_j^\alpha u\|_{L^2(\Omega)} = \|D^2 w\|_{L^2(\Omega)} = \|\partial_j^\alpha f\|_{L^2(\Omega)}.$$

In the general case we consider $Lu = f$ where $Lu = -\mathrm{div}\,(aDu) + b\cdot Du + cu$, $a_{ij}, b_j, c \in L^\infty(\Omega)$, and $f \in L^2(\Omega)$.
If $a \in C^1(\Omega)$ and $u \in H^1(\Omega)$ is a weak solution of $Lu = f$ we know, by Theorem 2.18, that $u \in H^2_{loc}(\Omega)$, $Lu = f$ holds pointwise a.e., and (2.24) holds.
Assume now in addition $a \in C^2(\Omega)$, $b, c \in C^1(\Omega)$, $f \in H^1(\Omega)$. Then *formally*

$$\partial_j f = \partial_j(Lu) = L(\partial_j u) + R_j(u),$$

where the error term $R_j(u)$ is defined as

$$R_j(u) := -\mathrm{div}\,((\partial_j a)Du) + (\partial_j b)\cdot Du + (\partial_j c)u. \tag{2.35}$$

Hence $w := \partial_j u$ is a formal solution of

$$Lw = \partial_j f - R_j(u) =: \tilde{f}_u. \tag{2.36}$$

Note that, since $u \in H^2_{loc}(\Omega)$ we have $w = \partial_j u \in H^1_{loc}(\Omega)$. Moreover, since $a \in C^2(\Omega)$, and $b, c \in C^1(\Omega)$ we have $D^2 a, Da, Da, Dc \in L^\infty_{loc}(\Omega)$ and hence the function $R_j(u)$ is well defined and $R_j(u) \in L^2_{loc}(\Omega)$.

On the other hand $L(\partial_j u)$ is not well defined, unless u admits third order weak derivatives. We will show below that $\partial_j u$ is a local weak solution of the formal PDE

$$L(\partial_j u) = \tilde{f}_u,$$

and hence by Theorem 2.18, $\partial_j u \in H^2_{loc}(\Omega)$, i.e. $u \in H^3_{loc}(\Omega)$. This is the content of the next theorem.

Theorem 2.19 (higher interior regularity).
Assume $\Omega \subset \mathbb{R}^d$ is open and bounded. We consider the formal uniformly elliptic differential operator $Lu = -\mathrm{div}\,(aDu) + b \cdot Du + cu$ with $a_{ij}, b_j, c \in L^\infty(\Omega)$.

Assume in addition $a \in C^{m+1}(\Omega)$, $b, c \in C^m(\Omega)$, $f \in H^m(\Omega)$ and $u \in H^1(\Omega)$ is a weak solution of $Lu = f$ in Ω (no boundary condition), i.e. $B_L[u, v] = (f, v)_{L^2(\Omega)}$ $\forall v \in H^1_0(\Omega)$.

Then $u \in H^{2+m}_{loc}(\Omega)$ and $\forall V$ open with $V \subset\subset \Omega$, there is a constant $C = C(\Omega, V, a, b, c) > 0$ such that

$$\|u\|_{H^{2+m}(V)} \leq C \ \left[\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)}\right]$$

Proof. We argue by induction on m.

For $m = 0$ we have $a \in C^1(\Omega) \cap L^\infty(\Omega)$, $b, c \in C^0(\Omega) \cap L^\infty(\Omega)$ $f \in L^2(\Omega)$ and the result follows from Theorem 2.18.

We prove now the first induction step: if the statement holds for $m = 0$, then the statement holds also for $m = 1$. Assume $a \in C^2(\Omega) \cap L^\infty(\Omega)$, $b, c \in C^1(\Omega) \cap L^\infty(\Omega)$, $f \in H^1(\Omega)$. Our goal is to show that $u \in H^3_{loc}(\Omega)$.

• Since $u \in H^1(\Omega)$ is a weak solution of $Lu = f$, by the case $m = 0$ we know that $u \in H^2_{loc}(\Omega)$, $Lu = f$ holds pointwise a.e. in Ω, and $\forall V$ open with $V \subset\subset \Omega$, there is a constant $C = C(\Omega, V, a, b, c) > 0$ such that

$$\|u\|_{H^2(V)} \leq C \ \left[\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right]. \tag{2.37}$$

Setting $R_j(u)$ as in (2.35) above, we will show that $\partial_j u$ is a local weak solution of the formal PDE

$$L(\partial_j u) = \partial_j f - R_j(u) =: \tilde{f}_u. \tag{2.38}$$

i.e. $\forall V$ open with $V \subset\subset \Omega$, $\partial_j u \in H^1(V)$ (since $u \in H^2_{loc}(\Omega)$) and

$$B_L[\partial_j u, v] = (\tilde{f}_u, v)_{L^2(V)} \qquad \forall v \in H^1_0(V).$$

Indeed for $v \in H^1_0(V)$, there is a sequence $n \mapsto v_n \in C^\infty_c(V)$ such that $\|v_n - v\|_{H^1_0} \to 0$. Then $B_L[\partial_j u, v] = \lim_{n\to\infty} B_L[\partial_j u, v_n]$. Moreover, since $a, b, c \in C^1(\Omega)$, and $v_n \in C^3_c(V)$ we can integrate by parts as follows:

$$\begin{aligned}
B_L[\partial_j u, v_n] &= \int_V \Big(Dv_n \cdot aD\partial_j u + v_n[b \cdot D\partial_j u + c\partial_j u]\Big)dx \\
&= -\int_V \Big(Dv_n \cdot (\partial_j a)Du + v_n[(\partial_j b) \cdot Du + (\partial_j c)u]\Big)dx \\
&\quad - \int_V \Big(D\partial_j v_n \cdot aDu + \partial_j v_n[b \cdot Du + cu]\Big)dx \\
&= -B_L[u, \partial_j v_n] - (R_j(u), v_n)_{L^2(V)} = -(f, \partial_j v_n)_{L^2(V)} - (R_j(u), v_n)_{L^2(V)} = (\tilde{f}_u, v_n)_{L^2(V)}
\end{aligned}$$

where in the last line we used $\partial_j v_n \in C_c^\infty(V) \Rightarrow B_L[u, \partial_j v_n] = (f, \partial_j v_n)_{L^2(V)}$. Finally

$$B_L[\partial_j u, v] = \lim_{n\to\infty} B_L[\partial_j u, v_n] = \lim_{n\to\infty} (\tilde{f}_u, v_n)_{L^2(V)} = (\tilde{f}_u, v)_{L^2(V)}.$$

Therefore $\partial_j u$ is a local weak solution of $L(\partial_j u) = \tilde{f}_u$.

• Since $\partial_j u \in H^1(\Omega)$ is a weak solution of $L\partial_j u = \tilde{f}_u$, in V, by the case $m = 0$ we know that $\partial_j u \in H^2_{loc}(V)$, $L\partial_j u = \tilde{f}_u$ holds pointwise a.e. in V, and $\forall W$ open with $W \subset\subset V$, there is a constant $C_1 = C_1(V, W, a, b, c) > 0$ such that

$$\|\partial_j u\|_{H^2(W)} \leq C_1 \ \left[\|\tilde{f}_u\|_{L^2(V)} + \|\partial_j u\|_{L^2(V)}\right].$$

We bound now the two terms on the right separately. Since $u \in H^1(\Omega)$ is a weak solution of $Lu = f$ in Ω, by Lemma 2.14(ii), we have

$$\|\partial_j u\|_{L^2(V)} \leq C_2 \ \left[\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right].$$

Moreover

$$\begin{aligned}
\|\tilde{f}_u\|_{L^2(V)} &\leq \|\partial_j f\|_{L^2(\Omega)} + \|Da\|_{L^\infty(V)}\|D^2 u\|_{L^2(V)} \\
&\quad + \|Du\|_{L^2(V)}\Big(\|D^2 a\|_{L^\infty(V)} + \|Db\|_{L^\infty(V)}\Big) + \|Dc\|_{L^\infty(V)}\|u\|_{L^2(V)} \\
&\leq \|\partial_j f\|_{L^2(\Omega)} + C_3 \ \left[\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right] \leq C_4 \left[\|f\|_{H^1(\Omega)} + \|u\|_{L^2(\Omega)}\right],
\end{aligned}$$

where we used $a, b, c \in C^2(\overline{V})$, the bound (2.37) and again Lemma 2.14(ii). Therefore $u \in H^3(W)$ and

$$\|u\|_{H^3(W)} \leq C_5 \left[\|f\|_{H^1(\Omega)} + \|u\|_{L^2(\Omega)}\right].$$

The claim for $m = 1$ now follows since W is arbitrary. The general step $m \Rightarrow m + 1$ is proved in the same way (exercise).

□

Theorem 2.20 (infinite interior regularity). *Assume $\Omega \subset \mathbb{R}^d$ is open and bounded. We consider the uniformly elliptic formal differential operator $Lu = -\mathrm{div}\,(aDu) + b \cdot Du + cu$ with $a_{ij}, b_j, c \in L^\infty(\Omega)$.*

Assume in addition $a, b, c, f \in C^\infty(\Omega)$ and $u \in H^1(\Omega)$ is a weak solution of $Lu = f$ in Ω (no boundary condition), i.e. $B_L[u, v] = (f, v)_{L^2(\Omega)}$ $\forall v \in H^1_0(\Omega)$.

Then $u \in C^\infty(\Omega)$ and $Lu = f$ holds pointwise in Ω.

To prove this result we will need the following generalized Sobolev inequalities.

Theorem 2.21 (generalized Sobolev inequalities). *Assume $\Omega \subset \mathbb{R}^d$ is open and bounded, with Lipschitz boundary. Let $k \geq 1$, and $u \in W^{k,p}(\Omega)$. The following hold.*

(i) If $1 \leq p < \frac{d}{k}$ then $u \in L^q(\Omega)$ $\forall 1 \leq q \leq p^*$ with

$$\frac{1}{p^*} := \frac{1}{p} - \frac{k}{d},$$

and there is a constant $C = C_{p,q,k,\Omega}$ such that

$$\|u\|_{L^q(\Omega)} \leq C \ \|u\|_{W^{k,p}(\Omega)}.$$

(ii) If $\frac{d}{k} < p$, we define

$$\gamma := \begin{cases} 1 - \left(\frac{d}{p} - \left\lfloor \frac{d}{p} \right\rfloor\right) & \text{if } \frac{d}{p} \notin \mathbb{N} \\ \text{any number } 0 < \gamma < 1 & \text{if } \frac{d}{p} \in \mathbb{N}. \end{cases}$$

Then for all $\forall 0 < \beta \leq \gamma$ $\exists \tilde{u} \in C^{k-1-\left\lfloor \frac{d}{p} \right\rfloor, \beta}(\overline{\Omega})$ and a constant $C = C_{p,\beta,k,\Omega} > 0$ such that $u = \tilde{u}$ a.e, and

$$\|\tilde{u}\|_{C^{k-1-\left\lfloor \frac{d}{p} \right\rfloor, \beta}(\overline{\Omega})} \leq C \ \|u\|_{W^{k,p}(\Omega)}.$$

Note: $u \in C^{m,\beta}(\overline{\Omega})$ means $u \in C^m(\overline{\Omega})$ and $\partial^\alpha u \in C^{0,\beta}(\Omega)$ $\forall |\alpha| = m$. Moreover

$$\|u\|_{C^{m,\beta}(\overline{\Omega})} := \sum_{0 \leq |\alpha| \leq m} \sup_{x \in \overline{\Omega}} |\partial^\alpha u(x)| + \sum_{|\alpha| = m} [\partial^\alpha u]_{C^{0,\beta}}.$$

Proof. We will consider only the case $k = 2$ for simplicity. The general case is proved in the same way.

(i) Assume $1 \leq p < \frac{d}{2}$. Then $u, Du \in W^{1,p}(\Omega)$, with $p < d$. By standard Sobolev inequality Theorem 1.17 it follows

$$u, Du \in L^{p_1}(\Omega), \qquad \frac{1}{p_1} := \frac{1}{p} - \frac{1}{d},$$

and $\|u\|_{W^{1,p_1}(\Omega)} \leq C_1 \ \|u\|_{W^{2,p}(\Omega)}$. Since $p < \frac{d}{2}$ we have $p_1 < d$, hence, again by Theorem 1.17, $u \in L^{p_2}(\Omega)$ with $\frac{1}{p_2} = \frac{1}{p_1} - \frac{1}{d} = \frac{1}{p} - \frac{2}{d} = \frac{1}{p^*}$ and

$$\|u\|_{L^{p_2}(\Omega)} \leq C_2 \ \|u\|_{W^{1,p_1}(\Omega)} \leq C_1 C_2 \|u\|_{W^{2,p}(\Omega)}.$$

The statement for $q < p^*$ now follows since Ω is bounded.

(ii) We distiguish three cases.
Case 1. Assume $p > \frac{d}{2}$ and $p > d$. In particular this means $0 < \frac{d}{p} < 1$, and hence $\frac{d}{p} \notin \mathbb{N}$ and $\left\lfloor \frac{d}{p} \right\rfloor = 0$. Our goal is to show that there is a $\tilde{u} \in C^{1,\gamma}(\overline{\Omega})$ with $u = \tilde{u}$ a.e., where $\gamma = 1 - \frac{d}{p}$.

Indeed, since $u, Du \in W^{1,p}(\Omega)$ and $d < p$, it follows, by standard Sobolev inequality Theorem 1.17, that $u, Du \in C^{0,\gamma}(\Omega)$ with $\gamma = 1 - \frac{d}{p}$. Precisely, there are functions $\tilde{u}, \tilde{v}_j \in C^{0,\gamma}(\Omega)$ such that $u = \tilde{u}$, $\partial_j u = \tilde{v}_j$ a.e. and

$$\|\tilde{u}\|_{C^{0,\gamma}(\Omega)} \leq C \ \|u\|_{W^{1,p}(\Omega)}, \quad \|\tilde{v}_j\|_{C^{0,\gamma}(\Omega)} \leq C \ \|\partial_j u\|_{W^{1,p}(\Omega)}.$$

It follows (excercise) that $\tilde{u} \in C^{1,\gamma}(\overline{\Omega})$ and $\|\tilde{u}\|_{C^{1,\gamma}(\overline{\Omega})} \leq C \ \|u\|_{W^{2,p}(\Omega)}$.

Case 2. Assume $d > p > \frac{d}{2}$. In particular this means $\frac{d}{p} \notin \mathbb{N}$ and $\left\lfloor \frac{d}{p} \right\rfloor = 1$. Our goal is to show that there is a $\tilde{u} \in C^{0,\gamma}(\overline{\Omega})$ with $u = \tilde{u}$ a.e., where $\gamma = 2 - \frac{d}{p}$.

Indeed, since $u, Du \in W^{1,p}(\Omega)$ and $p < d$, it follows, by standard Sobolev inequality Theorem 1.17, that

$$u, Du \in L^{p_1}(\Omega), \qquad \frac{1}{p_1} := \frac{1}{p} - \frac{1}{d},$$

and $\|u\|_{W^{1,p_1}(\Omega)} \leq C_1 \ \|u\|_{W^{2,p}(\Omega)}$. Since $p > \frac{d}{2}$ we have $p_1 > d$, hence there is a $\tilde{u} \in C^{0,\gamma}(\Omega)$, with $u = \tilde{u}$ a.e and $\gamma = 1 - \frac{d}{p_1} = 2 - \frac{d}{p}$, such that

$$\|\tilde{u}\|_{C^{0,\gamma}(\Omega)} \leq C_1 \ \|u\|_{W^{1,p_1}(\Omega)} \leq C_1 C_2 \|u\|_{W^{2,p}(\Omega)}.$$

It follows (excercise) that $\tilde{u} \in C^{0,\gamma}(\overline{\Omega})$ and $\|\tilde{u}\|_{C^{0,\gamma}(\overline{\Omega})} \leq C \; \|u\|_{W^{2,p}(\Omega)}$.

Case 3. Assume $p > \frac{d}{2}$ and $p = d$. In particular this means $\left\lfloor \frac{d}{p} \right\rfloor = \frac{d}{p} = 1 \in \mathbb{N}$ Our goal is to show that $\forall 0 < \gamma < 1$ there is a $\tilde{u} \in C^{0,\gamma}(\overline{\Omega})$ with $\tilde{u} = u$ a.e..

Indeed, since $u, Du \in W^{1,d}(\Omega)$ and Ω is bounded it follows $u, Du \in W^{1,d-\varepsilon}(\Omega)$ $\forall 0 < \varepsilon \leq d-1$. By Sobolev inequality we have then

$$u, Du \in L^{q_\varepsilon}(\Omega), \qquad q_\varepsilon := \frac{d(d-\varepsilon)}{\varepsilon},$$

Since ε is arbitrarily near to zero, we obtain $u, Du \in L^q(\Omega)$ $\forall d < q < \infty$. The result now follows again by standard Sobolev inequality, as in Case 1.

The statement for $\beta < \gamma$ follows since Ω is bounded.

□

Proof of Theorem 2.20.
Since $a, b, c, f \in C^\infty(\Omega)$, we have $a, b, c \in C^{m+1}(\Omega) \cap L^\infty_{loc}(\Omega)$ and $f \in H^m_{loc}(\Omega)$ $\forall m \geq 1$.
Since $u \in H^1(\Omega)$ is a weak solution of $Lu = f$ in Ω we know by Theorem 2.19, $u \in H^m_{loc}(\Omega)$ $\forall m \geq 0$. Fix now $x_0 \in \Omega$. Since Ω is open there exists $r > 0$ such that $\overline{B_r(x_0)} \subset \Omega$. Define $V := B_r(x_0)$. Then V is open and $V \subset\subset \Omega$. Hence $u \in H^m(V) = W^{m,2}(V)$ $\forall m \geq 1$. There exists $m_0 > 0$ such that $2 > \frac{d}{m}$ $\forall m \geq m_0$. Since ∂V is C^1 it follows by generalized Sobolev inequality Theorem 2.21(ii) that $u \in C^{m-1-\lfloor \frac{d}{m} \rfloor, \gamma}(\overline{V})$ $\forall m \geq m_0$ and hence $u \in C^\infty(\overline{V})$. Since x_0 is arbitrary $u \in C^\infty(\Omega)$. □

Remark. Note that we can repeat all the proofs above in unbounded domain Ω since we only work locally.

[10: 09.11.2023]

[11: 13.11.2023]

2.4.3 Regularity up to the boundary

Remember that if $u \in L^p(\Omega)$ then $D^h_i u \in L^p(\Omega_{|h|})$ $\forall i = 1, \ldots d$, where
$\Omega_\varepsilon := \{x \in \Omega | \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$, $\epsilon > 0$.
In the special case $\Omega = B^+_r(x_0) := \{x \in B_r(x_0) | \; x_d > 0\}$, the finite difference quotients in directions $i = 1, \ldots, d-1$ are well defined up to lower boundary of Ω i.e. $D^h_i u \in L^p(B^+_s(x_0))$ $\forall 0 < s < r - |h|$, and $i = 1, \ldots, d-1$. This remark motivates the following lemma, that extends Theorem 2.17.

Lemma 2.22. *Let $\Omega = B^+_r(x_0) = \{x \in B_r(x_0) | \, x_d > 0\}$ be the half-ball and $\Gamma := \{x \in B_r(x_0) | \, x_d = 0\}$ the corresponding lower boundary.*

(i) Set $1 \leq p < \infty$, $u \in W^{1,p}(B^+_r(x_0))$ and $i \in \{1, \ldots, d-1\}$. It holds

$$\|D^h_i u\|_{L^p(B^+_s(x_0))} \leq \|\partial_{x_i} u\|_{L^p(B^+_r(x_0))} \qquad \forall 0 < |h| \leq \frac{1}{2}(r-s).$$

(ii) Set $1 < p < \infty$, $u \in L^p(B^+_r(x_0))$. Assume $i \in \{1, \ldots, d-1\}$ and

$$\sup_{0 < |h| \leq \frac{r-s}{2}} \|D^h_i u\|_{L^p(B^+_s(x_0))} = C < \infty.$$

Then u admits a weak derivative in direction i and $\|\partial_{x_i} u\|_{L^p(B^+_s(x_0))} \leq C$.

Proof. Exercise (argue as in the proof of Theorem 2.17) □

Theorem 2.23 (H^2 regularity up to the boundary)**.** *Assume $\Omega \subset \mathbb{R}^d$ is open and bounded. We consider the formal differential operator $Lu = -\mathrm{div}\,(aDu) + b \cdot Du + cu$ with $a_{ij}, b_j, c \in L^\infty(\Omega)$, and L uniformly elliptic.*

Assume in addition $\partial\Omega$ is C^2, $a \in C^1(\overline{\Omega})$, $f \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$ is a weak solution of $Lu = f$ in Ω with $u_{|\partial\Omega} = 0$.

Then $u \in H^2(\Omega)$ and there is a constant $C = C(\Omega, a, b, c) > 0$ such that

$$\|u\|_{H^2(\Omega)} \leq C \ \left[\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right].$$

Remark 1. Compared to Theorem 2.18 the additional requirements are: $\partial\Omega$ is C^2, $a \in C^1(\overline{\Omega})$ and $u_{|\partial\Omega} = 0$ (i.e. $u \in H_0^1(\Omega)$). Note that for $u \in H_0^1(\Omega)$ weak solution of $Lu = f$ we already know, by Lemma 2.14(i), that

$$\|Du\|_{L^2(\Omega)} \leq C \left[\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right]$$

holds for some constant $C > 0$, without requiring any boundary regularity for Ω. We also know, by Theorem 2.18, that $u \in H^2_{loc}(\Omega)$. The problem is to replace $\|D^2u\|_{L^2(V)}$ with $\|D^2u\|_{L^2(\Omega)}$.

Remark 2. Since the boundary is C^2 we can flatten it locally via a coordinate change. Assume the boundary is already flat near $x_0 \in \partial\Omega$ i.e. $\exists r > 0$ such that $\Omega \cap B_r(x_0) = B_r^+(x_0) = \{x \in B_r(x_0) |\, x_d > 0\}$. Let $\Gamma := \{x \in B_r(x_0) |\, x_d = 0\}$ be the corresponding lower boundary. Since $u \in H_0^1(\Omega)$ we have $u \in H^1(B_r^+(x_0))$ and $Tu_{|\Gamma} = 0$. The main idea is to show that the boundary condition $Tu_{|\Gamma} = 0$ allows to extend the first and second derivative norm down to the lower boundary.

Proof. As noted in remark 1 above, we already know, by Theorem 2.18, that $u \in H^2_{loc}(\Omega)$ and the PDE $Lu = f$ holds pointwise a.e. in Ω. We argue in two steps.

Step 1. Assume $\Omega = B_r^+(x_0) := \{x \in B_r(x_0) |\, x_d > 0\}$, $b, c \in L^\infty(\Omega)$, $a \in C^1(\overline{\Omega})$ (hence in particular $a, Da \in L^\infty(\Omega)$), $f \in L^2(\Omega)$, and $u \in H^1(\Omega)$ is a weak solution of $Lu = f$ satisfying $Tu_{|\Gamma} = 0$, where $\Gamma := \{x \in B_r(x_0) |\, x_d = 0\}$ (i.e. u is zero on the lower boundary of Ω, but not necessarily on the upper one).

Let W be an open set such that $W \subset\subset B_r(x_0)$ and define $V := W \cap B_r^+(x_0)$. There exists a constant $C = C_{V,a,b,c} > 0$ such that

(i) $\|Du\|_{L^2(V)} \leq C \ \left[\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right]$,

(ii) $\|D^2u\|_{L^2(V)} \leq C \ \left[\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right]$.

In particular this means $u \in H^2(V)$ and $\|u\|_{H^2(V)} \leq C' \ \left[\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right]$.

Proof of (i) Since u is a weak solution of $Lu = f$ we have $B_L[u, v] = (f, v)_{L^2}$ $\forall v \in H_0^1(\Omega)$. Take $\zeta \in C_c^\infty(B_r(x_0); [0, 1])$ with $\zeta_{|V} = 1$, then, since $Tu_{|\Gamma} = 0$, it holds $v := \zeta^2 u \in H_0^1(\Omega)$ is a possible test function. The result follows arguing as in the proof of Lemma 2.14.

Proof of (ii) Let W' be open with $W \subset\subset W' \subset\subset B_r(x_0)$, and set $h_0 := \frac{1}{4}\mathrm{dist}\,(W', \partial B_r(x_0))$ and $V' := W' \cap \Omega$. Let now $\zeta \in C_c^\infty(W'; [0, 1])$, such that $\zeta_{|V} = 1$ and define $v := -\sum_{i=1}^{d-1} D_i^{-h}\zeta^2 D_i^h u$. Then argue as in the proof of Theorem 2.18 to deduce that $\partial_i Du \in L^2(V)$ $\forall i = 1, \ldots, d-1$ and

$$\|\partial_i Du\|_{L^2(V)} \leq C \ \left[\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right] \tag{2.39}$$

(cf. also Lemma 2.22 above).
Finally use that $Lu = f$ holds pointwise to argue

$$\begin{aligned}
a_{dd}\partial_d^2 u &= Lu + \sum_{jk}(\partial_j a_{jk})\partial_k u + \sum_{j<d}\sum_k a_{jk}\partial_j\partial_k u - b\cdot Du - cu \\
&= f + \sum_{jk}(\partial_j a_{jk})\partial_k u + \sum_{j<d}\sum_k a_{jk}\partial_j\partial_k u - b\cdot Du - cu.
\end{aligned}$$

Since $a_{dd} \geq \theta > 0$ we have

$$\theta\|\partial_d^2 u\|_{L^2(V)} \leq \|f\|_{L^2(V)} + \Big(\|Da\|_\infty + \|b\|_\infty\Big)\|Du\|_{L^2(V)} + \|c\|_\infty\|u\|_{L^2(V)} + \|a\|_\infty\sum_{j=1}^{d-1}\|\partial_i Du\|_{L^2(V)}.$$

The bound now follows from (i) and (2.39).

Step 2. We consider now the general case.

Since $\partial\Omega$ is C^2, for all $x_0 \in \partial\Omega$ there exists $r = r_{x_0} > 0$ sucht that (eventually after relabelling and rotation of the variables)

$$\Omega \cap B_r(x_0) = \{x = (x', x_d) \in B_r(x_0) |\, x_d > \gamma(x')\},$$

where $\gamma \in C^2(\mathbb{R}^{d-1};\mathbb{R})$.

To flatten the boundary near x_0 we introduce the coordinate change

$$y = \Phi(x) := (x', x_d - \gamma(x')).$$

Then the function Φ is invertible with $x = \Phi^{-1}(y) = (y', y_d + \gamma(y'))$, both Φ and Φ^{-1} are C^2 and

$$\Phi(\Omega \cap B_r(x_0)) = \{y = (y', y_d) \in \Phi(B_r(x_0)) |\, y_d > 0\}.$$

Define $y_0 := \Phi(x_0)$. The set $\Phi(B_r(x_0))$ is open, hence $\exists s > 0$ such that $B_s(y_0) \subset \Phi(B_r(x_0))$ and in particular $B_s^+(y_0) \subset \Phi(\Omega \cap B_r(x_0))$. We define now

$$\begin{aligned}
&\tilde U := B_s^+(y_0),\ \tilde V := B_{s/2}^+(y_0),\ \tilde\Gamma := \partial\tilde U \cap \partial\Phi(\Omega), \\
&U := \Phi^{-1}(B_s^+(y_0)),\ V := \Phi^{-1}(B_{s/2}^+(y_0)),\ \Gamma := \partial U \cap \partial\Omega.
\end{aligned}$$

• We show now that $u \in H^2(V)$ holds and

$$\|u\|_{H^2(V)} \leq C\ \left[\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right].$$

To prove this result we translate the problem in the new coordinates y.
Since $u \in H_0^1(\Omega)$, we have $u \in H^1(U)$ and $Tu_{|\partial U\cap\partial\Omega} = 0$ in the trace sense. Moreover u is a weak solution of $Lu = f$ in U, hence $B_L[u, v] = (f, v)_{L^2(U)}\ \forall v \in H_0^1(U)$, which can be reformulated as

$$\int_U Dv\cdot(aDu)\, dx \ = \ \int_U v\,[f - b\cdot Du - cu]\, dx \qquad \forall v \in H_0^1(U).$$

We change into the y coordinates in both integrals. For any function $F(x)$ we denote by $\tilde F(y) := F\circ\Phi^{-1}(y)$ the same function in the new coordinates. The Jacobian of this coordinate change is 1, hence

$$\int_{\tilde U} \widetilde{Dv}\cdot(\tilde a\widetilde{Du})\, dy \ = \ \int_{\tilde U} \tilde v\,[\tilde f - \tilde b\cdot\widetilde{Du} - \tilde c\tilde u]\, dy \qquad \forall v \in H_0^1(U). \tag{2.40}$$

Since Φ, Φ^{-1} are C^2 it holds (exercise)

$$\begin{aligned} u \in H^1_0(U), \qquad &\Leftrightarrow \qquad \tilde{u} \in H^1_0(\tilde{U}), \\ u \in H^1(U),\ Tu_{|\Gamma} = 0 \ &\Leftrightarrow \tilde{u} \in H^1(\tilde{U}), \quad T\tilde{u}_{|\tilde{\Gamma}} = 0 \\ u \in H^2_{loc}(U), \qquad &\Leftrightarrow \qquad \tilde{u} \in H^2_{loc}(\tilde{U}). \end{aligned}$$

Moreover $u \in H^1(U)$ is a weak solution of $Lu = f$ in $U \Leftrightarrow \tilde{u} \in H^1(\tilde{U})$ is a weak solution of $\tilde{L}\tilde{u} = \tilde{f}$ in $\tilde{U}$, where $\tilde{L}\tilde{u} := -\mathrm{div}\,(AD\tilde{u}) + B \cdot D\tilde{u} + C\tilde{u}$, with (exercise)

$$A := \widetilde{D\Phi}\ \tilde{a}\ \widetilde{D\Phi}^t, \qquad B := \widetilde{D\Phi}\ \tilde{b}, \qquad C := \tilde{c}.$$

Written explicitely in components

$$\begin{aligned} A_{i'j'}(y(x)) = A_{i'j'} \circ \Phi(x) &= \sum_{ij} \partial_{x_i}\Phi_{i'}(x)\ a_{ij}(x)\ \partial_{x_j}\Phi_{j'}(x) \\ B_{j'}(y(x)) = B_{j'} \circ \Phi(x) &= \sum_{j} \partial_{x_j}\Phi_{j'}(x)\ b_j(x). \end{aligned}$$

Hence we are reduced to consider $\tilde{u} \in H^1(B_s^+(y_0))$ weak solution of $\tilde{L}\tilde{u} = \tilde{f}$ in $B_s^+(y_0)$ and satisfying $T\tilde{u}_{|\tilde{\Gamma}} = 0$, so we are in the setting of Step 1. To apply the corresponding result we must ensure that $A \in C^1(\overline{\tilde{U}})$, $B, C \in L^\infty(\tilde{U})$, $\tilde{f} \in L^2(\tilde{U})$ and A is uniformly elliptic.

Since $b, c \in L^\infty(U)$, and $\Phi \in C(\overline{U})$ we have $\tilde{b}, \tilde{c} \in L^\infty(\tilde{U})$.

Since $f \in L^2(\Omega)$, $\Phi \in C^1(\overline{U})$ and the Jacobian equals 1, we have $\|\tilde{f}\|_{L^2(\tilde{U})} = \|f\|_{L^2(U)}$.

Since $a \in C^1(\overline{\Omega})$, and $\Phi \in C^2(\overline{U})$ we have $A \in C^1(\overline{\tilde{U}})$.

It remains to prove that A is uniformly elliptic. We compute, for any $\xi \in \mathbb{R}^d$,

$$\xi^t A(y(x))\xi = \xi^t D\Phi(x)\ a(x)\ D\Phi^t(x)\xi = \eta^t a(x)\eta \geq \theta|\eta(x)|^2,$$

where $\eta(x) := D\Phi^t(x)\xi$. We have $(D\Phi)^{-1} = (D\Phi^{-1})\circ\Phi$, hence $\xi = (D\Phi(x))^{-1}\eta(x) = (D\Phi^{-1}(y(x)))\eta(x)$, and

$$|\xi| = |(D\Phi(x))^{-1}\eta(x)| \leq \|(D\Phi^{-1})\|_{L^\infty(\tilde{U})}|\eta(x)| = C|\eta(x)|.$$

It follows,

$$\xi^t A(y(x))\xi \geq \theta|\eta(x)|^2 \geq \frac{\theta}{C^2}|\xi|^2,$$

a.e. in U which implies, since Φ is invertible, that A is uniformly elliptic in $\tilde{U}$.

Step 1 now ensures $\tilde{u} \in H^2(\tilde{V})$ and $\|\tilde{u}\|_{H^2(\tilde{V})} \leq C\ \left[\|\tilde{f}\|_{L^2(\tilde{U})} + \|\tilde{u}\|_{L^2(\tilde{U})}\right]$. Changing coordinates in each integral we obtain $\|u\|_{H^2(V)}\| \leq C'\|\tilde{u}\|_{H^2(\tilde{V})}$, $\|u\|_{L^2(V)} = \|\tilde{u}\|_{L^2(\tilde{V})}$ and $\|f\|_{L^2(V)} = \|\tilde{f}\|_{L^2(\tilde{V})}$, from which the result follows. This completes the proof inside V.

• Finally we show that $u \in H^2(\Omega)$ and

$$\|u\|_{H^2(\Omega)} \leq C\ \left[\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right].$$

For each $x \in \partial\Omega$ we construct $V'_x := \Phi^{-1}(B_{s/2}(\Phi(x)))$ as above (but this time we map the whole ball back, not only half of it). Then V'_x is an open neighborhood of x, and $\partial\Omega \subset \bigcup_{x\in\partial\Omega} V'_x$. By compactness of $\partial\Omega$ there are $N \in \mathbb{N}$, $x_1, , \dots, x_N \in \partial\Omega$ such that $\partial\Omega \subset \bigcup_{j=1}^N V'_{x_j}$. Finally we add an open subset $V_0 \subset\subset \Omega$ such that $\Omega = V_0 \cup \bigcup_{j=1}^N V_{x_j}$, where $V_x := \Phi^{-1}(B^+_{s/2}(\Phi(x)))$.

By the arguments above, we know that $u \in H^2(V_{x_j})$ $\forall j = 1, \dots, N$. Moreover, $u \in H^2(V_0)$, by interior regularity, hence

$$\|u\|_{H^2(\Omega)} \le \|u\|_{H^2(V_0)} + \sum_{j=1}^N \|u\|_{H^2(V_{x_j})} \le C \ \left[\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right].$$

This conclude the proof of the theorem. □

Remark. Suppose 0 is not in the real spectrum of L, $0 \notin \Sigma(L)$. Then the equation $Lu = f$ has a unique weak solution $u \in H^1_0(\Omega)$ for all $f \in L^2(\Omega)$, and the operator $T^{-1}_{L|L^2(\Omega)} \colon L^2(\Omega) \to H^1_0(\Omega)$ is well defined and bounded. It follows $\|u\|_{L^2(\Omega)} \le \|T^{-1}\|_{op} \|f\|_{L^2(\Omega)}$ and hence

$$\|u\|_{H^2(\Omega)} \le \ \left[\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right] \le C'\|f\|_{L^2(\Omega)}.$$

This means the operator T^{-1} is bounded as a map from $(L^2(\Omega), \|\cdot\|_{L^2})$ to $(H^2(\Omega), \|\cdot\|_{H^2})$.

[11: 13.11.2023]

[12: 16.11.2023]

Theorem 2.24 (higher regularity up to the boundary)**.** *Assume $\Omega \subset \mathbb{R}^d$ is open and bounded. We consider the formal differential operator $Lu = -\operatorname{div}(aDu) + b \cdot Du + cu$ with L uniformly elliptic.*

Assume in addition $\partial\Omega$ is C^{2+m}, $a \in C^{1+m}(\overline{\Omega})$, $b, c \in C^m(\overline{\Omega})$, $f \in H^m(\Omega)$ and $u \in H^1_0(\Omega)$ is a weak solution of $Lu = f$ in Ω.

Then $u \in H^{2+m}(\Omega)$ and there is a constant $C = C(\Omega, a, b, c, m) > 0$ such that

$$\|u\|_{H^{2+m}(\Omega)} \le C \ \left[\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)}\right].$$

Proof. We argue by induction on m.

For $m = 0$ we have $a \in C^1(\overline{\Omega})$, $b, c \in C^0(\overline{\Omega})$, $\partial\Omega$ is C^2, $f \in L^2(\Omega)$ and the result follows from Theorem 2.23.

We prove now the first induction step: if the statement holds for $m = 0$, then the statement holds also for $m = 1$. Assume $a \in C^2(\overline{\Omega})$, $b, c \in C^1(\overline{\Omega})$, $\partial\Omega$ is C^3 and $f \in H^1(\Omega)$. Our goal is to show that $u \in H^3(\Omega)$.

• Since $u \in H^1_0(\Omega)$ is a weak solution of $Lu = f$ with $u_{|\partial\Omega} = 0$, by the case $m = 0$ we know that $u \in H^2(\Omega)$, $Lu = f$ holds pointwise a.e. in Ω, and $\|u\|_{H^2(\Omega)} \le C\,[\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}]$. Moreover, by Theorem 2.19, $u \in H^3_{loc}(\Omega)$, and the PDE

$$L(\partial_j u) = f_j := \partial_j f + \operatorname{div}[(\partial_j a)Du] - (\partial_j b) \cdot Du - (\partial_j c)u, \tag{2.41}$$

holds pointwise a.e. Note that since $a \in C^2(\overline{\Omega})$, $b, c \in C^1(\overline{\Omega})$, $f \in H^1(\Omega)$ and $u \in H^2(\Omega)$, it holds $f_j \in L^2(\Omega)$, and not just $L^2_{loc}(\Omega)$, as was the case in the proof of Theorem 2.19. Therefore $\partial_j u \in H^1(\Omega)$ is a weak solution (not just local weak solution) of (2.41) in Ω.

• Suppose in addition that $\partial_j u \in H^1_0(\Omega)$. In this case Theorem 2.23 implies that $\partial_j u \in H^2(\Omega)$, and $\|\partial_j u\|_{H^2(\Omega)} \le C\,[\|f_j\|_{L^2(\Omega)} + \|\partial_j u\|_{L^2(\Omega)}]$, which would then provide the desired estimate.

• In general $\partial_j u \notin H^1_0(\Omega)$, but the statement is true if we consider only derivatives in the direction parallel to the boundary. Indeed, as in the proof of Theorem 2.23 Step 1, we consider

$$U := B_r^+(x_0), \quad V := B_{1/2}^+(0), \quad \Gamma := \{x \in \partial U \,|\, x_d = 0\},$$

and $u \in H^1(U)$ is a weak solution of $Lu = f$ in U satisfying $Tu_{|\Gamma} = 0$ in the trace sense. Our problem can be locally reduced to the above setting by restricting to a small neighborhood of some point $x_0 \in \partial\Omega$ and performing a coordinate chance $\Phi\colon \mathbb{R}^d \to \mathbb{R}^d$, that flattens locally the boundary. The PDE $Lu = f$ will be transformed into $\tilde{L}\tilde{u} = \tilde{f}$, (cf. proof of Theorem 2.23) where, since $\partial\Omega$ and hence also Φ is C^3, the new coefficients satify $A, B, C \in C^2(\overline{\Omega})$. Moreover, since $\Phi \in C^3(\Omega)$ it holds $u \in H^3(\Omega) \Leftrightarrow \tilde{u} \in H^3(\Phi(\Omega))$.

Claim. It holds $\partial_j u \in H^1(V)$ and $T\partial_j u_{|\Gamma} = 0$ for all $i = 1, \dots, d-1$.

Consequence. Since $\partial_j u \in H^1(V)$ is a weak solution of $L(\partial_j u) = f_j$ in Ω, with $T\partial_j u_{|\Gamma} = 0$, it follows (cf Step 1. in the proof of Theorem 2.23) that $\partial_j u \in H^2(V)$ and

$$\|D^2 \partial_j u\|_{L^2(V)} \leq C\,[\|f_j\|_{L^2(\Omega)} + \|\partial_j u\|_{L^2(\Omega)}] \leq C\,[\|f\|_{H^1(\Omega)} + \|u\|_{L^2(\Omega)}].$$

Finally, since $u \in H^3_{loc}(\Omega)$ and hence the equation $L(\partial_j u) = f_j$ holds pointwise a.e. for all $j = 1, \dots d$, we deduce the same result also for $\partial_d^3 u$.

Proof of the Claim. If $u \in C^1(\overline{U})$ we have $Tu_{|\Gamma} = u(x', 0) = 0$ $\forall |x'| < r$. It follows $T\partial_j u_{|\Gamma} = \partial_{x'_j} u(0, x') = 0$ for $j = 1, \dots, d-1$. Since $u \in H^2(V)$ and V has Lipschitz boundary, there is a sequence $n \mapsto u_n \in C^1(\overline{V})$ such that $\|u - u_n\|_{H^2(V)} \to 0$. It follows $\|Tu - Tu_n\|_{L^2(\partial V)} \to 0$ and $\|T\partial_j u - T\partial_j u_n\|_{L^2(\partial V)} \to 0$. Therefore Tu is weakly differentiable along Γ with $\partial_j Tu = T\partial_j u = 0$ for $j = 1, \dots, d-1$.

□

Theorem 2.25 (infinite regularity up to the boundary)**.** *Assume $\Omega \subset \mathbb{R}^d$ is open and bounded. We consider the formal differential operator $Lu = -\operatorname{div}(aDu) + b \cdot Du + cu$ with $a_{ij}, b_j, c \in L^\infty(\Omega)$, and L uniformly elliptic.*

Assume in addition $a, b, c, f \in C^\infty(\overline{\Omega})$, $\partial\Omega$ is C^∞ and $u \in H^1_0(\Omega)$ is a weak solution of $Lu = f$ in Ω with boundary condition $u_{|\partial\Omega} = 0$.

Then $u \in C^\infty(\overline{\Omega})$ and $Lu = f$ holds pointwise in Ω.

Proof. By Theorem 2.24 it holds $u \in H^m(\Omega)$ $\forall m \in \mathbb{N}$. It follows, by generalized Sobolev inequality, $u \in C^k(\overline{\Omega})$ $\forall k \in \mathbb{N}$.

□

2.5 Maximum principles

Assume $\Omega \subset \mathbb{R}^d$ is open and bounded, and $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

(i) If $x_0 \in \Omega$ is a local maximum of u, then $Du(x_0) = 0$ and $D^2 u(x_0) \leq 0$ as a quadratic form, i.e. $\sum_{ij=1}^d \xi_i \partial_i \partial_j u(x_0) \xi_j \leq 0$ $\forall \xi \in \mathbb{R}^d$.

(ii) If $D^2 u(x) > 0$ (as a quadratic form) $\forall x \in \Omega$ then there is no local maximum in $\Omega-$

The following two results were shown in Introduction to PDE.

Weak maximum principle: if $-\Delta u \leq 0$ on Ω, then $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$.

Strong maximum principle: if $-\Delta u \leq 0$ on Ω and Ω is connected, then exactly one of the following holds:

(a) $u(x) < \max_{\partial\Omega} u \ \forall x \in \Omega$, or

(b) u is constant on Ω.

Our goal is to derive these results with $-\Delta u$ replaced by Lu uniformly elliptic operator. In this section it will be convenient to switch to non-divergence formulation: $Lu = -\mathrm{Tr}\,(aD^2u) + b \cdot Du + cu$. Since we want to derive pointwise estimates, we will assume below $a, b, c \in C(\overline{\Omega})$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$.
Since we work with $\mathrm{Tr}\,(aD^2u)$ the followig result will be useful.

Lemma 2.26. *(comparing matrices) Let $A \in \mathbb{R}^{n\times n}_{sym}$ such that $A \geq 0$ as a quadratic form. The following statements hold.*

(i) $\mathrm{Tr}\, A \geq 0$.

(ii) $\forall B, C \in \mathbb{R}^{n\times n}$ *such that $B \geq C$ (i.e. $B - C \geq 0$ as a quadratic form) we have*

$$\mathrm{Tr}\, AB \geq \mathrm{Tr}\, AC.$$

Proof.
(i) Since $A \geq 0$ we have $A_{jj} = (e_j, Ae_j) \geq 0 \ \forall j = 1, \ldots, n$, and hence $\mathrm{Tr}\, A \geq 0$.

(ii) Since $A^T = A$ we can write $A = V^T DV$, where $V \in \mathbb{R}^{n\times n}$ is orthogonal $V^T V = \mathrm{Id}$, $D = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$ and $\lambda_1, \ldots \lambda_n \in \mathbb{R}$ are eigenvalues of the matrix A. Since $A \geq 0$ we have $\lambda_j \geq 0 \ \forall j = 1, \ldots, n$. Hence the matrix

$$\sqrt{A} := V^T D^{1/2} V, \qquad D^{1/2} := \mathrm{diag}\,(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$$

is well defined and symmetric. Using the decomposition $A = \sqrt{A}\sqrt{A}$, we compute

$$\mathrm{Tr}\,(B - C)A = \mathrm{Tr}\,(B - C)\sqrt{A}\sqrt{A} = \mathrm{Tr}\,\sqrt{A}(B - C)\sqrt{A}.$$

Since $B - C \geq 0$ it follows $\sqrt{A}(B - C)\sqrt{A} \geq 0$ and hence, by (i), $\mathrm{Tr}\,\sqrt{A}(B - C)\sqrt{A} \geq 0$. Indeed for any vector $v \in \mathbb{R}^n$ we have

$$(v, \sqrt{A}(B - C)\sqrt{A}v) = (\sqrt{A}v, (B - C)\sqrt{A}v) = (\tilde{v}, (B - C)\tilde{v}) \geq 0,$$

where we used $\tilde{v} := \sqrt{A}v$ and $\sqrt{A}^T = \sqrt{A}$. □

2.5.1 Weak maximum principle

Theorem 2.27 (weak maximum principle I)**.** *Assume $\Omega \subset \mathbb{R}^d$ is open and bounded. We consider the differential operator $Lu = -\mathrm{Tr}\,(aD^2u) + b \cdot Du + cu$ with $a_{ij}, b_j, c \in C(\overline{\Omega})$, and L uniformly elliptic. Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Then the following statements hold.*

(i) *(a) If $Lu < 0$ on Ω and $c = 0$, then u has no local maximum in Ω.*

(b) If $Lu > 0$ on Ω and $c = 0$, then u has no local minimum in Ω.

(ii) Assume $Lu \leq 0$ on Ω.

(a) If $\begin{array}{l} c = 0 \text{ or} \\ c \geq 0 \text{ and } u \geq 0 \end{array}$ *then* $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$.

(b) *If* $c \geq 0$ *then* $\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u_+$*, where* $u_+(x) := \max\{0, u(x)\} \geq 0$.

(iii) *Assume* $Lu \geq 0$ *on* Ω.

(a) *If* $\begin{array}{l} c = 0 \text{ or} \\ c \geq 0 \text{ and } u \leq 0 \end{array}$ *then* $\min_{\overline{\Omega}} u = \min_{\partial\Omega} u$.

(b) *If* $c \geq 0$ *then* $\min_{\overline{\Omega}} u \geq \min_{\partial\Omega}(-u_-)$*, where* $u_-(x) := -\min\{0, u(x)\} \geq 0$.

Notation. If $Lu \leq 0$ on Ω, u is called a subsolution of $Lu = 0$ on Ω. If $Lu \geq 0$ on Ω, u is called a supersolution of $Lu = 0$ on Ω.

Proof. The idea is that at a maximum (minimum) point $x_0 \in \Omega$ $Lu(x_0) = -\mathrm{Tr}\, aD^2u(x_0)$, since $Du(x_0) = 0$. Then uniform ellipticity garantees that $Lu(x_0)$ is "equivalent to" $D^2u(x_0)$.

$(i)(a)$ Since $c = 0$ we have $Lu = -\mathrm{Tr}\, aD^2u + b \cdot Du$. By contradiction, assume $Lu < 0$ on Ω and $x_0 \in \Omega$ is a local maximum. Then $Du(x_0) = 0$ and $D^2u(x_0) \leq 0$. Therefore

$$Lu(x_0) = -\mathrm{Tr}\, a(x_0)D^2u(x_0) + b(x_0) \cdot Du(x_0) = -\mathrm{Tr}\, a(x_0)D^2u(x_0) = \mathrm{Tr}\, AB,$$

where $A := a(x_0) \geq \theta \mathrm{Id}$, $B := -D^2u(x_0) \geq 0$ and $B^T = B$. It follows (cf. Lemma 2.26 below)

$$0 > Lu(x_0) = \mathrm{Tr}\, AB \geq \theta \mathrm{Tr}\, B \geq 0,$$

which gives a contradiction.

$(i)(b)$ Use $(i)(a)$ on the function $-u$.

$(ii)(a)$ *Case 1:* $c = 0$ and $Lu \leq 0$. We will construct below a function $v \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $Lv(x) < 0$ on Ω. Define $u_\varepsilon := u + \varepsilon v$, with $\varepsilon > 0$. Then, since $Lu(x) \leq 0$ and $Lv(x) < 0$ we have

$$Lu_\varepsilon(x) = Lu(x) + \varepsilon Lv(x) < 0 \qquad \forall x \in \Omega,$$

hence by (i) u_ε admits no local maximum. In particular this means $u_\varepsilon(x) \leq \max_{\partial\Omega} u_\varepsilon$ $\forall x \in \Omega$, $\forall \varepsilon > 0$. The result follows taking the limit $\varepsilon \to 0$.

Construction of v. In the special case $Lu = -\Delta u$ is suffices to take $v(x) := |x|^2/2$. Indeed by direct computation $-\Delta v = -d < 0$.
Consider now the general case $Lu = -\mathrm{Tr}\,(aD^2u) + b \cdot Du$. We take v to be a function of only one variable, say x_1, and of the form $v(x) := e^{\lambda x_1}$, where $\lambda \in \mathbb{R}$ is a parameter to choose. We compute

$$\begin{aligned} Dv(x) &= \lambda e^{\lambda x_1} e_1, \qquad D^2u = \lambda^2 e^{\lambda x_1} e_1 \otimes e_1, \\ Lv(x) &= [-a_{11}(x)\lambda^2 + b_1(x)\lambda]\, e^{\lambda x_1}. \end{aligned}$$

Therefore $Lv(x) < 0$ iff $[-a_{11}(x)\lambda^2 + b_1(x)\lambda] < 0$. By uniform ellipticity

$$\begin{aligned} -a_{11}(x)\lambda^2 + b_1(x)\lambda &\leq -\theta\lambda^2 + b_1(x)\lambda \leq -\theta\lambda^2 + \|b_1\|_{L^\infty}|\lambda| \\ &= |\lambda|\,[-\theta|\lambda| + \|b_1\|_{L^\infty}] < 0 \qquad \forall |\lambda| > \frac{\|b_1\|_{L^\infty}}{\theta}. \end{aligned}$$

This concludes the proof of Case 1.

Case 2: $c \geq 0$, $u \geq 0$ and $Lu \leq 0$. We can write $Lu = L_0u + cu$, where

$$L_0u := -\mathrm{Tr}\,(aD^2u) + b \cdot Du = Lu - cu.$$

Since $u, c \geq 0$ we have $-cu \leq 0$ and hence, using $Lu \leq 0$,

$$L_0 u(x) = Lu(x) - c(x)u(x) \leq 0.$$

The result follows from Case 1. This concludes the proof of $(ii)(a)$

$(ii)(b)$ Apply $(ii)(a)$ to the function $-u$.

$(iii)(a)$ Assume $Lu \leq 0$ and define $V := \{x \in \Omega |\, u(x) > 0\}$.
If $V = \varnothing$, then $u \leq 0$ on $\overline{\Omega}$, $u_{+|\partial\Omega} = 0$ and the statement holds.

Assume now $V \neq \varnothing$. Then $u_{|V} = u_{+|V}$, and since u is continuous, the set V is open. Therefore $u_{+|V} = u_{|V} \in C^2(V) \cap C(\overline{V})$ and, since $u > 0$ on V we have

$$L_0 u_+(x) = Lu(x) - c(x)u(x) \leq 0 \qquad \forall x \in V.$$

By $(ii)(a)$ we have $\max_{\overline{V}} u_+ = \max_{\partial V} u_+$. Since $u(x) \leq u_+(x)\ \forall x \in \Omega$ and $u_{+|\Omega\setminus V} = 0$, it follows

$$\max_{\overline{\Omega}} u \leq \max_{\overline{\Omega}} u_+ = \max_{\overline{V}} u_+ = \max_{\partial V} u_+.$$

We have $\partial V = (\partial V \cap \Omega) \cup (\partial V \cap \partial\Omega)$. Since $u_{+|\partial V\cap\Omega} = 0$ and $u_{+|\partial V \cap \partial\Omega} \geq 0$ it follows

$$\max_{\overline{\Omega}} u \leq \max_{\partial V} u_+ = \max_{\partial\Omega} u_+.$$

This concludes the proof of $(iii)(a)$.

$(iii)(b)$ Apply $(iii)(a)$ to the function $-u$. $\square$

Remark 1. If $u_{|\partial\Omega} \leq 0$ and $Lu \leq 0$ it follows $\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u_+ = 0$, i.e. $u(x) \leq 0\ \forall x \in \Omega$.

Remark 2. If $Lu = 0$ and $c \geq 0$ in Ω it follows

$$\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u|.$$

Indeed

$$Lu \leq 0 \Rightarrow u(x) \leq \max_{\partial\Omega} u_+ \leq \max_{\partial\Omega} |u|,$$
$$Lu \geq 0 \Rightarrow -\max_{\partial\Omega} |u| \leq -\max_{\partial\Omega} u_- \leq u(x).$$

The result follows.

[12: 16.11.2023]

[13: 20.11.2023]

2.5.2 Strong maximum principle

Theorem 2.28. *Assume $\Omega \subset \mathbb{R}^d$ is open, bounded and connected.*
We consider the differential operator $Lu = -\operatorname{Tr}(aD^2u) + b \cdot Du + cu$ with $a_{ij}, b_j, c \in C(\overline{\Omega})$, and a uniformly elliptic. Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and define

$$M := \max_{\overline{\Omega}} u, \qquad m := \min_{\overline{\Omega}} u.$$

Then the following statements hold.

(i) Assume one of these conditions holds.

(a) $c = 0$ *and* $Lu \leq 0$ *in* Ω*, or*

(b) $c \geq 0$, $Lu \leq 0$ *in* Ω *and* $M \geq 0$.

Then either $u(x) < M\ \forall x \in \Omega$ *or* $u \equiv M$ *is constant on* Ω.

(ii) Assume one of these conditions holds.

(a) $c = 0$ *and* $Lu \geq 0$ *in* Ω*, or*

(b) $c \geq 0$, $Lu \geq 0$ *in* Ω *and* $m \leq 0$.

Then either $u(x) > m\ \forall x \in \Omega$ *or* $u \equiv m$ *is constant on* Ω.

Strategy of the proof To prove (i) we define $V := \{x \in \Omega |\, u(x) < M\}$, $C := \{x \in \Omega |\, u(x) = M\}$. Then $\Omega = V \cup C$ and, since u is continuous, V is open. Our goal is to show that, if $V \neq \varnothing$, then $V = \Omega$.
By contradiction, assume $V \neq \varnothing$ and $V \subsetneq \Omega$. In particular this means $\partial V \cap \Omega \neq \varnothing$. Note also that $\partial V \cap \Omega \subset C$, hence every point in this set is a local maximum. The strategy is to show that there must be a point $x_1 \in \partial V \cap \Omega$ with $Du(x_1) \neq 0$, which contradicts the fact that this is a local maximum. To understand how this works we consider first a simple example.

Example Assume $d = 1$, $V = (a, b)$ such that $[a, b] \subset \Omega$, and $a \in \partial V \cap \Omega$ or $b \in \partial V \cap \Omega$. Assume in addition $Lu = -u''$.
If $b \in \partial V \cap \Omega$, then $u(x) < u(b) = M\ \forall x \in (a, b)$. It follows, that there exists a point $x \in (a, b)$ with $u'(x) > 0$. Hence, since $-u'' \leq 0$ on Ω, we have

$$u'(y) \geq u'(x) > 0 \qquad \forall y \in [x, b],$$

and therefore $u'(b) > 0$, which is impossible since b is a maximum point inside Ω.
In the same way we argue that $u'(a) < 0$ if $a \in \partial V \cap \Omega$, which again contradicts the fact that a is a maximum point.

In the general case, we need to find a point $x_1 \in \partial V \cap \Omega$ satisfying the requirements of the following lemma.

Theorem 2.29 (Hopf's lemma)**.** *Assume* $\Omega \subset \mathbb{R}^d$ *is open, bounded and connected.*
We consider the differential operator $Lu = -\mathrm{Tr}\,(aD^2u) + b \cdot Du + cu$ *with* $a_{ij}, b_j, c \in C(\overline{\Omega})$*, and* a *uniformly elliptic.*
Assume $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$*, and there is a point* $x_0 \in \partial\Omega$ *such that*

- Ω *satisfies interior ball regularity at* x_0*, i.e.* $\exists y \in \Omega$, $r > 0$ *with* $B_r(y) \subset \Omega$ *and* $x_0 \in \partial B_r(y)$*, and*
- $u(x) < u(x_0)\ \forall x \in \Omega$.

We consider $\partial_\nu u(x_0) := \nu_{x_0} \cdot Du(x_0)$*, where* ν_{x_0} *is the outward unit normal to* $\partial B_r(y)$ *in* x_0*. Then the following statements hold.*

(i) If $c = 0$ *and* $Lu \leq 0$ *in* Ω*, then* $\partial_\nu u(x_0) > 0$.

(ii) If $c \geq 0$, $Lu \leq 0$ *in* Ω *and* $u(x_0) \geq 0$*, then* $\partial_\nu u(x_0) > 0$.

Remark 1. From $u(x) < u(x_0)$ it follows that $\partial_\nu u(x_0) \geq 0$. The non trivial part is to prove that the inequality is strict $\partial_\nu u(x_0) > 0$.

Remark 2. If $\partial\Omega$ is C^2, interior ball regularity holds in every point $x \in \partial\Omega$.

We will prove this theorem below. Now, using Hopf's lemma, we can prove the strong maximum principle.

Proof of Theorem 2.28.
(i) We define $V := \{x \in \Omega |\, u(x) < M\}$, $C := \{x \in \Omega |\, u(x) = M\}$. Then $\Omega = V \cup C$ and, since u is continuous, V is open. Our goal is to show that, if $V \neq \varnothing$, then $V = \Omega$.
By contradiction, assume $V \neq \varnothing$ and $V \subsetneq \Omega$. In particular this means $\partial V \cap \Omega \neq \varnothing$. We look for a point $x_1 \in \partial V \cap \Omega$ with $Du(x_1) \neq 0$, which contradicts the fact that this must be a local maximum.

Indeed, since $\partial V \cap \Omega \neq \varnothing$ there is some point $x_0 \in \partial V \cap \Omega$. Since $x_0 \in \partial V$, for each $\varepsilon > 0$ there is $y_\varepsilon \in V$ such that $|y_\varepsilon - x_0| \leq \varepsilon$. Set now $\varepsilon_0 := \frac{\operatorname{dist}(x_0,\partial\Omega)}{4}$ and $y := y_{\varepsilon_0}$. We have $\operatorname{dist}(y, \partial V \cap \Omega) \leq |y - x_0| \leq \varepsilon_0$ and $\operatorname{dist}(y, \partial\Omega) \geq \operatorname{dist}(x_0, \partial\Omega) - |y - x_0| \geq 3\varepsilon_0$ and therefore

$$\operatorname{dist}(y, \partial V \cap \Omega) < \operatorname{dist}(y, \partial\Omega). \tag{2.42}$$

Since V is open, $B_r(y) \subset V$ for some $r > 0$. We define

$$R := \sup\{r > 0 |\, B_r(y) \subset V\}.$$

From (2.42) it follows that $\overline{B_R(y)} \subset \Omega$ and $\exists x_1 \in \partial V \cap \partial B_R(y)$.

Then $u \in C^2(B_R(y)) \cap C^1(\overline{B_R(y)})$, $Lu \leq 0$ on $B_R(y)$, $u(x) < u(x_1) = M$ $\forall x \in B_R(y)$, and, if $c \geq 0$ we have assumed $M = u(x_1) \geq 0$. Finally the set $B_R(y)$ satisfies interior ball regularity and all boundary points, hence in particular at x_1.
The result now follows from Hopf's lemma applied to the set $B_R(y)$.

(ii) Apply (i) to the function $-u$.

□

We now prove Hopf's lemma.

Proof of Theorem 2.29.

The case of $d = 1$ The ball $B_r(y)$ is replaced by $I = (y - r, y + r)$. Performing a translation we can reduce to the case $I = (-r, +r)$. To make the formulas more readable, in the following we write $a = -r$, $b = r$.
We distinguish three cases.

Case 1. Assume $x_1 = b$ and $a \in V$ i.e. $[a, b) \subset V$ and hence u $u(x) < u(b)$ $\forall x \in [a, b)$. We assume one of the following two conditions holds:

(α) $c = 0$, and $Lu = L_0 u = -a(x)u''(x) + b(x)u(x) \leq 0$, or

(β) $c \geq 0$, $Lu = L_0 u + c(x)u(x) \leq 0$ and $u(b) \geq 0$.

Our goal is to prove that $u'(b) > 0$.
Idea. Note that <u>if</u> $u'(b) > 0$ and $u(x) < u(b)$ $\forall a \leq x < b$ then, there exists a function $\bar{u} \in C^1(\bar{I})$ such that

$$u(x) \leq \bar{u}(x) \leq u(b) \ \forall x \in I, \quad \text{and} \quad 0 < \bar{u}'(b) < u'(b).$$

We can reformulate this statement by defining $w := \bar{u} - u$. Then $0 \le w(x) \le u(b) - u(x)\ \forall x \in I$, in particular $w(b) = 0$, and $w'(b) = \bar{u}'(b) - u'(b) < 0$.

Rigorous argument. Inspired by the idea above, assume $\exists w \in C^1(\bar{I})$ such that

$$0 \le w \le u(b) - u, \quad \text{and} \quad w'(b) < 0.$$

We claim that then $u'(b) > 0$. Indeed, setting $\bar{u} := u + w$, we have $\bar{u}(x) \le u(b)\ \forall x \in [a,b]$ and $\bar{u}(b) = u(b)$. It follows that $\bar{u}'(b) \ge 0$. We compute now

$$u'(b) + w'(b) = \bar{u}'(b) \ge 0 \ \Rightarrow\ u'(b) \ge -w'(b) > 0,$$

where in the last inequality we used $w'(b) < 0$.

Construction of w. It is not difficult to construct a function w satisfying $w \ge 0$, $w(b) = 0$ and $w'(b) < 0$. The hard part is to satify the constraint $w \le u(b) - u$.
Remember that, by weak maximum principle, and using $u(a) < u(b)$, we have:

$$\begin{aligned} &\text{if } c = 0, \quad Lu \le 0 && \Rightarrow && \max_{\bar{I}} u = \max\{u(a), u(b)\} = u(b), \\ &\text{if } c \ge 0, \quad Lu \le 0, \quad u(b) \ge 0 && \Rightarrow && \max_{\bar{I}} u \le \max\{u_+(a), u_+(b)\} = u(b). \end{aligned}$$

Claim. $\exists v \in C^2(I) \cap C^1(\bar{I})$, such that

$$v \ge 0, \qquad v(b) = 0, \qquad v'(b) < 0, \qquad Lv \le 0.$$

Proof. Take the ansatz $v(x) := (e^{-\lambda x} - e^{-\lambda b})$, with $\lambda > 0$ a parameter to choose later.
Since $\lambda > 0$ we have $v(x) > 0\ \forall x < b$ and $v(b) = 0$. Moreover $v'(x) = -\lambda e^{-\lambda x} < 0\ \forall x \in [a,b]$.
Finally, using $c(x) \ge 0$ and $a(x) \ge \theta \mathrm{Id}$, we get

$$\begin{aligned} Lv &= -a(x)v''(x) + b(x)v'(x) + c(x)v(x) = [-a(x)\lambda^2 - b(x)\lambda + c(x)]e^{-\lambda x} - c(x)e^{-\lambda b} \\ &\le [-a(x)\lambda^2 - b(x)\lambda + c(x)]e^{-\lambda x} \le [-\theta\lambda^2 + \|b\|_{L^\infty}\lambda + \|c\|_{L^\infty}]e^{-\lambda x} < 0 \end{aligned}$$

for λ large enough. This concludes the proof of the Claim.

Set now $w := \varepsilon v$, with $\varepsilon > 0$ a parameter to choose later. Then $w \in C^2(I) \cap C^1(\bar{I})$, $w \ge 0$, $w(b) = 0$, $w'(b) < 0$. It remains to check that $w \le u(b) - u$.
Set $\bar{u} := u + w$. Using $Lw \le 0$ we have

$$L\bar{u} = Lu + Lw \le 0.$$

Note that $\bar{u}(b) = u(b)$ and

$$\bar{u}(a) = u(a) + \varepsilon v(a) < u(b), \qquad \text{for } \varepsilon \quad \text{small enough.}$$

Hence, by weak maximum principle, if (α) or (β) holds we have

$$\bar{u}(x) \le u(b) \qquad \forall x \in [a,b].$$

It follows $w(x) = \bar{u}(x) - u(x) \le u(b) - u(x)$. This concludes the proof in Case 1.

Case 2. Assume $x_1 = a$ and $b \in V$ i.e. $(a,b] \subset V$. Using the function $v(x) := e^{\lambda x} - e^{\lambda a}$ we can repeat the above arguments to prove $u'(a) < 0$.

Case 3. Assume both $a, b \in \partial V$ i.e. $u(-r) = u(+r) = M$ and $u(x) < M \ \forall -r < x < r$. The argument above cannot work because $u(-r) + \varepsilon v(-r) > u(-r) = M$ for any $\varepsilon > 0$. Hence we need a function v such that $v(-r) = v(r) = 0$. We take the ansatz

$$v(x) := e^{-\lambda x^2} - e^{-\lambda r^2}.$$

This function satisfies $v \in C^\infty([-r, r])$, $v(x) < 0$ for $|x| < r$ and $v(\pm r) = 0$. Moreover setting $x_1 = r$ we argue $v'(x_1) = v'(r) = -2\lambda r e^{-\lambda r^2} < 0$ for any $\lambda > 0$. By direct computation

$$Lv(x) = \left[-4x^2\lambda^2 a(x) + 2a(x)\lambda - 2b(x)x\lambda + c(x)\right] e^{-\lambda x^2} - c(x)e^{-\lambda r^2}.$$

Note that $Lv \leq 0$ <u>cannot hold</u> on $(-r, r)$. Indeed for $x = 0$ we compute

$$Lv(0) = 2a(0)\lambda + c(0)(1 - e^{-\lambda r^2}) \geq 2\theta\lambda + c(0)(1 - e^{-\lambda r^2}) \geq 2\theta\lambda > 0.$$

The solution is to consider v only on the set $\{\frac{r}{2} < |x| < r\}$. On this set we have

$$Lv \leq \left[-r^2\lambda^2\theta + 2\|a\|_\infty\lambda + 2\|b\|_\infty r\lambda + \|c\|_\infty\right] e^{-\lambda x^2} < 0$$

for $\lambda > 0$ large enough. The price to pay is that we have additional points on the boundary $x = \pm r/2$. Since on these points we have $u(x) < M$ we can argue as in case 1 and 2.

The case of $d > 1$. By rotating and translating we can reduce to the case $y = 0$, and $x_0 = re_1$. Set

$$B := B_r(y) = B_r(0), \qquad R := B_r(0) \setminus \overline{B_{r/2}(0)}.$$

In the same spirit as $d = 1$ Case 3, we look for a function $v \in C^2(R) \cap C^1(\overline{R})$, and a parameter $\varepsilon > 0$, such that

$$\begin{cases} v(x) \geq 0 \text{ on } R, \\ \partial_\nu v(x_0) = \partial_1 v(re_1) < 0 \\ Lv \leq 0 \text{ on } R, \\ v(x_0) = 0, \text{ and } \ (u + \epsilon v)_{|\partial R} \leq u(x_0). \end{cases}$$

We take the ansatz

$$v(x) := (e^{-\lambda|x|^2} - e^{-\lambda r^2}),$$

with $\lambda > 0$ a parameter to choose later. Then $v \in C^\infty(\overline{B})$, $v \geq 0$ on B and $v_{|\partial B} = 0$. Hence $(u + \epsilon v)_{|\partial B} = u_{|\partial B} = u(x_0)$. Moreover $\partial_\nu v(x_0) = \partial_1 v(re_1) = -2\lambda r e^{-\lambda r^2} < 0$. It remains to check that $Lv \leq 0$ and $(u + \epsilon v)_{|\partial B_{r/2}(0)} \leq u(x_0)$. We compute

$$\partial_j v = -2\lambda x_j e^{-\lambda|x|^2}, \qquad \partial_i \partial_j v = [4\lambda^2 x_i x_j - 2\lambda\delta_{ij}]e^{-\lambda|x|^2},$$

hence, using $a \geq \theta$, and $r/2 \leq |x| \leq r$, we get

$$\begin{aligned} Lv(x) &= [-4\lambda^2(x, a(x)x) + 2\lambda \mathrm{Tr}\, a(x) - 2\lambda b(x) \cdot x + c(x)]e^{-\lambda|x|^2} - c(x)e^{-\lambda r^2} \\ &\leq [-4\lambda^2(x, a(x)x) + 2\lambda \mathrm{Tr}\, a(x) - 2\lambda b(x) \cdot x + c(x)]e^{-\lambda|x|^2} \\ &\leq [-4\lambda^2\theta|x|^2 + 2\lambda(d\|a\|_{L^\infty} + |x|\|b\|_{L^\infty}) + \|c\|_{L^\infty}]e^{-\lambda|x|^2} \\ &\leq [-\lambda^2\theta r^2 + 2\lambda(d\|a\|_{L^\infty} + r\|b\|_{L^\infty}) + \|c\|_{L^\infty}]e^{-\lambda|x|^2} < 0 \end{aligned}$$

for λ large enough. Finally for $x \in \partial B_{r/2}(0)$, we have

$$\bar{u}(x) = u(x) + \varepsilon(e^{-\lambda\frac{r^2}{4}} - e^{-\lambda r^2}) \leq \max_{\partial B_{r/2}(0)} u + \varepsilon\alpha.$$

where $\alpha := (e^{-\lambda\frac{r^2}{4}} - e^{-\lambda r^2}) > 0$. Since $\max_{\partial B_{r/2}(0)} u < u(x_0)$ there exists ε satisfying

$$0 < \varepsilon < \frac{\max_{\partial B_{r/2}(0)} u}{\alpha}.$$

For this choice of ε, we have $\bar{u}_{|\partial B_{r/2}(0)} \leq u(x_0)$. This concludes the construction of v and hence the proof of the theorem.

□

[13: 20.11.2023]

[14: 23.11.2023]

2.6 Harnack's inequality

Theorem 2.30 (Harnack's inequality on balls)**.** *Assume $\Omega \subset \mathbb{R}^d$ is open and bounded. We consider the formal differential operator $Lu = -\mathrm{div}\,(aDu)$ with $a_{ij} \in L^\infty(\Omega)$, and a uniformly elliptic.*

Then $\exists C = C(a, d) > 1$ such that for all $u \in H^1(\Omega)$ weak solution of $Lu = 0$ in Ω with $u \geq 0$ a.e. in Ω we have

$$\sup_{B_R(x_0)} u \leq C \inf_{B_R(x_0)} u, \qquad \forall R > 0, x_0 \in \Omega \;\; s.t. \;\; B_{4R}(x_0) \subset \Omega, \tag{2.43}$$

where by sup *and* inf *we mean the essential* sup *and essential* inf .

Remark 1 Harnack's inequality is a quantitative version of the strong maximum principle. Indeed, assuming $a \in C^1(\Omega)$ and $u \in C^2(\Omega, [0, \infty)) \cap H^1_0(\Omega)$ is a weak solution of $-\mathrm{div}\,(aDu) = 0$, we have $Lu = -\mathrm{Tr}\, aD^2u - Da \cdot Du = 0$. Since $\inf_{B_{2R}(x_0)} u \geq 0$ and $B_{2R}(x_0)$ is connected, the strong maximum principle ensures that either $u(x) > 0$ for all $x \in B_{2R}(x_0)$ or $u = 0$ on the ball hence

$$\min_{\overline{B_R(x_0)}} u = 0 \qquad \Rightarrow \qquad \max_{\overline{B_R(x_0)}} u = 0.$$

Harnack inequality provides a more quantitative estimate of the relation between sup and inf.

Remark 2 Remember that if $u \in C^2(\Omega)$ is a solution of $\Delta u = 0$ then

$$u(x) = \fint_{B_R(x)} u(y)dy = \frac{1}{|B_R(x)|} \int_{B_r(x)} u(y)dy \qquad \forall B_R(x) \subset\subset \Omega. \tag{2.44}$$

Assume $u \geq 0$ and $B_{3R}(x_0) \subset\subset \Omega$. Then $\forall x, y \in B_R(x_0)$ we have

$$u(x) = \frac{1}{|B_R(x)|} \int_{B_R(x)} u(z)dz \leq \frac{1}{|B_R(x)|} \int_{B_{2R}(y)} u(z)dz = \frac{|B_{2R}(y)|}{|B_R(x)|} u(y) = 2^d u(y),$$

which implies $u(x) \leq 2^d u(y)$ $\forall x, y \in B_R(x_0)$ and hence Harnack's inequality holds with $C = 2^d$.

Proof of Harnack's inequality in the case of smooth coefficents (sketch).
Assume $a_{ij} \in C^\infty(\Omega)$ and $u \in H_0^1(\Omega)$ is a weak solution of $Lu = -\mathrm{div}\,(aDu) = 0$. By improved regularity $u \in C^\infty(\Omega)$. Our goal is to find a constant $C = C(a,d) > 1$ such that

$$u(x) \le C\ u(y) \qquad \forall x, y \in B_R(x_0). \tag{2.45}$$

For $\varepsilon > 0$ consider the function $u_\varepsilon := u + \varepsilon$. Since $u \ge 0$ we have $u_\varepsilon := u + \varepsilon \ge \varepsilon > 0$ and since $Du = Du_\varepsilon$, we have $\mathrm{div}\,(aDu_\varepsilon) = \mathrm{div}\,(aDu) = 0$. We argue, for each $x, y \in B_R(x_0)$

$$u_\varepsilon(x) = \frac{u_\varepsilon(x)}{u_\varepsilon(y)} u_\varepsilon(y) = e^{v(x)-v(y)} u_\varepsilon(y) \le e^{|v(x)-v(y)|} u_\varepsilon(y).$$

where we defined $v := \ln u_\varepsilon$. Since $u_\varepsilon > 0$ and $u_\varepsilon \in C^\infty(\Omega)$ the function v is well defined and $v \in C^\infty(\Omega)$. Note that

$$|v(x) - v(y)| = |\int_0^1 Dv(y + \tau(x-y)) \cdot (x-y) d\tau| \le |x-y| \sup_{B_R(x_0)} |Dv|.$$

The hard part is to show that $\sup_{B_R(x_0)} |Dv|$ is bounded by a constant independent of u_ε. The main trick is to remark that

$$Lu = 0 \qquad \Rightarrow \qquad Lv = Dv \cdot aDv.$$

Indeed $Dv = \frac{1}{u_\varepsilon} Du$ and hence, using also $Lu = 0$,

$$Lv = -\mathrm{div}\,(aDv) = -\mathrm{div}\,\left(\frac{1}{u_\varepsilon} aDu\right) = -\frac{1}{u_\varepsilon} Lu - \sum_{jk} a_{jk} \partial_k u \partial_j \frac{1}{u_\varepsilon} = \sum_{jk} a_{jk} \partial_k u \partial_j u \frac{1}{u_\varepsilon^2} = Dv \cdot aDv.$$

Extracting information on Dv from the <u>nonlinear</u> PDE above requires some work (see Evans). Here we consider directly the more general case $a \in L^\infty(\Omega)$.

□

We will see the proof Harnack's inequality only for $d \ge 3$ (for $d = 1, 2$ there are simpler arguments, see exercise sheet). The idea is to replace in (2.45) $u_\varepsilon(x)$ with $\left(⨍_{B_{2R}(x_0)} u^p\right)^{\frac{1}{p}}$ for some $p > 0$, $u_\varepsilon(y)$ with $\left(⨍_{B_{2R}(x_0)} u^{-p}\right)^{-\frac{1}{p}}$ and $\sup |D \ln u_\varepsilon|$ with $\|D \ln u\|_{L^2(B_{2R}(\Omega))}$. We will need the following results and definitions.

Definition 2.31 (sub and supersolution)**.** *Assume $\Omega \subset \mathbb{R}^d$ is open and bounded. We consider the formal differential operator $Lu = -\mathrm{div}\,(aDu)$ with $a_{ij} \in L^\infty(\Omega)$, and a uniformly elliptic.*

(i) $u \in H^1(\Omega)$ is a weak subsolution of $Lu = 0$ if

$$B_L[u, v] \le 0 \qquad \forall v \in H_0^1(\Omega) \text{ with } v \ge 0.$$

(ii) $u \in H^1(\Omega)$ is a weak supersolution of $Lu = 0$ if

$$B_L[u, v] \ge 0 \qquad \forall v \in H_0^1(\Omega) \text{ with } v \ge 0.$$

Theorem 2.32 (weak Harnack inequality)**.** *Assume $\Omega \subset \mathbb{R}^d$ is open and bounded, $d \geq 3$. We consider the formal differential operator $Lu = -\operatorname{div}(aDu)$ with $a_{ij} \in L^\infty(\Omega)$, and a uniformly elliptic. The following hold.*

(i) *$\forall p > 0$ $\exists C_1 = C_1(a, p, d) > 0$ such that $\forall u \in H^1(\Omega; [0, \infty))$ weak subsolution of $Lu = 0$ in Ω it holds*

$$\sup_{B_R(x_0)} u \leq C_1 \left(\fint_{B_{2R}(x_0)} u^p \right)^{\frac{1}{p}} \qquad \forall B_{4R}(x_0) \subset \Omega. \tag{2.46}$$

(ii) *$\forall q > 0$ $\exists C_2 = C_2(a, q, d) > 0$ such that $\forall u \in H^1(\Omega; [0, \infty))$ weak supersolution of $Lu = 0$ in Ω it holds*

$$\inf_{B_R(x_0)} u \geq C_2 \left(\fint_{B_{2R}(x_0)} u^{-q} \right)^{-\frac{1}{q}} \qquad \forall B_{4R}(x_0) \subset \Omega. \tag{2.47}$$

Remark. Assume $u \in C^2(\Omega; [0, \infty))$.
If $-\Delta u \leq 0$, then $u(x) \leq \fint_{B_r(x)} u(y)dy$ $\forall B_r(x) \subset\subset \Omega$. Assuming $B_{2R}(x_0) \subset\subset \Omega$, we argue for all $x \in B_R(x_0)$,

$$u(x) \leq \fint_{B_R(x)} u\, dy \leq 2^d \fint_{B_{2R}(x_0)} u\, dy,$$

which gives the weak Harnack's inequality (i) for $p = 1$.

Assume now $u \geq \varepsilon > 0$ on Ω, and $-\Delta u \geq 0$. Then $1/u \in C^2(\Omega; (0, \infty))$ and $-\Delta(1/u) \leq 0$. Indeed

$$-\Delta \frac{1}{u} = \frac{\Delta u}{u^2} - 2\frac{|Du|^2}{u^3} \leq 0.$$

Hence

$$\sup_{B_R(x_0)} \frac{1}{u} \leq 2^d \fint_{B_{2R}(x_0)} u^{-1} dy.$$

It follows

$$\inf_{B_R(x_0)} u \geq 2^{-d} \left(\fint_{B_{2R}(x_0)} u^{-1} dy \right)^{-1},$$

which gives the weak Harnack's inequality (ii) for $q = 1$.

Theorem 2.33 (bound on log variation)**.** *Assume $\Omega \subset \mathbb{R}^d$ is open and bounded. We consider the formal differential operator $Lu = -\operatorname{div}(aDu)$ with $a_{ij} \in L^\infty(\Omega)$, and a uniformly elliptic. The following hold.*

For all $d \geq 1$ there exists a constant $C_3 = C_3(a, d) > 0$ such that, if $u \in H^1(\Omega; [0, \infty))$ is a weak solution of $Lu = 0$ we have

$$\int_{B_{2R}(x_0)} |D \ln(u + \varepsilon)|^2 dx \;\leq\; C_3 R^{d-2} \qquad \forall B_{4R}(x_0) \subset \Omega, \forall \varepsilon > 0. \tag{2.48}$$

Definition 2.34 (functions of bounded mean oscillation)**.**
Assume $\Omega \subset \mathbb{R}^d$ is open. We say that f has bounded mean oscillation $f \in \mathrm{BMO}(\Omega)$ if $f \in L^1(\Omega)$ and

$$[f]_{\mathrm{BMO}(\Omega)} := \sup_{Q\subset\Omega; Q\, cube} \fint_Q |f - f_Q| \; dx < \infty, \tag{2.49}$$

where

$$f_Q := \fint_Q f \; dx.$$

Remark $[\cdot]_{\mathrm{BMO}(\Omega)}$ is a seminorm since $[f]_{\mathrm{BMO}(\Omega)} = 0 \Rightarrow f = const$. The space $\mathrm{BMO}(\Omega)$ is a Banach space with the norm $\|f\|_{\mathrm{BMO}(\Omega)} := \|f\|_{L^1(\Omega)} + [f]_{\mathrm{BMO}(\Omega)}$.

Theorem 2.35 (John-Nierenberg)**.** *Let $Q_0 \subset \mathbb{R}^d$ be a cube. Therere exist three contants $A, \sigma, C_4 > 0$ independent of Q_0 such that $\forall f \in \mathrm{BMO}(\Omega)$ the following hold.*

(i) $\forall t > 0$ *we have* $\frac{|\{x\in Q_0|\ |f(x)-f_{Q_0}|>t\}|}{|Q_0|} \leq Ae^{-\frac{\sigma t}{[f]_{\mathrm{BMO}(\Omega)}}}$,

(ii) $\fint_{Q_0} e^{\gamma|f-f_{Q_0}|}dx \leq C_4 \qquad \forall 0 < \gamma \leq \frac{1}{2}\frac{\sigma}{[f]_{\mathrm{BMO}(\Omega)}}$,

(iii) $\fint_{Q_0}\fint_{Q_0} e^{\gamma(f(x)-f(y))}dxdy \leq C_4 \qquad \forall 0 < \gamma \leq \frac{1}{2}\frac{\sigma}{[f]_{\mathrm{BMO}(\Omega)}}$.

Remark With some work one can replace the cubes with balls (see also Sheet 9)

Proof.
(i) see chapter 6 in the book by M. Giaquinta, L. Martinazzi.

(ii) Using (i) we argue

$$\begin{aligned}\fint_{Q_0} e^{\gamma|f-f_{Q_0}|}dx &= \frac{1}{|Q_0|}\sum_n \int_{Q_0} e^{\gamma|f-f_{Q_0}|}\mathbf{1}_{|f-f_{Q_0}|\in[n,n+1)}dx \\ &\leq \frac{1}{|Q_0|}\sum_n e^{\gamma(n+1)}|\{x\in Q_0|\ |f(x)-f(y)|>n\}| \leq A\sum_n e^{\gamma(n+1)}e^{-\frac{\sigma n}{[f]_{\mathrm{BMO}(\Omega)}}} < \infty\end{aligned}$$

for all $\gamma < \frac{\sigma}{[f]_{\mathrm{BMO}(\Omega)}}$.

(iii) Use (ii) together with $f(x) - f(y) \leq |f(x) - f(y)| \leq |f(x) - f_{Q_0}| + |f(y) - f_{Q_0}|$.

□

Proof of Harnack's inequality Thm 2.30. Fix $\varepsilon > 0$ and consider $u_\varepsilon := u + \varepsilon$. Since $Du = Du_\varepsilon$ the function $u_\varepsilon \in H^1(\Omega; [\varepsilon, \infty))$ is a weak solution of $Lu = 0$, hence it is also a weak subsolution and by weak Harnack's inequality (i) we have

$$\sup_{B_R(x_0)} u_\varepsilon \leq C_1 \left(\fint_{B_{2R}(x_0)} u^p\right)^{\frac{1}{p}} \qquad \forall p > 0$$

Moreover u_ε is a weak supersolution of $Lu = 0$, and hence by weak Harnack's inequality (ii) we have

$$\inf_{B_R(x_0)} u_\varepsilon \geq C_2 \left(\fint_{B_{2R}(x_0)} u^{-q}\right)^{-\frac{1}{q}} \qquad \forall q > 0.$$

Set $v := \ln u_\varepsilon$ and $\overline{v} := \fint_{B_{2R}(x_0)} v\,dx$. Using Poincaré inequality and Thm 2.33, we get

$$\int_{B_{2R}(x_0)} |v-\overline{v}|^2\,dx \le C_P R^2 \int_{B_{2R}(x_0)} |Dv|^2\,dx \le C_P C_3 R^d.$$

It follows

$$\fint_{B_{2R}(x_0)} |v-\overline{v}|\,dx = \tfrac{1}{|B_{2R}(x_0)|}\int_{B_{2R}(x_0)} |v-\overline{v}|\,dx \le \frac{1}{|B_{2R}(x_0)|^{\frac{1}{2}}}\|v-\overline{v}\|_{L^2(B_{2R}(x_0))} \le \frac{\sqrt{C_P C_3}R^{d/2}}{|B_{2R}(x_0)|^{\frac{1}{2}}} = C'$$

for some constant C' independent of R and x_0. It follows that $v \in \mathrm{BMO}(\Omega)$ with $[v]_{\mathrm{BMO}(\Omega)} \le C'$, and hence, using Thm 2.35, there exists $\gamma > 0$ such that

$$\fint_{B_{2R}(x_0)}\fint_{B_{2R}(x_0)} e^{\gamma(v(x)-v(y))}dxdy \le C_4^2$$

for some constant $C_4 > 0$ independent of v, R, x_0. This can be also written as

$$\left(\fint_{B_{2R}(x_0)} u_\varepsilon^\gamma dx\right)^{\frac{1}{\gamma}} = \left(\fint_{B_{2R}(x_0)} e^{\gamma v(x)}dx\right)^{\frac{1}{\gamma}} \le C_4^{2/\gamma}\left(\fint_{B_{2R}(x_0)} e^{-\gamma v(y)}dy\right)^{-\frac{1}{\gamma}} = C_4^{2/\gamma}\left(\fint_{B_{2R}(x_0)} u_\varepsilon^{-\gamma}dy\right)^{-\frac{1}{\gamma}}.$$

Putting all this together we argue

$$\sup_{B_R(x_0)} u_\varepsilon \le C_1\left(\fint_{B_{2R}(x_0)} u^\gamma\right)^{\frac{1}{\gamma}} \le C_1 C_4^{2/\gamma}\left(\fint_{B_{2R}(x_0)} u_\varepsilon^{-\gamma}dy\right)^{-\frac{1}{\gamma}} \le \frac{C_1 C_4^{2/\gamma}}{C_2}\inf_{B_R(x_0)} u_\varepsilon,$$

which completes the proof of Harnack's inequality. $\square$

Proof of the bound on $|D\ln u_\varepsilon|$ Theorem 2.33.
Remember that, if $u \in C^\infty(\Omega;[0,\infty))$ is a strong solution of $Lu = 0$ then $v := \ln u_\varepsilon = \ln(u+\varepsilon)$ is a strong solution of $Lv = Dv \cdot aDv$.
To obtain some kind of weak formulation set $\zeta \in C_c^\infty(B_{4R}(x_0);[0,1])$ with $\zeta_{|B_{2R}(x_0)} = 1$. Then

$$\int_\Omega \zeta^2 Lv\;dx = \int_\Omega \zeta^2 Dv\cdot aDv\;dx.$$

The second integral can be written as

$$\int_\Omega \zeta^2 Dv\cdot aDv\;dx = \int_\Omega \zeta^2\frac{Du}{u_\varepsilon}\cdot a\frac{Du}{u_\varepsilon}\;dx = \int_\Omega \zeta^2\frac{Du}{u_\varepsilon^2}\cdot aDu\;dx.$$

Integrating by parts in the first integral we obtain

$$\int_\Omega \zeta^2 Lv\;dx = \int_\Omega D\zeta^2\cdot aDv\;dx = \int_\Omega D\zeta^2\cdot a\frac{Du}{u_\varepsilon}\;dx.$$

Putting all this together we get

$$0 = \int_\Omega\left(\frac{1}{u_\varepsilon}D\zeta^2 - \zeta^2\frac{Du}{u_\varepsilon^2}\right)\cdot a\frac{Du}{u_\varepsilon}\;dx = \int_\Omega D\left(\frac{\zeta^2}{u_\varepsilon}\right)aDu\;dx = B_L[u,\zeta^2/u_\varepsilon].$$

This suggests to try use in the general case ζ^2/u_ε as test function.

Remember that $u_\varepsilon := u+\varepsilon$, satisfies $u_\varepsilon \in H^1(\Omega;[\varepsilon,\infty))$ and is a weak solution of $Lu_\varepsilon = 0$. Since $u_\varepsilon \geq \varepsilon > 0$, the function $v := \ln u_\varepsilon$ is well defined and $v \in H^1(\Omega)$, with $Dv = \frac{1}{u_\varepsilon} Du$. In addition $u_\varepsilon^{-1} \in H^1(\Omega)$ with $Du_\varepsilon^{-1} = -u_\varepsilon^{-2} Du$.
Therefore $w := \zeta^2 u_\varepsilon^{-1} \in H_0^1(\Omega)$ is a possible test function and hence $B_L[u_\varepsilon, \zeta^2 u_\varepsilon^{-1}] = 0$. It follows

$$\int_\Omega (\zeta Dv) \cdot a(\zeta Dv))dx = 2 \int_\Omega D\zeta \cdot a(\zeta Dv)dx.$$

By uniform ellipticity and Young's inequality, it follows

$$\begin{aligned}\theta \|\zeta Dv\|^2_{L^2(\Omega)} &\leq \int_\Omega (\zeta Dv) \cdot a(\zeta Dv))dx \leq 2\|a\|_\infty \int_\Omega |D\zeta| \, (\zeta |Dv|)dx \\ &\leq 2\|a\|_\infty \left[\frac{\delta}{2} \|\zeta Dv\|^2_{L^2(\Omega)} + \frac{1}{2\delta} \|D\zeta\|^2_{L^2(\Omega)} \right].\end{aligned}$$

Choosing $\delta = \theta/(2\|a\|_\infty)$ we get

$$\frac{\theta}{2} \|\zeta Dv\|^2_{L^2(\Omega)} \leq \frac{\|a\|_\infty}{\delta} \|D\zeta\|^2_{L^2(\Omega)}.$$

Since $\zeta_{B_{2R}(x_0)} = 1$ and $\operatorname{supp} \zeta \subset B_{4R}(x_0)$ we can choose the function such that $\sup |D\zeta| \leq 2/R$. Hence

$$\int_{B_{2R}(x_0)} \frac{|Du|^2}{(u+\varepsilon)^2} dx \leq \|\zeta Dv\|^2_{L^2(\Omega)} \leq \frac{2\|a\|_\infty}{\delta\theta} \|D\zeta\|^2_{L^2(\Omega)} \leq \frac{8\|a\|_\infty}{\theta\delta} R^{-2} |B_{4R}(x_0)| = C_3 R^{d-2}.$$

This concludes the proof. Note that, since the constant C_3 is independent of ε, by monotone convergence we have

$$\int_{B_{2R}(x_0)} \frac{|Du|^2}{u^2} dx = \lim_{\varepsilon\to 0} \int_{B_{2R}(x_0)} \frac{|Du|^2}{(u+\varepsilon)^2} dx \leq C_3 R^{d-2}.$$

$\square$

[14: 23.11.2023]

[15: 27.11.2023]

The key tools to prove the weak Harnack's inequality in dimension $d \geq 3$ are Sobolev inequality and Moser iteration. Remember that if $f \in H_0^1(\Omega) = W_0^{1,2}(\Omega)$ and $d \geq 3$ we have $f \in L^{2^*}(\Omega)$ with

$$\frac{1}{2^*} = \frac{1}{2} - \frac{1}{d} = \frac{d-2}{2d}$$

and there exists a constant $C_S = C_S(d,\Omega) > 0$ such that

$$\|f\|_{L^{2^*}(\Omega)} \leq C_S \|Df\|_{L^2(\Omega)}. \tag{2.50}$$

Theorem 2.36 (Moser iteration)**.** *Assume $\Omega \subset \mathbb{R}^d$ is open and bounded, $d \geq 3$. We consider the formal differential operator $Lu = -\operatorname{div}(aDu)$ with $a_{ij} \in L^\infty(\Omega)$, and a uniformly elliptic. Let $u \in H^1(\Omega;[0,\infty))$ be a weak subsolution of $Lu = 0$. Fix $x_0 \in \Omega$. For $r > 0$ such that $B_r(x_0) \subset \Omega$ and $p > 0$ we define*

$$\Phi(p,r) := \left(\int_{B_r(x_0)} u^p \right)^{\frac{1}{p}}. \tag{2.51}$$

Set

$$\mu := \frac{2^*}{2} = \frac{d}{d-2},$$

where 2^* *is the Sobolev exponent associated to* $p = 2$. *Note that* $\mu > 1$. *Then there is a constant* $C = C(a, d) > 0$ *such that*

$$\Phi(\mu p, s) \le \left(\frac{Cp}{r-s}\right)^{\frac{2}{p}} \Phi(p, r) \qquad \forall 0 < s < r,\ \forall p \ge 2. \tag{2.52}$$

Note that $\Phi(p, r)$ is well defined and non negative $\forall B_r(x_0) \subset \Omega$, but it may take value $+\infty$. We show now how weak Harnack's inequality follows from Moser's iteration.

Proof of weak Harnack's inequality Thm. 2.32 (i) for $p \ge 2$.
Let $u \in H^1(\Omega; [0, \infty))$ be a weak subsolution of $Lu = 0$. Our goal is to prove

$$\sup_{B_{R_0}(x_0)} u \le C_1 \left(\fint_{B_{2R_0}(x_0)} u^p\right)^{\frac{1}{p}} \qquad \forall B_{2R_0}(x_0) \subset \Omega.$$

Note that for $p \ge 2$ we only need to require $B_{2R_0}(x_0) \subset \Omega$.

Fix $\rho > 0$ such that $B_\rho(x_0) \subset\subset \Omega$. We have (exercise, see also FA)

$$\sup_{B_\rho(x_0)} u = \lim_{p \to \infty} \left(\int_{B_r(x_0)} u^p\right)^{\frac{1}{p}} = \lim_{p \to \infty} \Phi(p, \rho). \tag{2.53}$$

Take now $R > \rho > 0$ such that $B_R(x_0) \subset \Omega$ (possible since $B_\rho(x_0) \subset\subset \Omega$). We interpolate between R and ρ as follows: for each $n \in \mathbb{N}$ we define

$$R_n := \rho + \frac{R-\rho}{2^n}, \qquad p_n := \mu^n p.$$

Then $p_{n+1} = \mu p_n$, $p_0 = p$, $R_0 = R$, $R_{n+1} < R_n$, $\lim_{n\to\infty} R_n = \rho$, and $R_n - R_{n+1} = (R-\rho)2^{-n-1}$. We argue, using Moser iteration (2.52),

$$\begin{aligned}
\Phi(p_{n+1}, R_{n+1}) = \Phi(\mu p_n, R_{n+1}) &\le \left(\frac{Cp_n}{R_n - R_{n+1}}\right)^{\frac{2}{p_n}} \Phi(p_n, R_n) \\
&= \left(\frac{Cp\, 2^{n+1}\mu^n}{R-\rho}\right)^{\frac{2}{p_n}} \Phi(p_n, R_n) = \left(\frac{2Cp}{R-\rho}\right)^{\frac{2}{p_n}} (2\mu)^{\frac{2n}{p_n}} \Phi(p_n, R_n) \\
&\le \prod_{j=0}^{n} \left(\frac{2Cp}{R-\rho}\right)^{\frac{2}{p_j}} (2\mu)^{\frac{2j}{p_j}}\ \Phi(p_0, R_0) = \left(\frac{2Cp}{R-\rho}\right)^{2\sum_{j=0}^n \frac{1}{p_j}} (2\mu)^{2\sum_{j=0}^n \frac{j}{p_j}}\ \Phi(p, R).
\end{aligned}$$

Note that

$$\begin{aligned}
\sum_{j=0}^n \frac{1}{p_j} &= \frac{1}{p}\sum_{j=0}^n \frac{1}{\mu^j} = \frac{1}{p(1-\frac{1}{\mu})}(1 - \frac{1}{\mu^{n+1}}) \to_{n\to\infty} \frac{1}{p(1-\frac{1}{\mu})} = \frac{d}{2p} \\
\sum_{j=0}^n \frac{j}{p_j} &= \frac{1}{p}\sum_{j=0}^n \frac{j}{\mu^j} = -\frac{\mu}{p}\partial_\mu \sum_{j=1}^n \frac{1}{\mu^j} = \frac{1}{\mu p(1-\frac{1}{\mu})^2}\left[1 - \frac{1}{\mu^{n+1}} - \frac{n+1}{\mu^n}(1-\frac{1}{\mu})\right] \\
&\to_{n\to\infty} \frac{1}{p\mu(1-\frac{1}{\mu})^2} = \frac{1}{p}\frac{d-2}{d}\frac{d^2}{4} = \frac{d}{p}\frac{d-2}{4},
\end{aligned}$$

hence

$$\left(\frac{2Cp}{R-\rho}\right)^{2\sum_{j=0}^{n}\frac{1}{p_j}} \to_{n\to\infty} \left(\frac{2Cp}{R-\rho}\right)^{\frac{d}{p}}, \qquad (2\mu)^{2\sum_{j=0}^{n}\frac{j}{p_j}} \to_{n\to\infty} (2\mu)^{\frac{d-2}{2}\frac{d}{p}} = (2^*)^{\frac{d-2}{2}\frac{d}{p}}.$$

Using this results together with (2.53) we argue

$$\begin{aligned}\sup_{B_r(x_0)} u &= \lim_{n\to\infty}\left(\int_{B_r(x_0)} u^{p_n}\right)^{\frac{1}{p_n}} \le \limsup_{n\to\infty}\left(\int_{B_{R_n}(x_0)} u^{p_n}\right)^{\frac{1}{p_n}} \\ &= \limsup_{n\to\infty}\Phi(p_n,R_n) \le \frac{C'}{(R-\rho)^{\frac{d}{p}}}\left(\int_{B_R(x_0)} u^p\right)^{\frac{1}{p}} \le C''\left(\frac{R}{R-\rho}\right)^{\frac{d}{p}}\left(\fint_{B_R(x_0)} u^p\right)^{\frac{1}{p}}\end{aligned}$$

where

$$C' := \left(2Cp\,2^{*\frac{d-2}{2}}\right)^{\frac{d}{p}}, \qquad C'' := C'\left(\frac{|B_R(x_0)|}{R^d}\right)^{\frac{1}{p}}.$$

The result now follows setting $\rho = R_0$ and $R = 2R_0$. □

Proof of Moser iteration Thm 2.36. Remember the definition of $\Phi(p,r)$ (2.51). Our goal is to prove that there is a constant $C = C(a,d) > 0$ such that

$$\Phi(\mu p, s) \le \left(\frac{Cp}{r-s}\right)^{\frac{2}{p}}\Phi(p,r) \qquad \forall 0 < s < r. \tag{2.54}$$

for all $B_r(x_0) \subset \Omega$ and $u \ge 0$, such that $u \in H^1(\Omega)$ is a subsolution of $Lu = 0$, i.e.

$$\int_\Omega Dv\cdot aDu\;dx \le 0 \qquad \forall v \in H_0^1(\Omega;[0,\infty))$$

We construct now an appropriate test function v. For this we distinguish two cases.

Case 1. Assume $0 < \varepsilon \le u \le M$ for some fixed $M > \varepsilon > 0$.
Then the function u^α satisfies $u^\alpha \in H^1(\Omega)$ $\forall \alpha \in \mathbb{R}$ with $Du^\alpha = \alpha u^{\alpha-1}Du$. We will assume from now on $\alpha > 0$.
For $0 < s < r$ take $\zeta \in C_c^\infty(B_r(x_0);[0,1])$ such that $\zeta_{|B_s(x_0)} = 1$ We define

$$v := \zeta^2 u^\alpha,$$

The function v satisfies $v \in H_0^1(\Omega;[0,\infty))$, hence we can use it as test function. We compute

$$0 \ge B_L[u,v] = \int_\Omega Dv\cdot aDu\;dx = \int_\Omega(\alpha u^{\alpha-1}\zeta^2 Du + 2u^\alpha\zeta D\zeta)\cdot aDu\;dx,$$

and hence

$$\alpha\int_\Omega u^{\alpha-1}\zeta^2\;Du\cdot aDu\;dx \le -2\int_\Omega u^\alpha\zeta\;D\zeta\cdot aDu\;dx \le 2\|a\|_\infty\int_\Omega u^\alpha\;|D\zeta|\,\zeta|Du|\;dx.$$

By uniform ellipticity $a \ge \theta\,\mathrm{Id}$, we get

$$\|\zeta u^{\frac{\alpha-1}{2}}Du\|^2_{L^2(\Omega)} = \int_\Omega u^{\alpha-1}\zeta^2|Du|^2dx \le \frac{2\|a\|_\infty}{\alpha\theta}\int_\Omega u^\alpha\zeta\;|D\zeta|\,|Du|\;dx.$$

Inserting the decomposition $u^\alpha = u^{\frac{\alpha-1}{2}} u^{\frac{\alpha+1}{2}}$, we get

$$\begin{aligned}\|\zeta u^{\frac{\alpha-1}{2}} Du\|^2_{L^2(\Omega)} &\le \frac{2\|a\|_\infty}{\alpha\theta} \int_\Omega (\zeta\ u^{\frac{\alpha-1}{2}} |Du|)\ (u^{\frac{\alpha+1}{2}} |D\zeta|)\ dx \\ &\le \frac{2\|a\|_\infty}{\alpha\theta} \|\zeta u^{\frac{\alpha-1}{2}} Du\|_{L^2(\Omega)}\ \|u^{\frac{\alpha+1}{2}} D\zeta\|_{L^2(\Omega)},\end{aligned}$$

and hence

$$\|\zeta u^{\frac{\alpha-1}{2}} Du\|_{L^2(\Omega)} \le \frac{2\|a\|_\infty}{\alpha\theta} \|u^{\frac{\alpha+1}{2}} D\zeta\|_{L^2(\Omega)}. \tag{2.55}$$

It follows, using $u^{\frac{\alpha-1}{2}} Du = \frac{2}{\alpha+1} D(u^{\frac{\alpha+1}{2}})$,

$$\|\zeta D u^{\frac{\alpha+1}{2}}\|_{L^2(\Omega)} = \frac{\alpha+1}{2} \|\zeta u^{\frac{\alpha-1}{2}} Du\|_{L^2(\Omega)} \le \frac{\alpha+1}{\alpha} \frac{\|a\|_\infty}{\theta} \|u^{\frac{\alpha+1}{2}} D\zeta\|_{L^2(\Omega)}.$$

We assume in the following $\alpha \ge 1$, therefore $\frac{\alpha+1}{\alpha} \le 2$. We will also set

$$p := \alpha + 1 \ge 2.$$

Inserting this above we obtain

$$\|\zeta D u^{\frac{p}{2}}\|_{L^2(\Omega)} \le \frac{2\|a\|_\infty}{\theta} \|u^{\frac{p}{2}} D\zeta\|_{L^2(\Omega)},$$

and hence

$$\begin{aligned}\|D(\zeta u^{\frac{p}{2}})\|_{L^2(\Omega)} &\le \|\zeta D u^{\frac{p}{2}}\|_{L^2(\Omega)} + \|u^{\frac{p}{2}} D\zeta\|_{L^2(\Omega)} \\ &\le \left[1 + \frac{2\|a\|_\infty}{\theta}\right] \|u^{\frac{p}{2}} D\zeta\|_{L^2(\Omega)} \\ &\le \left[1 + \frac{2\|a\|_\infty}{\theta}\right] \|u^{\frac{p}{2}} D\zeta\|_{L^2(\Omega)} \le \left[1 + \frac{2\|a\|_\infty}{\theta}\right] \left(\int_{B_r(x_0)} u^p |D\zeta|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Since $u^{p/2} \in H^1(\Omega)$ and $\zeta \in C_c^\infty(\Omega)$, it holds $\zeta u^{\frac{p}{2}} \in H_0^1(\Omega)$, hence, by Sobolev inequality (2.50) we have $\zeta u^{\frac{p}{2}} \in L^{2^*}(\Omega)$ and

$$C_S \|D(\zeta u^{\frac{p}{2}})\|_{L^2(\Omega)} \ge \|\zeta u^{\frac{p}{2}}\|_{L^{2^*}(\Omega)} = \left(\int_\Omega \zeta^{2^*} u^{p\mu} dx\right)^{\frac{1}{2\mu}}. \tag{2.56}$$

It follows

$$\begin{aligned}\Phi(\mu p, s) = \left(\int_{B_s(x_0)} u^{\mu p}\right)^{\frac{1}{\mu p}} &\le \left(\int_\Omega \zeta^{2^*} u^{\mu p}\right)^{\frac{1}{\mu p}} = \|\zeta u^{\frac{p}{2}}\|^{\frac{2}{p}}_{L^{2^*}(\Omega)} \le C_S^{\frac{2}{p}} \|D(\zeta u^{\frac{p}{2}})\|^{\frac{2}{p}}_{L^2(\Omega)} \\ &\le C_S^{\frac{2}{p}} \left[1 + \frac{2\|a\|_\infty}{\theta}\right]^{\frac{2}{p}} \left(\int_{B_r(x_0)} u^p |D\zeta|^2\right)^{\frac{1}{p}}.\end{aligned}$$

Since $\zeta_{B_s(x_0)} = 1$ and $\operatorname{supp} \zeta \subset B_r(x_0)$ we can choose ζ such that $|D\zeta| \le 2/(r-s)$ hence

$$\left(\int_{B_r(x_0)} u^p |D\zeta|^2\right)^{\frac{1}{p}} \le \left(\frac{2}{r-s}\right)^{\frac{2}{p}} \left(\int_{B_r(x_0)} u^p\right)^{\frac{1}{p}} = \left(\frac{2}{r-s}\right)^{\frac{2}{p}} \Phi(p, r).$$

Finally we obtain

$$\Phi(\mu p, s) \leq \left(\frac{C}{r-s}\right)^{\frac{2}{p}} \Phi(p, r)$$

with

$$C := 2C_S \left[1 + \frac{2\|a\|_\infty}{\theta}\right].$$

Note that there is no p factor next to C. The p factor will appear when we consider the general case $u \geq 0$. This completes the proof in the case $\varepsilon \leq u \leq M$.

Case 2. Assume now $u \geq 0$. Then $u \in H^1(\Omega) \not\Rightarrow u^\alpha \in H^1(\Omega)$ so we need to modify our strategy.

Note that, since we assumed $\alpha \geq 1$ we never needed the assumption $u \geq \varepsilon$ and we used the assumption $u \leq M$ only in two steps:

- $u^\alpha \in H^1(\Omega) \Rightarrow u^\alpha \zeta^2 H_0^1(\Omega; [0, \infty))$ is an admissible test function, and hence (2.55) holds.
- $u^{\frac{\alpha+1}{2}} \in H^1(\Omega) \Rightarrow \zeta u^{\frac{\alpha+1}{2}} \in H_0^1(\Omega)$ and we can apply Sobolev inequality to get (2.56).

We distinguish now three cases:

For $p = 2$, we have $\alpha = 1$, hence $u = u^\alpha = u^{\frac{\alpha+1}{2}} \in H^1(\Omega)$, so the arguments used in Case 1 hold.

For $p > 2$, if $u^p \notin L^1(B_r(x_0))$, then $\Phi(r, p) = \infty$ and the (2.52) holds trivially.

Assume now $p > 2$ and $u^p \in L^1(B_r(x_0))$. Set $\alpha := p - 1$. We will prove the following two statements:

(a) $\|\zeta u^{\frac{\alpha-1}{2}} Du\|_{L^2(\Omega)} \leq \frac{2\|a\|_\infty}{\theta} \|u^{\frac{\alpha+1}{2}} D\zeta\|_{L^2(\Omega)}$ $\forall \zeta \in C_c^\infty(B_r(x_0))$.

(b) $u^{\frac{p}{2}} = u^{\frac{\alpha+1}{2}} \in H^1_{loc}(B_r(x_0))$.

Note that the inequality in (a) differs from (2.55) by a factor α.

We see first how Moser iteration follows from (a) and (b).
Since $u^{\frac{p}{2}} \in H^1_{loc}(B_r(x_0))$ and $\zeta \in C_c^\infty(B_r(x_0))$ we have $\zeta u^{\frac{p}{2}} \in H_0^1(\Omega)$ and by Sobolev inequality we obtain (2.56). Putting this together with (a) we obtain the result. Note that instead of $\frac{1+\alpha}{\alpha}$ we have $(1 + \alpha) = p$, which accounts for the additional factor p in the final bound.

$\square$

Proof of (b), assuming (a) holds .
Since $u^p \in L^1(B_r(x_0))$ we have $u^{\frac{p}{2}} \in L^2(B_r(x_0))$.
We show that $D(u^{\frac{p}{2}}) = \frac{p}{2} u^{\frac{p-2}{2}} Du = \frac{p}{2} u^{\frac{\alpha-1}{2}} Du \in L^2_{loc}(B_r(x_0))$.
For this let K be a compact set with $K \subset B_r(x_0)$, and let $\zeta \in C_c^\infty(B_r(x_0); [0, 1])$ such that $\zeta_{|K} = 1$. We argue, using (a),

$$\begin{aligned}\|D(u^{\frac{p}{2}})\|_{L^2(K)} &\leq \|\zeta D(u^{\frac{p}{2}})\|_{L^2(\Omega)} = \frac{p}{2} \|\zeta u^{\frac{\alpha-1}{2}} Du\|_{L^2(\Omega)} \\ &\leq C \|u^{\frac{p}{2}} D\zeta\|_{L^2(\Omega)} \leq C \|D\zeta\|_{L^\infty} \|u^{\frac{p}{2}}\|_{L^2(B_r(x_0))} < \infty.\end{aligned}$$

Hence $D(u^{\frac{p}{2}}) \in L^2_{loc}(B_r(x_0))$ and (b) follows. $\square$

[15: 27.11.2023]

[16: 30.11.2023]

Proof of (a) .
For $M > 0$ we introduce the cut-off function $h_M : (0,\infty) \to (0,\infty)$ defined as

$$h_M(t) := \begin{cases} t^\alpha & \text{if } 0 \le t \le M \\ M^\alpha + \alpha M^{\alpha-1}(t-M) & \text{if } t > M \end{cases}$$

where $\alpha = p - 1 > 1$. This function satisfies $h_M \in C^\infty([0,\infty))$, $0 < h_M'(t) \le \alpha M^{\alpha-1}$ $\forall t > 0$, and $h_M'(0) = 0$. We have $|h_M(u)|^2 = 1 + u^2 \in L^1(\Omega)$ since Ω is bounded and $Dh_M(u) = h_M'(u)Du \in L^2(\Omega)$ since h_M' is bounded. Hence $h_M(u) := h_M \circ u \in H^1(\Omega)$ and $v := \zeta^2 h_M(u) \in H_0^1(\Omega; [0,\infty))$ is an admissible test function for all $\zeta \in C_c^\infty(B_r(x_0))$. Since u is a subsolution of $Lu = 0$ we have

$$0 \ge B_L[u,v] = \int_\Omega Dv \cdot aDu \, dx = \int_\Omega (\zeta^2 h_M'(u)Du + 2h_M(u)\zeta D\zeta) \cdot aDu \, dx.$$

Using $\zeta^2 \ge 0$, $h_M' \ge 0$ and uniform ellipticity, we argue

$$\theta \int_\Omega \zeta^2 h_M'(u) \, |Du|^2 \, dx \le \int_\Omega \zeta^2 h_M'(u) \, Du \cdot aDu \, dx \le 2\|a\|_\infty \int_\Omega |\zeta||Du| \, h_M(u)|D\zeta| \, dx.$$

The function h_M satisfies

$$0 \le h_M(t) \le t h_M'(t),$$

hence

$$\begin{aligned} \|\zeta h_M'(u)^{\frac12} Du\|_{L^2(\Omega)}^2 &\le \frac{2\|a\|_\infty}{\theta} \int_\Omega (|\zeta| h_M'(u)^{\frac12}|Du|) \, (u h_M'(u)^{\frac12}|D\zeta|) \, dx \\ &\le \frac{2\|a\|_\infty}{\theta} \|\zeta h_M'(u)^{\frac12} Du\|_{L^2(\Omega)} \, \|u h_M'(u)^{\frac12} D\zeta\|_{L^2(\Omega)}. \end{aligned}$$

It follows,

$$\|\zeta h_M'(u)^{\frac12} Du\|_{L^2(\Omega)} \le \frac{2\|a\|_\infty}{\theta} \|u h_M'(u)^{\frac12} D\zeta\|_{L^2(\Omega)}.$$

The function h_M' satisfies

$$h_M'(t)t^2 \le \alpha t^{\alpha+1} = \alpha t^p,$$

hence $|u|h_M'(u)^{\frac12} \le \sqrt{\alpha} u^{\frac{\alpha+1}{2}}$ and the inequality becomes

$$\|\zeta h_M'(u)^{\frac12} Du\|_{L^2(\Omega)} \le \frac{2\|a\|_\infty \sqrt{\alpha}}{\theta} \|u^{\frac{\alpha+1}{2}} D\zeta\|_{L^2(\Omega)}.$$

Note that $\|u^{\frac{\alpha+1}{2}} D\zeta\|_{L^2(\Omega)} \le \|D\zeta\|_{L^\infty} \|u^{\frac{\alpha+1}{2}}\|_{L^2(\Omega)} = \|D\zeta\|_{L^\infty} \|u^p\|_{L^1(\Omega)} < \infty$ hence by dominated convergence it follows, using $\lim_{M\to\infty} h_M'(t) = \alpha t^{\alpha-1}$,

$$\lim_{M\to\infty} \int_\Omega u^2 h_M'(u)|D\zeta|^2 dx = \alpha \int_\Omega u^p |D\zeta|^2 dx < \infty.$$

Finally we obtain

$$\sqrt{\alpha}\|\zeta u^{\frac{\alpha+1}{2}} Du\|_{L^2(\Omega)} \le \frac{2\|a\|_\infty \sqrt{\alpha}}{\theta} \|u^{\frac{\alpha+1}{2}} D\zeta\|_{L^2(\Omega)},$$

which proves (a).

$\square$

Until now we have proved, using Moser iteration, the weak Harnack's inequalty (i) for $p \ge 2$. We consider now the case $0 < p < 2$.

Proof of weak Harnack's inequality Thm. 2.32 (i) for $0 < p < 2$.
Let $u \in H^1(\Omega; [0, \infty))$ be a weak subsolution of $Lu = 0$. Our goal is to prove

$$\sup_{B_R(x_0)} u \leq C_1 \left(\fint_{B_{2R}(x_0)} u^p \right)^{\frac{1}{p}} \qquad \forall B_4(x_0) \subset \Omega.$$

From the proof of weak Harnack inequality for $p = 2$ we know

$$\sup_{B_\rho(x_0)} u \leq \frac{C}{(R-\rho)^{\frac{d}{2}}} \left(\int_{B_R(x_0)} u^2 \right)^{\frac{1}{2}} = \frac{C}{(R-\rho)^{\frac{d}{2}}} \|u\|_{L^2(\Omega)} < \infty \qquad \forall B_R(x_0) \subset \Omega,\ 0 < \rho < R. \tag{2.57}$$

Assume $B_{2R_0}(x_0) \subset \Omega$ and define $\phi\colon [0, R_0] \to \mathbb{R}$ by

$$\phi(r) := \sup_{B_r(x_0)} u.$$

From (2.57) ϕ is well defined. Moreover $0 \leq \phi(\rho) \leq \phi(R_0)$. Using (2.57) again we have

$$\phi(\rho) = \sup_{B_\rho(x_0)} u \leq \frac{C}{(R-\rho)^{\frac{d}{2}}} \left(\int_{B_R(x_0)} u^2 dx \right)^{\frac{1}{2}} \qquad \forall 0 < \rho < R \leq R_0.$$

We compute, using $2 - p > 0$,

$$u^2 = u^p u^{2-p} \leq u^p \left(\sup_{B_R(x_0)} u \right)^{2-p} = u^p\ \phi(R)^{2-p}.$$

Hence, using the inequality $ab \leq \delta a^q + C_\delta b^{q'}$ with $0 < \delta$, $1/q = 1 - p/2$, and $1/q' = p/2$, we get

$$\begin{aligned} \phi(\rho) \leq\ & \phi(R)^{1-\frac{p}{2}} \frac{C}{(R-\rho)^{\frac{d}{2}}} \left(\int_{B_R(x_0)} u^p dx \right)^{\frac{1}{2}} \leq\ \delta\ \phi(R) + \frac{C_\delta C^{\frac{2}{p}}}{(R-\rho)^{\frac{d}{p}}} \left(\int_{B_R(x_0)} u^p dx \right)^{\frac{1}{p}} \\ \leq\ & \delta\ \phi(R) + \frac{C_\delta C^{\frac{2}{p}}}{(R-\rho)^{\frac{d}{p}}} \left(\int_{B_{R_0}(x_0)} u^p dx \right)^{\frac{1}{p}} = \delta\ \phi(R) + \frac{A}{(R-\rho)^{\frac{d}{p}}}, \end{aligned}$$

where

$$A := C_\delta C^{\frac{2}{p}} \left(\int_{B_{R_0}(x_0)} u^p dx \right)^{\frac{1}{p}} < \infty.$$

Claim. Let $\phi\colon [0, R_0] \to \mathbb{R}$ be a function satisfying: $\phi \geq 0$, ϕ is bounded and

$$\phi(\rho) \leq \delta \phi(R) + \frac{A}{(R-\rho)^{\frac{d}{p}}}, \qquad \forall 0 < \rho < R \leq R_0,$$

with $0 < \delta < \min\{1, 2^{-\frac{d}{p}}\}$. Then there exists a constant $C' > 0$ independent of A, such that

$$\phi(\rho) \leq \frac{C' A}{(R-\rho)^{\frac{d}{p}}} \qquad \forall 0 < \rho < R \leq R_0.$$

Consequence. We argue

$$\sup_{B_\rho(x_0)} u \ \leq \ \frac{C'C_\delta}{(R-\rho)^{\frac{d}{p}}} \left(\int_{B_{R_0}(x_0)} u^p dx\right)^{\frac{1}{p}}.$$

setting $\rho := R_0/2$ $R = R_0$ we obtain the result.
Proof of the Claim. Fix $0 < \rho < R \leq R_0$, and set

$$\rho_0 := \rho, \quad \rho_{n+1} := \rho_n + \frac{(R-\rho)}{2^{n+1}},\ n \geq 0.$$

This sequence is increasing and

$$\rho_{n+1} = \rho + (R-\rho)\sum_{j=1}^{n+1} \frac{1}{2^j} = \rho + (R-\rho)(1 - \frac{1}{2^{n+1}}).$$

In particular $\lim_{n\to\infty} \rho_n = R$. We argue

$$\begin{aligned}\phi(\rho) = \phi(\rho_0) \leq \delta\ \phi(\rho_1) + \frac{A}{(\rho_1-\rho_0)^{\frac{d}{p}}} &\leq \delta^{n+1}\phi(\rho_{n+1}) + A\sum_{j=0}^{n} \frac{\delta^j}{(\rho_{j+1}-\rho_j)^{\frac{d}{p}}} \\ &= \delta^{n+1}\phi(\rho_{n+1}) + \frac{A2^{\frac{d}{p}}}{(R-\rho)^{\frac{d}{p}}} \sum_{j=0}^{n} (\delta 2^{\frac{d}{p}})^j.\end{aligned}$$

Since $(\delta 2^{\frac{d}{p}}) < 1$ we have $\sum_{j=0}^n (\delta 2^{\frac{d}{p}})^j \to_{n\to\infty} \frac{1}{1-\delta 2^{\frac{d}{p}}}$. Hence, using that ϕ is bounded, we obtain in the limit $n \to \infty$

$$\phi(\rho) \leq \frac{A2^{\frac{d}{p}}}{(R-\rho)^{\frac{d}{p}}} \frac{1}{1-\delta 2^{\frac{d}{p}}} = \frac{C'A}{(R-\rho)^{\frac{d}{p}}}.$$

This completes the proof of the claim. □

Proof of weak Harnack's inequality Thm. 2.32 (ii).
Assume $u \in H^1(\Omega)$ is a weak supersolution of $Lu = 0$ and $u \geq 0$.
Then $u_\varepsilon := u + \varepsilon$ is also a weak supersolution of $Lu_\varepsilon = 0$, $u_\varepsilon \geq \varepsilon$, and hence $1/u_\varepsilon \in H^1(\Omega)$ and $1/u_\varepsilon^2 \in H^1(\Omega)$.
We will show now that $1/u_\varepsilon$ is a weak subsolution. Indeed, let $\varphi \in C_c^\infty(\Omega; [0,\infty))$. We compute

$$\int_\Omega D\left(\frac{1}{u_\varepsilon}\right) \cdot aD\varphi\ dx = -\int_\Omega \frac{1}{u_\varepsilon^2} Du \cdot aD\varphi\ dx = -\int_\Omega Du \cdot aD\left(\frac{\varphi}{u_\varepsilon^2}\right)\ dx - 2\int_\Omega (Du \cdot aDu)\,\frac{\varphi}{u_\varepsilon^3}\ dx.$$

By uniform ellipticity, and since $\varphi \geq 0$, we have

$$\int_\Omega (Du \cdot aDu)\,\frac{\varphi}{u_\varepsilon^3}\ dx \geq 0.$$

Moreover, $1/u_\varepsilon \in H^1(\Omega)$ and hence $\varphi/u_\varepsilon \in H^1_0(\Omega; [0,\infty))$ is a possible test function. It follows, since u_ε is also a weak supersolution,

$$\int_\Omega Du \cdot aD\left(\frac{\varphi}{u_\varepsilon^2}\right)\ dx = B_L[u, \frac{\varphi}{u_\varepsilon^2}] \geq 0,$$

and hence

$$\int_\Omega D\left(\frac{1}{u_\varepsilon}\right)\cdot aD\varphi\,dx \le 0 \qquad \forall\varphi\in C_c^\infty(\Omega;[0,\infty)).$$

This implies $1/u_\varepsilon$ is a weak subsolution of $Lu=0$. By weak Harnack's inequality Thm. 2.32 (i) we have $\forall p>0$,

$$\sup_{B_R(x_0)}\frac{1}{u+\varepsilon}\le C_1\left(\fint_{B_{2R}(x_0)}(u+\varepsilon)^{-p}\right)^{\frac{1}{p}} \qquad \forall B_{4R}(x_0)\subset\Omega.$$

Hence

$$\inf_{B_R(x_0)}(u+\varepsilon)\ge C_1^{-1}\left(\fint_{B_{2R}(x_0)}(u+\varepsilon)^{-p}\right)^{-\frac{1}{p}} \qquad \forall B_{4R}(x_0)\subset\subset\Omega.$$

The result now follows taking $\varepsilon\to 0$. $\square$

[16: 30.11.2023]

[17: 4.12.2023]

Until now we have seen Harack's inequality on balls. We can prove the same result on any compact subset of Ω. This is the content of the next corollary.

Corollary 2.37 (Harnack's inequality on compact subsets). *Assume $\Omega\subset\mathbb{R}^d$ is open and bounded. We consider the formal differential operator $Lu=-\operatorname{div}(aDu)$ with $a_{ij}\in L^\infty(\Omega)$, and a uniformly elliptic.*

For all V open and connected with $V\subset\subset\Omega$ there exists a constant $C_V=C(a,d,V)>1$ such that

$$\sup_V u\le C_V\ \inf_V u, \tag{2.58}$$

for all $u\in H^1(\Omega)$ weak solution of $Lu=0$ in Ω with $u\ge 0$ a.e. in Ω.
Remember that by sup *and* inf *we mean the essential* sup *and essential* inf .

Proof. Since $\overline{V}$ is compact and $V\subset\subset\Omega$ we can find $N>0$, $x_1,\dots x_N\in\Omega$ and $r_1,\dots,r_N>0$ such that

- $\overline{V}\subset\cup_{j=1}^N B_{r_j}(x_j)$,
- $B_{4r_j}(x_j)\subset\Omega\ \forall j$.

Since V is connected we can also choose the balls such that $|B_{r_j}(x_j)\cap B_{r_{j+1}}(x_{j+1})|>0\ \forall j=1,\dots,N-1$. Set $B_j:=B_{r_j}(x_j)$. Harnack's inequality on balls Thm. 2.30 ensures that

$$\sup_{B_j}u\le C_0\ \inf_{B_j}u \qquad \forall j=1,\dots,N, \tag{2.59}$$

where the constant $C_0>1$ is independent on r_j and x_j.

We show

$$\sup_{\cup_{j=1}^N B_j}u\le C_0^N\ \inf_{\cup_{j=1}^N B_j}u. \tag{2.60}$$

Indeed, by (2.59), we can write $\cup_{j=1}^N B_j=\Omega_0\cup N$ where $|N|=0$ and

$$\inf_{B_j}u\le u(x)\le\sup_{B_j}u \qquad \forall x\in B_j\cap\Omega_0, \qquad \forall j=1,\dots,N. \tag{2.61}$$

Assume now $N = 2$. Since $|B_1 \cap B_2| > 0$ there is a point $z_1 \in B_1 \cap B_2 \cap \Omega_0$ and using (2.61) we have

$$\inf_{B_1} u \leq u(z_1) \leq \sup_{B_2} u.$$

For any two points $x \in B_1 \cap \Omega_0$ and $y \in B_2 \cap \Omega_0$ we argue

$$u(x) \leq \sup_{B_1} u \leq C_0 \inf_{B_1} u \leq C_0 u(z_1) \leq C_0 \sup_{B_2} u \leq C_0^2 \inf_{B_2} u.$$

The same holds exchanging the roles of B_1 and B_2. If $x, y \in B_j \cap \Omega_0$ we have

$$u(x) \leq C_0 u(y) \leq C_0^2 u(y),$$

where in the last step we used $C_0 > 1$. Putting these results together we get (2.60) in the case $N = 2$. The case $N > 2$ is proved in the same way.

Finally we argue, since $V \subset \cup_{j=1}^N B_j$,

$$\sup_V u \leq \sup_{\cup_{j=1}^N B_j} u \leq C_0^N \inf_{\cup_{j=1}^N B_j} u \leq C_0^N \inf_V u,$$

which completes the proof.

□

Corollary 2.38 (de Giorgi)**.** *Assume $\Omega \subset \mathbb{R}^d$ is open and bounded. We consider the formal differential operator $Lu = -\mathrm{div}\,(aDu)$ with $a_{ij} \in L^\infty(\Omega)$, and a uniformly elliptic.*
Le $u \in H^1(\Omega)$ be a weak solution of $Lu = 0$ in Ω. The following hold.

(i) $u \in L^\infty_{loc}(\Omega)$

(ii) $\exists \alpha \in (0,1)$, *and* $\overline{u} \in C^{0,\alpha}_{loc}(\Omega)$ *such that* $u = \overline{u}$ *a.e. in* Ω.

Remark. $\overline{u} \in C^{0,\alpha}_{loc}(\Omega)$ means for all $V \subset\subset \Omega$ there is a constant $C_V > 0$ and a coefficient $\alpha > 0$ such that

$$\sup_{x \neq y \in V} \frac{|\overline{u}(x) - \overline{u}(y)|}{|x - y|^\alpha} \leq C_V.$$

Not that $(ii) \Rightarrow (i)$. We will prove first (i) and use the result to prove (ii).

Proof. For $d = 1, 2$ we can prove (ii) directly.

Indeed, for $d = 1$ $u \in H^1_0(\Omega)$ implies, by Sobolev embedding, $u \in C^{0,\frac{1}{2}}(\overline{\Omega})$.

For $d = 2$, any weak solution of $Lu = 0$ satisfies, by interior H^2 regularity, $u \in H^2_{loc}(\Omega)$. In particular $u \in H^2(B_r(x_0))$ for all $B_r(x_0) \subset\subset \Omega$. By Sobolev embedding we have $u \in W^{1,q}(B_r(x_0))$ $\forall 1 < q < \infty$ and hence, again by Sobolev embedding, $u \in C^{0,\gamma}(B_r(x_0))$ for some $\gamma > 0$, independent of $B_r(x_0)$. It follows $u \in C^{0,\gamma}_{loc}(\Omega)$. In the following we consider the case $d \geq 3$.

(i) If $u \geq 0$ is a weak subsolution of $Lu = 0$, weak Harnack's inequality ensures

$$\sup_{B_R(x_0)} u \leq C_1 \left(\fint_{B_{2R}(x_0)} u^2 dx \right)^{\frac{1}{2}} < \infty \qquad \forall B_{2R}(x_0) \subset \Omega$$

and hence $u \in L^\infty_{loc}(\Omega)$. The problem is that here u may be negative. The most natural idea would be to consider $|u| \in H^1_0(\Omega)$. But u weak solution of $Lu = 0 \not\Rightarrow |u|$ weak subsolution. To avoid this problem we will use an approximation of $|u|$ constructed as follows.

Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined via $f(t) := \sqrt{t^2+1}$. It holds $f \in C^2(\mathbb{R})$, $|f'| \leq 1$ and $0 < f'' < 1$. Therefore, since $u \in H^1(\Omega)$, we have $f(u), f'(u) \in H^1(\Omega)$ with

$$D(f(u)) = f'(u)Du, \quad D(f'(u)) = f''(u)Du.$$

We show now that $f(u)$ is a weak subsolution of $Lu = 0$. For this, it is sufficent to show $B_L[f(u), \varphi] \leq 0 \ \forall \varphi \in C^\infty_c(\Omega; [0,\infty))$. We argue

$$B_L[f(u), \varphi] = \int_\Omega D(f(u)) \cdot aD\varphi \, dx = \int_\Omega f'(u) Du \cdot aD\varphi \, dx.$$

Since $f'(u) \in H^1(\Omega)$ and $\varphi \in C^\infty_c(\Omega)$ the function $f'(u)\varphi \in H^1_0(\Omega)$ is a possible test function for u. We write

$$f'(u)\varphi = D(f'(u)\varphi) - f''(u)Du \, \varphi.$$

Hence

$$B_L[f(u), \varphi] = \int_\Omega f'(u)Du \cdot aD\varphi \, dx = B_L[u, f'(u)\varphi] - \int_\Omega f''(u)Du \cdot aDu \, dx = -\int_\Omega f''(u) Du \cdot aDu \, dx,$$

where we used that $B_L[u, f'(u)\varphi] = 0$ since u is a weak solution of $Lu = 0$. Finally, since $f'' > 0$, $\varphi \geq 0$ and $Du \cdot aDu \geq \theta |Du|^2 \geq 0$ we have

$$B_L[f(u), \varphi] \leq 0.$$

which proves that $f(u)$ is a non-negative subsolution of $Lu = 0$ and hence $f(u) \in L^\infty_{loc}(\Omega)$. The result now follows from $|u| \leq f(u)$ a.e. in Ω.

(ii) For $x \in \Omega$ and $r > 0$ such that $B_r(x) \subset\subset \Omega$ we define the oscillation of u on $B_r(x_0)$ by:

$$\omega(x, r) := \sup_{B_r(x_0)} u - \inf_{B_r(x_0)} u.$$

This function is well defined since $u \in L^\infty_{loc}(\Omega)$. We show now that there exists $0 < \sigma < 1$ such that

$$\omega(x, R/4) \leq \sigma \omega(x, R) \qquad \forall B_R(x) \subset\subset \Omega. \tag{2.62}$$

This inequality implies local Hölder continuity (see exercise sheet).

Fix $x \in \Omega$ and assume $B_R(x) \subset\subset \Omega$. For each $\rho \leq R$ we define

$$M(\rho) := \sup_{B_\rho(x)} u, \quad m(\rho) := \inf_{B_\rho(x)} u.$$

Consider the two functions $u_1 := M(R) - u$, $u_2 := u - m(R)$. For each $j = 1, 2$, $u_j \geq 0$ on $B_R(x)$ and $u_j \in H^1(B_R(x))$ is a weak solution of $Lu_j = 0$ in $B_R(x)$. Then, by Harnack inequality, there is a constant $C_0 > 1$ such that

$$\sup_{B_\rho(x)} u_j \leq \ C_0 \inf_{B_\rho(x)} u_j, \qquad \forall 0 < \rho \leq R/4,$$

which gives the two inequalities, for $0 < \rho \le R/4$,

$$\begin{aligned} M(R) - m(\rho) &\le \ C_0\ [M(R) - M(\rho)] \\ M(\rho) - m(R) &\le \ C_0\ [m(\rho) - m(R)]. \end{aligned}$$

Summing both terms we obtain $w(R) + w(\rho) \le \ C_0\ [w(R) - w(\rho)]$ and hence

$$w(\rho) \le \frac{C_0 - 1}{C_0 + 1} w(R) = \ \nu\ w(R) \qquad \forall 0 < \rho \le R/4,$$

where $0 < \nu < 1$. □

3 Semilinear ellipic PDEs

We consider PDEs of the form $-\mathrm{div}\,(aDu) + g(u, Du) = f$, where a is uniformly elliptic.

3.1 Weak formulation

Definition 3.1. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded, $a \in L^\infty(\Omega; \mathbb{R}^{d\times d}_{sym})$, $g\colon \mathbb{R} \to \mathbb{R}$ given functions and $f \in H^{-1}(\Omega)$ a given operator.*
We say that u is a weak solution of the Dirichlet boundary value problem

$$\begin{cases} -\mathrm{div}\,(aDu) + g(u) = f & in\ \Omega \\ u_{|\partial\Omega} = 0 \end{cases}$$

if

$$\int_\Omega Dv \cdot aDu\ dx + \int_\Omega v g(u)\ dx = F(v) \qquad \forall v \in H^1_0(\Omega).$$

as long as the integral above are all well defined.

Remark. The only problem is to ensure that $g(u)v \in L^1(\Omega)$ for all $u, v \in H^1_0(\Omega)$. This is the content of the next lemma.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded.*

(i) Assume $d \ge 3$ and $f \in L^m(\Omega)$ with $m \ge \frac{2d}{d+2}$.
Then $fv \in L^1(\Omega)$ for all $v \in H^1_0(\Omega)$ and $\exists c_1 = c_1(\Omega, m, d) > 0$ such that

$$\|fv\|_{L^1(\Omega)} \le \ c_1\ \|f\|_{L^m(\Omega)} \|v\|_{H^1(\Omega)}.$$

(ii) Assume $g\colon \mathbb{R} \to \mathbb{R}$ satisfies $|g(t)| \le \ C|t|^\alpha\ \forall t \in \mathbb{R}$, with $C > 0$ some constant and

- $0 \le \alpha < \infty$ *if* $d = 1, 2$,
- $0 \le \alpha \le \frac{d+2}{d-2}$ *if* $d \ge 3$.

Then there exists $m = m(d, \alpha) > 1$ and constants $C_1, C_2 > 0$ depending on Ω, α, d, such that

(a) $u \in H^1_0(\Omega) \Rightarrow g(u) \in L^m(\Omega)$ and $\|g(u)\|_{L^m(\Omega)} \le C_1 \|u\|^\alpha_{H^1_0(\Omega)}$,

(b) $\|g(u)v\|_{L^1(\Omega)} \le C_2 \|g(u)\|_{L^m(\Omega)} \|v\|_{H^1_0(\Omega)} \le C_1 C_2 \|u\|^\alpha_{H^1_0(\Omega)} \|v\|_{H^1_0(\Omega)}\ \forall u, v \in H^1_0(\Omega)$.

Proof.
(i) Assume $d \geq 3$. Since $v \in H_0^1(\Omega)$, we have, by Sobolev embedding, $v \in L^{2^*}(\Omega)$ with $2^* = \frac{2d}{d-2}$ and hence

$$\|fv\|_{L^1(\Omega)} \leq \|f\|_{L^q(\Omega)}\|v\|_{L^{2^*}(\Omega)} \leq C_S \ \|f\|_{L^q(\Omega)}\|v\|_{H^1(\Omega)},$$

where $q := 2^{*\prime} = \frac{2d}{d+2} \leq m$. By Hölder inequality,

$$\|f\|_{L^q(\Omega)} \leq \ |\Omega|^{\frac{1}{q}-\frac{1}{m}}\|f\|_{L^m(\Omega)}$$

and the result follows.

(ii) Assume first $d \geq 3$. Using (i), $g(u)v \in L^1(\Omega)$ if $g(u) \in L^m(\Omega)$, for some $m \geq \frac{2d}{d+2}$. Since $|g(u)| \leq C|u|^\alpha$, that means we need $|u|^{\alpha m} \in L^1(\Omega)$.
Since $u \in H_0^1(\Omega)$, by Sobolev inequality we have $|u|^{2^*} \in L^1(\Omega)$.
Therefore, $vg(u) \in L^1(\Omega)$, if there exists $m \geq 0$ such that $m\alpha \leq 2^*$ and $m \geq \frac{2d}{d+2}$, i.e.

$$\frac{2d}{d+2} \leq m \leq \frac{2d}{\alpha(d-2)}.$$

Since $\alpha \leq \frac{d+2}{d-2}$, we have $\frac{2d}{d+2} \leq \frac{2d}{\alpha(d-2)}$, and hence a solution m exists always. Moreover

$$m \geq \frac{2d}{d+2} = 1 + \frac{d-2}{d+2} > 1 \qquad \forall d \geq 3,$$

and

$$\|g(u)\|_{L^m(\Omega)} \leq \ C\||u|^\alpha\|_{L^m(\Omega)} = C \ \|u\|^\alpha_{L^{\alpha m}(\Omega)} \leq C|\Omega|^{\frac{1}{\alpha m}-\frac{1}{2^*}}\|u\|^\alpha_{L^{2^*}(\Omega)} \leq CC_S|\Omega|^{\frac{1}{\alpha m}-\frac{1}{2^*}}\|u\|^\alpha_{H^1(\Omega)},$$

which proves (a). Finally we argue

$$\begin{aligned}\|g(u)v\|_{L^1(\Omega)} &\leq \|g(u)\|_{L^q(\Omega)}\|v\|_{L^{2^*}(\Omega)} \leq C_S \ \|g(u)\|_{L^q(\Omega)}\|v\|_{H^1(\Omega)} \\ &\leq C_S|\Omega|^{\frac{1}{q}-\frac{1}{m}}\|g(u)\|_{L^m(\Omega)}\|v\|_{H^1(\Omega)},\end{aligned}$$

which proves (b) in the case $d \geq 3$.

Assume $d = 2$. We argue $u \in W^{1,2}(\Omega) \Rightarrow u \in W^{1,2-\varepsilon}(\Omega)$ $\forall 0 < \varepsilon \leq 1$. By Sobolev embedding we obtain $u \in L^q(\Omega)$ $\forall 1 \leq q < \infty$ and

$$\|u\|_{L^q(\Omega)} \leq C_{S,q}\|u\|_{H^1(\Omega)}.$$

Setting $m = 2$ we argue

$$\|g(u)\|_{L^2(\Omega)} \leq \ C\||u|^\alpha\|_{L^2(\Omega)} = C \ \|u\|^\alpha_{L^{2\alpha}(\Omega)} \leq CC^\alpha_{S,2\alpha}\|u\|^\alpha_{H^1(\Omega)},$$

which proves (a). Finally $|g(u)v\|_{L^1(\Omega)} \leq \|g(u)\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}$ which proves (b).

Assume $d = 1$. We argue $u \in W^{1,2}(\Omega) \Rightarrow u \in C^{0,\alpha}(\Omega)$ by Sobolev embedding. It follows

$$|u(x)| \leq \fint_\Omega |u(y)|dy + \fint_\Omega |u(x) - u(y)|dy \leq \frac{1}{|\Omega|^{\frac{1}{2}}}\|u\|_{L^2(\Omega)} + (\operatorname{diam}\Omega)^\alpha[u]_{C^{0,\alpha}(\Omega)}$$

and hence $u \in L^\infty(\Omega)$ with $\|u\|_{L^\infty(\Omega)} \leq C'\|u\|_{H^1_0(\Omega)}$. Setting $m = 2$ we argue

$$\|g(u)\|_{L^2(\Omega)} \leq C|\Omega|^{\frac{1}{2}}\|u\|^\alpha_{L^\infty(\Omega)},$$

which proves (a). Finally $|g(u)v\|_{L^1(\Omega)} \leq \|g(u)\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}$ which proves (b). □

[17: 4.12.2023]

[18: 7.12.2023]

3.2 Stampacchia's theorem and some applications

Theorem 3.3 (Stampacchia)**.** *Let H be a real Hilbert space, and*

$$a\colon \quad \begin{array}{ll} H\times H & \to \mathbb{R} \\ (x,y) & \to a(x,y) \end{array}$$

such that

(i) $\forall x\in H$ *the map* $a(x,\cdot)\colon H\to\mathbb{R}$ *is linear and continuous i.e.* $a(x,\cdot)\in H^*$,

(ii) $\exists\alpha>0$ *such that* $|a(x_1,y)-a(x_2,y)|\le\alpha\|x_1-x_2\|\,\|y\|\ \forall x_1,x_2,y\in H$,

(iii) $\exists\beta>0$ *such that* $a(x_1,x_1-x_2)-a(x_2,x_1-x_2)\ge\beta\|x_1-x_2\|^2\ \forall x_1,x_2\in H$.

Then $\forall T\in H^*\ \exists! u_T\in H$ *such that*

$$T(v)=a(u_T,v)\qquad\forall v\in H.$$

Proof. The result follows from the following Claim (cf Ex 1.4)

Let H be a Hilbert space, and $A\colon H\to H$ a map (in general nonlinear) satisfying

(i) $\exists\alpha>0$ such that $\|A(x)-A(y)\|\le\alpha\|x-y\|\ \forall x,y\in H$,

(ii) $\exists\beta>0$ such that $(A(x)-A(y),x-y)\ge\beta\|x-y\|^2\ \forall x,y\in H$.

Then A is invertible i.e. $\forall f\in H\ \exists! u_f\in H$ such that $A(u_f)=f$.

□

We will see now two applications of this result.

Theorem 3.4. *Let $\Omega\subset\mathbb{R}^d$ be open and bounded, $a\in L^\infty(\Omega;\mathbb{R}^{d\times d}_{sym})$ uniformly elliptic and $g\colon\mathbb{R}\to\mathbb{R}$ Lipschitz continuous and non-decreasing.*

Then $\forall F\in H^1_0(\Omega)^*\ \exists! u=u_F\in H^1_0(\Omega)$ *weak solution of*

$$\begin{cases} -\operatorname{div}(aDu)+g(u)=F & \text{in } \Omega,\\ u_{|\partial\Omega}=0. \end{cases}\tag{3.1}$$

Remark. Note that if g is linear and non-decreasing we have $g(u)=Cu$, with $C\le 0$. By the first existence theorem 2.8, the linear PDE $-\operatorname{div} aDu+Cu=F$ has a unique weak solution for all $F\in H^{-1}(\Omega)$. The theorem above extends this result to nonlinear g.

Proof.
• The weak formulation for (3.1) is well defined. Indeed, since g is Lipschitz continuous,

$$|g(t)-g(s)|\le L_g\ |t-s|\qquad\forall t,s\in\mathbb{R},$$

where $L_g>0$ is the Lipschitz constant. Hence, using also $|\Omega|<\infty$,

$$|g(u)|\le|g(0)|+|g(u)-g(0)|\le|g(0)|+L_g|u|\in L^2(\Omega)\qquad\forall u\in L^2(\Omega).$$

• We define

$$
\begin{array}{lll}
B\colon & H_0^1(\Omega)\times H_0^1(\Omega) & \to \mathbb{R} \\
 & (u,v) & \to B[u,v] := \int_\Omega Dv\cdot aDu\ dx + \int_\Omega vg(u)\ dx.
\end{array}
$$

Then $u \in H_0^1(\Omega)$ is a weak solution of (3.1) iff $B[u,v] = T(v)\ \forall v \in H_0^1(\Omega)$.
We prove now that B satisfies the assumptions of Stampacchia's theorem, and hence the weak solution exists and is unique.

For this purpose we write $B[u,v] = B_0[u,v] + B_1[u,v]$ where

$$
B_0[u,v] := \int_\Omega Dv\cdot aDu\ dx, \qquad B_1[u,v] := \int_\Omega vg(u)\ dx.
$$

We check now that B satisfies $(i)(ii)(iii)$ in the assumptions of Theorem 3.3.

(i) B_0 is bilinear and continous, while B_1 is linear in the second variable. It remains to check that $B_1[u,\cdot]$ is also continuous. Since $v \mapsto B_1[u,v]$ is linear we only need to check the map is bounded:

$$
|B_1[u,v]| \le \int_\Omega |g(u)|\ |v|\ dx \ \le\ \|g(u)\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \le \|g(u)\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)}
$$

and hence the map $v \mapsto B_1[u,v] \in H^{-1}(\Omega)$ with $\|B_1[u,\cdot]\|_{op} \le \|g(u)\|_{L^2(\Omega)} < \infty$.

(ii) Set $u_1, u_2, v \in H_0^1(\Omega)$. We compute, using $|g(u_1) - g(u_2)| \le C|u_1 - u_2|$,

$$
\begin{aligned}
|B[u_1,v] - B[u_2,v]| &\le \left|\int_\Omega Dv\cdot aD(u_1-u_2)\ dx\right| + \int_\Omega |v|\ |g(u_1)-g(u_2)|\ dx \\
&\le \|a\|_{L^\infty(\Omega)}\|Du_1 - Du_2\|_{L^2(\Omega)}\|Dv\|_{L^2(\Omega)} + C\|u_1-u_2\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\
&\le\ \alpha\ \|u_1-u_2\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)},
\end{aligned}
$$

for some $\alpha > 0$.

(iii) Set $u_1, u_2 \in H_0^1(\Omega)$. We compute

$$
B[u_1, u_1-u_2] - B[u_2, u_1-u_2] = \int_\Omega D(u_1-u_2)\cdot aD(u_1-u_2)\ dx + \int_\Omega [g(u_1)-g(u_2)](u_1-u_2)\ dx.
$$

Since g is non-decreasing we have

$$
\int_\Omega [g(u_1)-g(u_2)](u_1-u_2)\ dx \ge 0.
$$

By uniform ellipticity $a \ge \theta\mathrm{Id}$, and Poincaré inequality, we conclude

$$
\begin{aligned}
B[u_1, u_1-u_2] - B[u_2, u_1-u_2] \ &\ge\ \int_\Omega D(u_1-u_2)\cdot aD(u_1-u_2)\ dx \\
&\ge \theta\|D(u_1-u_2)\|^2_{L^2(\Omega)} \ge \beta\|u_1-u_2\|^2_{H_0^1(\Omega)},
\end{aligned}
$$

for some $\beta > 0$. Hence B satisfies the assumptions of Theorem 3.3 which garantees $\forall F \in H_0^1(\Omega)^*$ $\exists! u = u_F \in H_0^1(\Omega)$ weak solution of (3.1). This concludes the proof of the theorem.

□

This can be extended to the case when g is just locally Lpischitz continuous. To prove it, we will need the following result from Functional Analysis.

Theorem 3.5 (Vitali). *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space with $\mu(\Omega) < \infty$, and $n \mapsto f_n \in L^p(\Omega)$, with $1 \le p < \infty$, a sequence of functions satisfying:*

(i) $f_n \to f$ pointwise a.e.,

(ii) $\forall \varepsilon > 0\ \exists \delta = \delta_\varepsilon > 0$ such that the following holds:

$$\int_E |f_n|^p dx < \varepsilon \qquad \forall n \in \mathbb{N}, \forall E \subset \Omega \text{ measurable set with } \mu(E) < \delta.$$

Then $f \in L^p(\Omega)$ and $f_n \to f$ in $L^p(\Omega)$.

Proof. See for example Boccardo-Croce, or Brezis. □

Theorem 3.6. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded, $a \in L^\infty(\Omega; \mathbb{R}^{d\times d}_{sym})$ uniformly elliptic and $g\colon \mathbb{R} \to \mathbb{R}$ locally Lipschitz continuous and non-decreasing.*

Then $\forall F \in H^1_0(\Omega)^*\ \exists! u = u_F \in H^1_0(\Omega)$ weak solution of

$$\begin{cases} -\mathrm{div}\,(aDu) + g(u) = F & \text{in } \Omega, \\ u_{|\partial\Omega} = 0. \end{cases} \tag{3.2}$$

in the following sense: $g(u) \in L^1(\Omega)$ and

$$\int_\Omega Dv \cdot aDu\ dx + \int_\Omega vg(u)\ dx = F(v) \qquad \forall v \in H^1_0(\Omega) \cap L^\infty(\Omega). \tag{3.3}$$

Remark. We need $v \in L^\infty(\Omega), g(u) \in L^1(\Omega)$ to ensure that $\int_\Omega |vg(u)|\ dx < \infty$ holds and hence the weak formulation makes sense.

Proof. We can assume $g(0) = 0$. Indeed, if $g(0) \neq 0$ we can write

$$g(t) = g(t) - g(0) + g(0) = \tilde{g}(t) + g(0),$$

where $\tilde{g}(t) := g(t) - g(0)$ is locally Lipschitz continuous, non-decreasing and satisfies $\tilde{g}(0) = 0$. Moreover $u \in H^1_0(\Omega)$ is a weak solution of (3.1) iff it is weak solution of $-\mathrm{div}\,(aDu) + \tilde{g}(u) = F - g(0)$ in Ω with $u_{|\partial\Omega} = 0$.

Existence. We construct a weak solution by approximation. In order to make g Lipschitz continous, we introduce the cut-off function

$$T_k\colon \quad \begin{array}{l} \mathbb{R} \to \mathbb{R}, \\ s \to T_k(s) \end{array}, \qquad T_k(s) := \begin{cases} k & s > k \\ s & |s| \le k \\ -k & s < -k, \end{cases}$$

and define

$$g_k := g \circ T_k \qquad \text{i.e. } g_k(s) = \begin{cases} g(k) & s > k \\ g(s) & |s| \le k \\ g(-k) & s < -k. \end{cases}$$

Note that $g_k(0) = 0\ \forall k$ and

$$\lim_{k\to\infty} g_k = g \qquad \text{pointwise.}$$

Since g is locally Lipschitz continuous and non-decreasing, it follows (exercise) that g_k is Lipschitz continuous and non-decreasing. By Theorem 3.4, for all $F \in H_0^1(\Omega)^*$ and $k \geq 1$ there exists a unique function $u_k \in H_0^1(\Omega)$ weak solution of $-\operatorname{div} aDu_k + g_k(u_k) = F$, i.e.

$$\int_\Omega Dv \cdot aDu_k \, dx + \int_\Omega v g_k(u_k) \, dx = F(v) \qquad \forall v \in H_0^1(\Omega).$$

Claim. $\exists u \in H_0^1(\Omega)$ and a subsequence $j \mapsto u_{k_j}$ such that

(a) $u_{k_j} \rightharpoonup u$ weakly in $H_0^1(\Omega)$,

(b) $g(u) \in L^1(\Omega)$ and $g_{k_j}(u_{k_j}) \to g(u)$ strongly in $L^1(\Omega)$.

Consequence. It follows from (a) that

$$\lim_{j\to\infty} \int_\Omega Dv \cdot aDu_{k_j} \, dx \; = \; \int_\Omega Dv \cdot aDu \, dx, \qquad \forall v \in H^1(\Omega).$$

It follows from (b) that

$$\lim_{j\to\infty} \int_\Omega v g_{k_j}(u_{k_j}) \, dx \; = \; \int_\Omega v g(u) \, dx \qquad \forall v \in L^\infty(\Omega).$$

Hence, since u_{k_j} is a weak solution of $-\operatorname{div} aDu_{k_j} + g_{k_j}(u_{k_j}) = F$, it holds, $\forall v \in H_0^1(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega Dv \cdot aDu \, dx + \int_\Omega v g(u) \, dx = \lim_{j\to\infty} \int_\Omega [Dv \cdot aDu_{k_j} + v g_{k_j}(u_{k_j})] \, dx = \lim_{j\to\infty} F(v) = F(v),$$

and therefore u is a weak solution for (3.2).

Proof of Claim (a). We show that the sequence $n \mapsto u_k$ is bounded in $H_0^1(\Omega)$, and hence (since $H_0^1(\Omega)$ is reflexive) there exists a weakly convergence subsequence.

Indeed, since $u_k \in H_0^1(\Omega)$ we can take as test function $v = u_k$. We obtain, using also the uniform ellipticity of a,

$$\theta \|Du_k\|^2_{L^2(\Omega)} \leq (Du_k, aDu_k)_{L^2(\Omega)} = F(u_k) - \int_\Omega u_k g_k(u_k) \, dx.$$

Note that $u_k g_k(u_k) = (u_k - 0)(g_k(u_k) - g_k(0)) \geq 0$, since g_k is non-decreasing. It follows, using also Poincaré inequality, that there exists a constant $c_1 > 0$ such that

$$c_1 \|u_k\|^2_{H_0^1(\Omega)} \leq \theta \|Du_k\|^2_{L^2(\Omega)} \leq F(u_k) - \int_\Omega u_k g_k(u_k) \, dx \leq F(u_k) \leq \|F\|_{op} \, \|u_k\|_{H_0^1(\Omega)},$$

and hence

$$\sup_k \|u_k\|_{H_0^1(\Omega)} \leq \frac{\|F\|_{op}}{c_1} < \infty.$$

This concludes the proof of Claim (a).

Proof of Claim (b). By (a) we have $u_{k_j} \rightharpoonup u$ in $H_0^1(\Omega)$ and hence, by Rellich, $u_{k_j} \to u$ in $L^2(\Omega)$. It follows that there is a subsequence $l \to u_{k_{j_l}}$ such that $u_{k_{j_l}} \to u$ pointwise a.e. in Ω. Since $g_k(s) = g(s)$ $\forall k > |s|$ we have $g_{k_{j_l}}(u_{k_{j_l}}) \to g(u)$ pointwise a.e. in Ω.
We want to apply Vitali's Theorem 3.5 with $p = 1$ to the sequence $f_n := g_n(u_n)$, where we write, to simplify the notation, u_n instead of $u_{k_{j_l}}$. Since $g_n(u_n) \to g(u)$ pointwise a.e., we already have condition (i) in Theorem 3.5. We check now the validity of condition (ii).

Let $E \subset \Omega$ be a measurable set and $\varepsilon > 0$. Our goal is to find $\delta_\varepsilon > 0$ independent of E such that

$$|E| < \delta_\varepsilon \;\Rightarrow\; \int_E |g_n(u_n)| dx < \varepsilon \;\forall n.$$

For any $M > 1$ note that

$$|u_n| \leq M \;\Rightarrow\; |g_n(u_n)| \leq \max\{g(M), |g(-M)|\} =: C_{M,g}.$$

Indeed, using that g_n is nondecreasing, $g_n(t) \leq g(t)$ $\forall t > 0$ and $g_n(t) \geq g(t)$ $\forall t < 0$ we argue

$$\begin{aligned}
&0 \leq g_n(u_n) \leq g_n(M) \leq g(M) \qquad \forall 0 < u_n \leq M,\\
&g(-M) \leq g_n(-M) \leq g_n(u_n) \leq 0 \qquad \forall -M \leq u_n \leq 0.
\end{aligned}$$

Inserting this bounds in the integral we obtain

$$\begin{aligned}
\int_E |g_n(u_n)| dx &= \int_{E \cap \{|u_n| \leq M\}} |g_n(u_n)| dx + \int_{E \cap \{|u_n| > M\}} |g_n(u_n)| dx\\
&\leq C_{M,g}|E| + \int_{E \cap \{|u_n| > M\}} |g_n(u_n)| dx.
\end{aligned}$$

When $|u_n| > M$ we have $1 < \frac{|u_n|}{M}$ and hence

$$\int_{E \cap \{|u_n| > M\}} |g_n(u_n)| dx \leq \frac{1}{M} \int_{E \cap \{|u_n| > M\}} |u_n g_n(u_n)| dx = \frac{1}{M} \int_{E \cap \{|u_n| > M\}} u_n g_n(u_n) dx,$$

where we used that g_n is non-decreasing and $g_n(0) = 0$, and hence $g(u_n)u_n \geq 0$. Since u_n is a weak solution for $-\operatorname{div} aDu_n + g_n(u_n) = F$ and also a possible test function, we compute

$$\int_\Omega u_n g_n(u_n) dx = F(u_n) - (Du_n, aDu_n)_{L^2(\Omega)} \leq F(u_n) \leq \|F\|_{op} \, \|u_n\|_{H_0^1(\Omega)} \leq \frac{\|F\|_{op}^2}{c_1},$$

and hence

$$\int_E |g_n(u_n)| dx \leq C_{M,g}|E| + \frac{1}{M} \frac{\|F\|_{op}^2}{c_1}$$

For $\varepsilon > 0$ choose $M = M_\varepsilon$ such that

$$\frac{1}{M_\varepsilon} \frac{\|F\|_{op}^2}{c_1} = \frac{\varepsilon}{4}$$

and choose δ_ε such that

$$C_{M_\varepsilon,g} \delta_\varepsilon = \frac{\varepsilon}{4}.$$

We obtain

$$|E| < \delta_\varepsilon \qquad \Rightarrow \qquad \int_E |g_n(u_n)| dx < \varepsilon \qquad \forall n \in \mathbb{N},$$

i.e. condition (ii) in Theorem 3.5 holds too. It follows $g(u) \in L^1(\Omega)$ and $g_n(u_n) \to g(u)$ in $L^1(\Omega)$. This completes the proof of existence.

$\square$

[18: 7.12.2023]

[19: 11.12.2023]

Proof of uniqueness. Assume $u_1, u_2 \in H_0^1(\Omega)$ satisfy $g(u_1), g(u_2) \in L^1(\Omega)$ and

$$(Du_1, aDv)_{L^2(\Omega)} + \int_\Omega g(u_1) v dx = F(v) = (Du_2, aDv)_{L^2(\Omega)} + \int_\Omega g(u_2) v dx \qquad \forall v \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

It follows

$$(D(u_1 - u_2), aDv)_{L^2(\Omega)} + \int_\Omega [g(u_1) - g(u_2)] v dx \;=\; 0 \qquad \forall v \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

We cannot replace as test function $v = u_1 - u_2$ since this function is not necessarily in $L^\infty(\Omega)$. Instead set

$$v_k := T_k \circ (u_1 - u_2) = \begin{cases} k & u_1 - u_2 \geq k \\ u_1 - u_2 & |u_1 - u_2| \leq k \\ -k & u_1 - u_2 \leq -k. \end{cases}$$

It holds (exercise) $v_k \in L^\infty(\Omega) \cap H_0^1(\Omega)$ $\forall k$, and

$$Dv_k = \mathbf{1}_{|u_1 - u_2| < k} D(u_1 - u_2).$$

Inserting $v = v_k$ above and using

$$\int_\Omega D(u_1 - u_2) \cdot aDv_k = \int_\Omega \mathbf{1}_{|u_1 - u_2| < k} D(u_1 - u_2) \cdot aD(u_1 - u_2) = \int_\Omega Dv_k \cdot aDv_k,$$

we argue

$$\begin{aligned} \theta \|Dv_k\|^2_{L^2(\Omega)} &\leq \int_\Omega D(u_1 - u_2) \cdot aDv_k \; dx \\ &= -\int_\Omega [g(u_1) - g(u_2)] T_k(u_1 - u_2) dx \leq 0, \end{aligned}$$

where in the last step we used that T_k and g are non-decreasing. It follows that $v_k = 0$ holds a.e. in Ω and hence $u_1 = u_2$ a.e. in Ω. This concludes the proof of unicity and of the theorem. □

Example 1. Set $g(t) := t|t|^{p-2}$, with $p \geq 2$. This function is locally Lipschitz and non-decreasing. Hence the unique weak solution $u \in H_0^1(\Omega)$ of (3.2) must satisfy $g(u) \in L^1(\Omega)$, i.e. $u \in H_0^1(\Omega) \cap L^{p-1}(\Omega)$.

Example 2. Set $g(t) := e^t - 1$. This function is locally Lipschitz and non-decreasing. Hence the unique weak solution $u \in H_0^1(\Omega)$ of (3.2) must satisfy $e^u - 1 \in L^1(\Omega)$.

3.3 Subsolution and supersolution method

Let $\Omega \subset \mathbb{R}^d$ be open and bounded. We consider now the linear operator $Lu := -\operatorname{div} aDu$ and the nonlinear PDE

$$\begin{cases} Lu = g(u) + F & \text{in } \Omega \\ u_{|\partial\Omega} = 0 \end{cases} \tag{3.4}$$

with $F \in H^1(\Omega)^*$ and $g\colon \mathbb{R} \to \mathbb{R}$ non-decreasing.

Note that, contrary to the previous chapter, we study $Lu - g(u) = F$. For g linear this corresponds to study $Lu - Cu = F$, with $C > 0$. This equation is not always solvable (cf Section 2.2.3). We will see that, in some cases, we can at least garantee existence, though not uniqueness, of a weak solution. The idea is to compare the PDE with the solutions of some appropriately chosen linearized equation. To make this rigorous we will use sub and supersolutions.

Definition 3.7. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded.*
Remember that $B_L[u,v] = (Dv, aDu)_{L^2(\Omega)} = \int_\Omega Dv \cdot aDu \; dx \; \forall u, v \in H_0^1(\Omega)$.

(i) The function $\underline{u} \in H_0^1(\Omega)$ is a (weak) subsolution for (3.4) if

$$B_L[\underline{u}, v] \leq \int_\Omega vg(\underline{u}) \; dx + F(v) \qquad \forall v \in H^1(\Omega; [0, \infty)).$$

(ii) The function $\overline{u} \in H_0^1(\Omega)$ is a (weak) supersolution for (3.4) if

$$B_L[\overline{u}, v] \geq \int_\Omega vg(\overline{u}) \; dx + F(v) \qquad \forall v \in H^1(\Omega; [0, \infty)).$$

Here we assume g is regular enough to ensure the weak formulation above is well defined.

Remark. If $\underline{u}, \overline{u} \in C^2(\Omega)$, $a \in C^1(\Omega; \mathbb{R}^{d \times d}_{sym})$ and $F = (f, \cdot)_{L^2(\Omega)}$, with $f \in C(\Omega) \cap L^2(\Omega)$, then we can replace the integrals above with pointwise inequalities

$$\begin{aligned} L\underline{u} &\leq g(\underline{u}) + f \\ L\overline{u} &\geq g(\overline{u}) + f. \end{aligned} \tag{3.5}$$

Theorem 3.8. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded, $a \in L^\infty(\Omega; \mathbb{R}^{d \times d}_{sym})$ uniformly elliptic, $F \in H_0^1(\Omega)^*$ and $g \colon \mathbb{R} \to \mathbb{R}$ a function satisfying:*

- *g is continuous and $g(0) = 0$,*
- *$|g(t)| \leq \; C \; |t|^\alpha \; \forall t \in \mathbb{R}$, where $C > 0$ is some constant and $\alpha \in [0, \infty)$ for $d = 1, 2$, while $\alpha \in [0, \frac{d+2}{d-2}]$ for $d \geq 3$.*

Assume $\underline{u} \in H_0^1(\Omega)$ is a weak subsolution for (3.4), and $\overline{u} \in H_0^1(\Omega)$ is a weak supersolution for (3.4), satisfying

$$\underline{u} \; \leq \; \overline{u} \qquad a.e. \; in \; \Omega,$$

and at least one of the conditions below holds.

(i) g is non-decreasing.

(ii) g is Lipschitz continuous.

(iii) $g_{|\mathbb{R}_+}$ is non-decreasing and $0 \leq \; \underline{u} \; \leq \; \overline{u}$ a.e. in Ω.

(iv) g is locally Lipschitz continuous, $g_{|\mathbb{R}_+}$ is non-decreasing and $\exists M > 0$ such that $-M \leq \; \underline{u} \; \leq \; \overline{u}$ a.e. in Ω.

Then $\exists u \in H_0^1(\Omega)$ weak solution of (3.4) satisfying $\underline{u} \; \leq \; u \; \leq \; \overline{u}$ a.e. in Ω.

Remark 1. The assumptions on α garantee, using Lemma 3.2, that the weak formulation is well defined.

Remark 2. The weak solution obtained from the theorem above is not unique in general (cf. Lemma 3.10 below).

Proof. Sketch (cf Ex 10.1 and 10.2)

Case 1. Assume g is non-decreasing. We approximate u by a sequence $k \mapsto u_k \in H_0^1(\Omega)$ constructed as follows.
Set $u_0 := \underline{u}$. For $k \geq 1$ take $u_k \in H_0^1(\Omega)$ to be the unique weak solution of the linear PDE

$$\begin{cases} Lu_k = F_k & \text{in } \Omega \\ u_{|\partial\Omega} = 0 \end{cases}$$

where

$$F_k := F + \Big(g(u_{k-1}) \ , \ \cdot \ \Big)_{L^2(\Omega)} \in H_0^1(\Omega)^*.$$

The function u_k exists and is unique by the first existence theorem 2.8. Moreover (exercise)

$$\underline{u} \leq u_1 \leq u_2 \leq \cdots \leq \overline{u},$$

and $u_k \to u$ in some appropriate sense, where $u \in H_0^1(\Omega)$ is a weak solution of (3.4) satisfying $\underline{u} \ \leq \ u \ \leq \ \overline{u}$ a.e. in Ω.

Case 2. Assume g is Lipschitz continuous. Then $\exists C > 0$ such that $|g(t) - g(s)| \leq C|t - s|$ $\forall t, s \in \mathbb{R}$. It follows $h(t) := g(t) + Ct$ is non-decreasing. We use then Case 1 to construct a weak solution of $Lu + Cu = h(u) + F$ and from there a weak solution of (3.4) satisfying $\underline{u} \ \leq \ u \ \leq \ \overline{u}$ a.e. in Ω

Case 3. Use the same strategy as in Case 1.

Case 4. Use the same strategy as in Case 2 and 3.

□

Lemma 3.9 (Example 1)**.** *Let $\Omega \subset \mathbb{R}^d$, with $d \leq 6$, be open, bounded and connected, with smooth boundary. Assume in addition $f \in C^\infty(\overline{\Omega}; [0, \infty))$ and $\exists x_0 \in \Omega$ such that $f(x_0) > 0$.*
Then the nonlinear PDE

$$\begin{cases} -\Delta u = u^2 - f & \text{in } \Omega \\ u_{|\partial\Omega} = 0 \end{cases} \tag{3.6}$$

has at least one non-positive weak solution $u \in H_0^1(\Omega; (-\infty, 0])$.

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Proof.
• The weak formulation is well defined. Indeed $g(s) = s^2$, i.e. $\alpha = 2$ and

$$2 \leq \frac{d+2}{d-2} \qquad \forall 3 \leq d \leq 6.$$

• The map $s \mapsto g(s)$ satisfies $g(0) = 0$, is non-decreasing on $\mathbb{R}_+$ and locally Lipschitz on $\mathbb{R}$. Moreover we look for a non-positive solution, hence we need to use Case (iv) of Theorem 3.8.

• We look for a supersolution satisfying $\overline{u} \leq 0$. The choice $\overline{u} := 0$ works. Indeed, using $f \geq 0$ and $g(0) = 0$, we get

$$\int_\Omega Dv \cdot D\overline{u}\, dx = 0 \geq -\int_\Omega vf\, dx = \int_\Omega v[g(\overline{u}) - f]\, dx, \qquad \forall v \in H_0^1(\Omega; [0, \infty)).$$

Equivalently we can argue pointwise

$$-\Delta\overline{u} = 0 \geq -f = \overline{u}^2 - f.$$

• We look for a subsolution $\underline{u} \in H_0^1(\Omega)$ satisfying $\underline{u} \leq \overline{u} = 0$. Note that $u^2 - f \geq -f$ so we consider the linear PDE

$$\begin{cases} -\Delta\psi = -f & \text{in } \Omega \\ \psi_{|\partial\Omega} = 0. \end{cases}$$

By the first existence theorem 2.8, this PDE has a unique weak solution $\psi \in H_0^1(\Omega)$. Since $f \in C^\infty(\overline{\Omega})$ and $\partial\Omega$ is smooth, by Theorem 2.25 we have $\psi \in C^\infty(\overline{\Omega})$ and the equation $-\Delta\psi = -f$ holds pointwise. Hence we have

$$-\Delta\psi = -f \leq 0 \qquad \text{in } \Omega, \qquad \text{and} \qquad -\Delta\psi(x_0) = -f(x_0) < 0.$$

Therefore, by the strong maximum principle (since Ω is connected) $\psi < 0$ in Ω.

We define $\underline{u} := \psi$. This is a subsolution. Indeed

$$-\Delta\underline{u} = -f \leq -f + \underline{u}^2.$$

Moreover this function satisfies $\underline{u} < 0 = \overline{u}$. Finally, since $\underline{u} \in C(\overline{\Omega})$, there exists $M > 0$ such that $-M \leq \underline{u}(x)$ $\forall x \in \Omega$, and by Case (iv) of Theorem 3.8 there exists at least one weak solution $u \in H_0^1(\Omega)$ such that $\underline{u} \leq u \leq 0$.

□

Lemma 3.10 (Example 2)**.** *Let $\Omega \subset \mathbb{R}^d$, with $d \leq 6$, be open, bounded and connected, with smooth boundary. Assume in addition $0 < \theta < 1$.*
Then the nonlinear PDE

$$\begin{cases} -\Delta u = |u|^\theta & \text{in } \Omega \\ u_{|\partial\Omega} = 0 \end{cases} \tag{3.7}$$

has at least one strictly positive weak solution $u \in H_0^1(\Omega; (0, \infty))$.

Remark. Since $u = 0$ is also a weak solution, the PDE above has at least two different solutions.

Proof.
• We have $g(s) = |s|^\theta$, with $0 < \theta < 1$ hence the weak formulation is well defined in any dimension.

• Since g is non decreasing on $\mathbb{R}_+$ it is enough to find a sub and supersolution such that $0 < \underline{u} \leq \overline{u}$. We note that, if $\theta = 0$ we have $-\Delta u = 1$ which is a linear non-homegeneous PDE. If $\theta = 1$ and $u \geq 0$ we have $-\Delta u = |u| = u$ which is a linear homegeneous PDE. Since $u_{|\partial\Omega} = 0$ we expect $0 < u < 1$ near $\partial\Omega$ and hence

$$u^1 \leq u^\theta \leq u^0 = 1.$$

Therefore we will use the case $\theta = 0$ to look for a supersolution $\overline{u}$ and the case $\theta = 1$ to look for a subsolution $\underline{u}$.

• Supersolution: consider the PDE

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u_{|\partial\Omega} = 0. \end{cases}$$

By the first existence theorem 2.8, this PDE has a unique weak solution $\psi \in H_0^1(\Omega)$. By Theorem 2.25 we have $\psi \in C^\infty(\overline{\Omega})$, $\psi_{|\partial\Omega} = 0$, and the equation $-\Delta\psi = 1$ holds pointwise. Hence we have

$$-\Delta\psi = 1 > 0 \qquad \text{in } \Omega,$$

and by the strong maximum principle (since Ω is connected) $\psi > 0$ in Ω.

We define $\overline{u} := c_1\psi$, where $c_1 > 0$ is some constant to be chosen later. $\overline{u}$ is a supersolution if

$$c_1 = -\Delta\overline{u} \geq c_1^\theta \psi^\theta.$$

For this it is enough to choose $c_1 \geq \|\psi\|_{L^\infty}^{\frac{\theta}{1-\theta}}$.

• Subsolution: consider the PDE

$$\begin{cases} -\Delta u = u & \text{in } \Omega \\ u_{|\partial\Omega} = 0. \end{cases}$$

The spectrum of $-\Delta$ is at most countable and strictly positive (cf. Theorem 2.12 and the remarks after it)

$$\Sigma = \{\lambda_n\}_{n\in\mathbb{N}}, \qquad 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots$$

Claim. $\exists \varphi \in H_0^1(\Omega; (0,\infty))$ strictly positive eigenvector for λ_1, i.e. φ is a weak solution of $-\Delta\varphi = \lambda_1\varphi$.

Proof. Cf Evans for the general case. In the special case $d = 1$ and $\Omega = (-1, 1)$ one can construct explicitely all eigenvalues and eigenvectors for $-\Delta$ (exercise).

We define $\underline{u} := c_2\psi$, where $c_2 > 0$ is some constant to be chosen later. $\underline{u}$ is a subsolution if

$$\lambda_1\underline{u} = -\Delta\underline{u} \leq \underline{u}^\theta, \qquad \text{iff} \quad \underline{u}^{1-\theta} \leq \lambda_1^{-1} \qquad \text{iff} \qquad \|\underline{u}\|_{L^\infty} \leq \lambda_1^{-\frac{1}{1-\theta}}.$$

For this it is enough to choose $c_2 \leq \lambda_1^{-\frac{1}{1-\theta}} \|\psi\|_{L^\infty}^{-1}$.

• We have now a super and a subsolution. We still need to check that $\underline{u} \leq \overline{u}$. We define $w := \overline{u} - \underline{u}$. Then $w \in C^\infty(\overline{\Omega})$ and $w_{|\partial\Omega} = 0$. We compute

$$-\Delta w = (-\Delta\overline{u}) - (-\Delta\underline{u}) = c_1 - c_2\lambda_1\varphi.$$

We choose $c_1 \geq \lambda_1 c_2 \|\varphi\|_{L^\infty}$. Then $-\Delta w \geq 0$ and by the strong maximum principle $w \geq 0$ on Ω and hence $\overline{u} \geq \underline{u}$.

• We have constructed a subsolution and a supersolution such that $0 < \underline{u} \leq \overline{u}$. Since g is non-decreasing on $\mathbb{R}_+$, by Case (*iii*) of Theorem 3.8, there exists at least one strictly positive weak solution $u \in H_0^1(\Omega)$. □

4 Fixed point methods

Existence of solutions for NLPDEs can be sometimes proved by finding a fixed point. The following theorems summarize the most important fixed point results.

Theorem 4.1 (Banach-Cacciopoli).
Let (X, d) be a complete metric space, and $F\colon X \to X$ a contraction, i.e. $\exists 0 < \lambda < 1$ such that

$$d(F(x), F(y)) \leq \ \lambda\ d(x, y) \qquad \forall x, y \in X.$$

Then there exists a unique fixed point for F, i.e. $\exists! x_0 \in X$ such that $F(x_0) = x_0$.

Proof. Cf Analysis 2. □

Theorem 4.2 (Brouwer).
Let $K \subset \mathbb{R}^d$ be convex, closed and bounded.
Let $F\colon K \to K$ be a continuous function.

Then F admits a fixed point, i.e $\exists x_0 \in K$ such that $F(x_0) = x_0$.

Theorem 4.3 (Schauder I).
Let X be a real Banach space, $K \subset X$ convex and compact.
Let $F\colon K \to K$ be a continuous function.

Then F admits a fixed point, i.e $\exists x_0 \in K$ such that $F(x_0) = x_0$.

Theorem 4.4 (Schauder II).
Let X be a real Banach space, $A \subset X$ convex, closed and bounded.
Let $F\colon X \to X$ be a function satisfying

- *F is continous and compact (cf Def. 4.7 below),*
- *$F(A) \subset A$.*

Then F admits a fixed point in A, i.e $\exists x_0 \in A$ such that $F(x_0) = x_0$.

Theorem 4.5 (Schaefer).
Let X be a real Banach space and $F\colon X \to X$ a function satisfying

- *F is continous and compact (cf. Def. 4.7 below),*
- *the set $\mathcal{A} := \{x \in X \,|\, x = \lambda F(x) \ \text{ for some } 0 \leq \lambda \leq 1\}$ is bounded.*

Then F admits a fixed point, i.e $\exists x_0 \in X$ such that $F(x_0) = x_0$.

The rest of this section is devoted to prove these fixed point theorems. We start with some remarks and preliminary results.

Remarks.

- Banach-Cacciopoli is the only result that garantees not only existence but also uniqueness of the fixed point.
- Brouwer is a special case of Schauder I, since $K \subset \mathbb{R}^d$ is compact iff K is closed and bounded.
- The advantage of Schaefer is that we do not need to look for a convex subset of X
- The assumptions in Schauder I and II are all important: see the examples below.

Example 1. Let $X = l^2(\mathbb{R}) = \{\{x_j\}_{j=1}^{\infty} |\ \sum_{j=1}^{\infty} x_j^2 < \infty\}$.
We consider $K := \overline{B_1(0)}$ and $F\colon X \to X$ defined as

$$F(x) := \left\{ \frac{1 - \|x\|^2}{2}, x_1, x_2, \dots \right\}.$$

The set K is convex, closed and bounded, but not compact. The function F is well defined, continuous, and satisfies $F(K) \subset K$ but is not compact (exercise).

This function has no fixed point. Indeed, we have $F(x) = x$ iff

$$x_1 = F(x)_1 = \frac{1 - \|x\|^2}{2}, \qquad x_k = F(x)_k = x_{k-1}\ \forall k \geq 2.$$

This holds iff $x_k = x_1\ \forall k \geq 1$, and hence $x_1 = 0$ (since the sequence must be square summable). It follows $0 = x_1 = \frac{1-0}{2} = 1/2$ which gives a contradiction.

Example 2. Let $X = \mathbb{R}^2$, $K := \overline{B_2(0)} \setminus B_1(0)$, and $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$F(x_1, x_2) := (-x_2, x_1).$$

The set K is compact, but not convex. The function F is well defined and continuous (actually it is the rotation by $\pi/2$).

This function has the unique fixed point $x = (0, 0)$ which does not belong to K.

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To prove Brouwer's fixed point theorem we will need the following result from FA.

Lemma 4.6 (projection on a convex and closed subset)**.** *Let $(H, \langle \cdot, \cdot \rangle)$ be a real Hilbert space. Let $A \subset H$ be a closed and convex subset with $A \neq \varnothing$.*
Then $\exists!$ map $P_A\colon H \to A$ such that $\|P_A(x) - x\|_H = \operatorname{dist}(x, A)$. Moreover P_A satisfies

(i) P_A is the unique map such that $\langle x - P_A(x), y - P_A(x) \rangle \leq 0\ \forall y \in A, x \in H$.

(ii) P_A is Lipschitz continuous with Lipschitz constant $L_{P_A} = 1$.

Remark. In general P_A is not a linear function, unless A is a linear subspace of H.

Proof. Existence/uniqueness of P_A, as well as (i) have been seen in Functional Analysis.

(ii) Fix $x_1, x_2 \in H$. It follows from (i) that

$$\langle x_1 - P_A(x_1), y - P_A(x_1) \rangle \leq 0 \qquad \forall y \in A,$$
$$\langle x_2 - P_A(x_2), y - P_A(x_2) \rangle \leq 0 \qquad \forall y \in A.$$

Setting $y = P(x_2)$ in the first inequality and $y = P(x_1)$ in the second, we obtain

$$\langle x_1 - P_A(x_1), P_A(x_2) - P_A(x_1) \rangle \leq 0,$$
$$\langle P_A(x_2) - x_2, P_A(x_2) - P_A(x_1) \rangle \leq 0.$$

Summing the two lines we get

$$\langle x_2 - x_1, P_A(x_2) - P_A(x_1) \rangle \ \geq\ \|P_A(x_2) - P_A(x_1)\|_H^2$$

and hence $\|P_A(x_2) - P_A(x_1)\|_H \leq \|x_2 - x_1\|_H\ \forall x_1, x_2 \in X$.

$\square$

Proof of Brouwer's fixed point theorem 4.2. Let $K \subset \mathbb{R}^d$ be convex, closed and bounded. Let $F\colon K \to K$ a continuous function. Our goal is to show that there exists $x_0 \in X$ such that $F(x_0) = x_0$.

• We can assume $K = \overline{B_R(0)}$. Indeed, suppose $K \neq \overline{B_R(0)}$.

Since K is bounded, $\exists R > 0$ such that $K \subset \overline{B_R(0)}$.
Since K is convex and closed, by Lemma 4.6, $\exists! P_K\colon \mathbb{R}^d \to K$ such that $|P_K(x) - x| = \operatorname{dist}(x, K)$.
We define $\tilde{F}\colon \overline{B_R(0)} \to K \subset \overline{B_R(0)}$ by

$$\tilde{F}(x) := F(P_K(x)).$$

This function is continuous, since both F and P_K are continuous.

Assume the theorem holds for the case $K = \overline{B_R(0)}$. Then $\tilde{F}$ admits a fixed point, i.e. $\exists x_0 \in \overline{B_R(0)}$ such that $\tilde{F}(x_0) = x_0$.
Since $\tilde{F}(\overline{B_R(0)}) \subset K$ we must have $x_0 \in K$. Therefore $P_K(x_0) = x_0$ and $x_0 = \tilde{F}(x_0) = F(P_K(x_0)) = F(x_0)$, hence x_0 is a fixed point for F.

• We can assume $R = 1$. Indeed, suppose $R \neq 1$.

We define $\tilde{F}\colon \overline{B_1(0)} \to \overline{B_1(0)}$ by

$$\tilde{F}(x) := \frac{1}{R} F(Rx).$$

This function is well defined and continuous.

Assume the theorem holds for the case $K = \overline{B_1(0)}$. Then $\tilde{F}$ admits a fixed point, i.e. $\exists x_0 \in \overline{B_1(0)}$ such that $\tilde{F}(x_0) = x_0$. Therefore $Rx_0 = F(Rx_0)$ and hence $x_1 := Rx_0$ is a fixed point for F.

• We can assume $F\colon \mathbb{R}^d \to \overline{B_1(0)}$. Indeed, set $\overline{B} := \overline{B_1(0)}$ and suppose $F\colon \overline{B} \to \overline{B}$. We define $\tilde{F}\colon \mathbb{R}^d \to \overline{B}$ by

$$\tilde{F}(x) := F(P_{\overline{B}}(x)),$$

where $P_{\overline{B}}$ is well defined since $\overline{B}$ is convex and closed. Since F and $P_{\overline{B}}$ are continous, $\tilde{F}$ is continuous.

Assume the theorem holds for the case $F\colon \mathbb{R}^d \to \overline{B}$. Then $\tilde{F}$ admits a fixed point i.e. $\exists x_0 \in \mathbb{R}^d$ such that $\tilde{F}(x_0) = x_0$. Since $\tilde{F}(\mathbb{R}^d) \subset \overline{B}$ it follows $x_0 \in \overline{B}$, $P_{\overline{B}}(x_0) = x_0$ and hence $\tilde{F}(x_0) = F(x_0) = x_0$. Therefore x_0 is a fixed point for F.

• We can assume $F\colon \mathbb{R}^d \to \overline{B}$ is smooth. Indeed, assume F is continuous and define $F_\varepsilon := \rho_\varepsilon * F$, where $\{\rho_\varepsilon\}_{\varepsilon>0}$ is a family of standard mollifiers, i.e. $\rho_\varepsilon := \varepsilon^{-d}\rho(\varepsilon^{-1}x)$, with $\rho \in C_c^\infty(B; [0,\infty))$ and $\int \rho \, dx = 1$.
It holds $F_\varepsilon \in C^\infty(\mathbb{R}^d; \overline{B})$ $\forall \varepsilon$ and $F_\varepsilon \to F$ uniformly on $\overline{B}$.

Assume the theorem holds for the case $F\colon \mathbb{R}^d \to \overline{B}$ smooth function. Then for each $\varepsilon > 0$ $\exists x_\varepsilon \in \overline{B}$ such that $F_\varepsilon(x_\varepsilon) = x_\varepsilon$. The family $\{x_\varepsilon\}_{\varepsilon>0}$ is bounded hence there is a sequence $n \to \varepsilon_n$ with $\varepsilon_n \to 0$ and $x_{\varepsilon_n} \to x \in \overline{B}$. By uniform convergence we get $F_{\varepsilon_n}(x_{\varepsilon_n}) \to F(x)$, and hence $F(x) = x$.

• We assume now $F\colon \mathbb{R}^d \to \overline{B}$ is a smooth function. Our goal is to prove that there exists a point $x_0 \in \overline{B}$ such that $F(x_0) = x_0$.

By contradiction assume $F(x) \neq x$ $\forall x \in \overline{B}$. Then we can construct a function $g\colon \overline{B} \to \partial B$, where $g(x)$ is the unique intersection with ∂B of the half-line starting in $F(x)$ (the starting point is not included) and passing through x. This function has the following properties.

- $g(x) = x$ iff $x \in \partial B$, by construction.
- $g \in C^\infty(\overline{B}; \partial B)$. Indeed $g(x) = F(x) + \lambda(x)[x - F(x)]$, where $\lambda(x)$ is the unique positive solution of

$$|F(x) + \lambda(x)[x - F(x)]|^2 = 1, \qquad \text{i.e.} \qquad \lambda^2 a(x) + 2\lambda b(x) - c(x) = 0,$$

with

$$a(x) := |x - F(x)|^2, \qquad b(x) := F(x) \cdot [x - F(x)], \qquad c(x) := 1 - |F(x)|^2.$$

Note that $a(x) > 0$ on $\overline{B}$, since $x \neq F(x)$ $\forall x \in \overline{B}$, and $c(x) \geq 0$ since $F(\mathbb{R}^d) \subset \overline{B}$. Hence

$$\lambda(x) = \frac{b(x) + \sqrt{b(x)^2 + a(x)c(x)}}{a(x)},$$

It holds $a, b, c \in C^\infty(\overline{B})$ and $b(x)^2 + a(x)c(x) > 0$ on $\overline{B}$ (exercise), therefore $\lambda \in C^\infty(\overline{B}; (0, \infty))$, which implies $g \in C^\infty(\overline{B}; \partial B)$.

The claim below states that no such function exists, hence providing a contradiction.

Claim 1. There exists no function $g \in C^\infty(\overline{B}; \partial B)$ such that $g(x) = x \Leftrightarrow x \in \partial B$.

Proof. We could use topological arguments. Here we use a different proof, using Banach fixed point theorem. By contradiction, assume $\exists g \in C^\infty(\overline{B}; \partial B)$ such that $g(x) = x \Leftrightarrow x \in \partial B$. We will show that g satisfies the following statements:

(a) $\det Dg(x) = 0$ $\forall x \in B$.

(b) g satisfies $\int_{\overline{B}} \det Dg \; dx = |\overline{B}|$.

But these two statements are incompatible, which gives the contradiction.

Proof of (a). Since $g(\overline{B}) \subset \partial B$ we have $|g(x)|^2 = 1$ and hence $Dg(x)^t g(x) = 0$ $\forall x \in B$. It follows that $\ker Dg(x)^t \neq \{0\}$ and hence $\det Dg(x) = 0$ $\forall x \in B$.

Proof of (b). Note that if we had $g(x) = x$, then $Dg = \mathrm{Id}$ and $\int_{\overline{B}} \det Dg \; dx = \int_{\overline{B}} 1 \; dx = |\overline{B}|$. We will show that g is not far from the identity. For this purpose we define, for $t \in [0, 1]$ the map

$$\begin{aligned} \varphi_t \colon \quad & \overline{B} \to \overline{B} \\ & x \mapsto \varphi_t(x) := (1 - t)x + t g(x). \end{aligned}$$

- φ_t is well defined since $|\varphi_t(x)| \leq (1 - t)|x| + t|g(x)| \leq 1 - t + t = 1$. Morever $\varphi_0(x) = x$ and $\varphi_1(x) = g(x)$, hence φ_t interpolates between x and $g(x)$.
- $\varphi_t \in C^\infty(\overline{B}; \overline{B})$, since both g and the identity are smooth. Moreover

$$\det D\varphi_t = \det[\mathrm{Id} + t(Dg - \mathrm{Id})] = 1 + a_1(x)t + \cdots + a_d(x)t^d,$$

where $a_j \in C^\infty(\overline{B}; \mathbb{R})$, therefore

$$\int_{\overline{B}} \det D\varphi_t \; dx = |\overline{B}| + \sum_{j=1}^{d} t^j \alpha_j, \qquad \text{with } \alpha_j := \int_{\overline{B}} a_j \; dx.$$

• $\det D\varphi_0 = 1$ and $t \mapsto \det D\varphi_t$ is continuous, hence $\exists t_1 > 0$ such that $\det D\varphi_t > 0$ $\forall t \in [0, t_1]$. Therefore

$$\int_{\overline{B}} |\det D\varphi_t| \, dx = \int_{\overline{B}} \det D\varphi_t \, dx = |\overline{B}| + \sum_{j=1}^{d} t^j \alpha_j, \qquad \forall t \in [0, t_1].$$

Claim 2. $\exists t_2 > 0$ such that φ_t is bijective $\forall t \in [0, t_2]$.

Consequence. Set $\delta := \min\{t_1, t_2\}$. Then, $\forall t \in [0, \delta]$ we have

$$|\overline{B}| + \sum_{j=1}^{d} t^j \alpha_j = \int_{\overline{B}} \det D\varphi_t \, dx = \int_{\overline{B}} |\det D\varphi_t| \, dx = \int_{\varphi_t(\overline{B})} \, dx = |\varphi_t(\overline{B})| = |\overline{B}|,$$

where we used $\varphi_t(\overline{B}) = \overline{B}$ by surjectivity. Hence $\alpha_j = 0$ $\forall j = 1, \ldots, d$ and

$$\int_{\overline{B}} \det D\varphi_t \, dx = |\overline{B}|, \qquad \forall t \in [0, 1].$$

Proof of Claim 2. Here we use Banach fixed point theorem. Fix some $t < 1$, $y \in \overline{B}$. Our goal is to show that, if $t \le t_2$ there exists a unique $x \in \overline{B}$ such that $\varphi_t(x) = y$.

We have $\varphi_t(x) = y$ iff $(1-t)x + tg(x) = y$ iff $x = \psi(x)$, where

$$\begin{aligned} \psi\colon \quad & \overline{B} \to \mathbb{R}^d \\ & x \mapsto \psi(x) := \tfrac{1}{1-t} y - \tfrac{t}{1-t} g(x). \end{aligned}$$

To apply Banach fixed point theorem, we extend this function to $\tilde{\psi}\colon \mathbb{R}^d \to \mathbb{R}^d$ defined by $\tilde{\psi}(x) := \psi(P(x))$, where $P = P_{\overline{B}}$.
This function is a contraction if t_2 is small enough. Indeed

$$\begin{aligned} |\tilde{\psi}(x) - \tilde{\psi}(x')| &= \frac{t}{1-t} |g(P(x)) - g(P(x'))| \le \frac{t}{1-t} \|Dg\|_{L^\infty(\overline{B})} |P(x) - P(x')| \\ &\le \frac{t}{1-t} \|Dg\|_{L^\infty(\overline{B})} |x - x'| \le \frac{t_2}{1-t_2} \|Dg\|_{L^\infty(\overline{B})} |x - x'| = \theta |x - x'|, \end{aligned}$$

where $\theta < 1$ if t_2 is small enough. It follows that $\exists ! x \in \mathbb{R}^d$ such that $\tilde{\psi}(x) = x$. We distinguish now two cases.

- If $x \in \overline{B}$, then $P(x) = x$ and hence $x = \psi(P(x)) = \psi(x)$.
- If $x \in \overline{B}^c$, then $P(x) \in \partial B$ and hence $g(P(x)) = P(x)$, by the assumptions on g. It follows

  $$y = (1-t)x + tg(P(x)) = (1-t)x + tP(x) \notin \overline{B},$$

  since $x \notin \overline{B}$ and $P(x) \in \partial B$. This contradicts $y \in \overline{B}$. Hence this case cannot occur.

This completes the proof of Claim 2 and of the theorem.

□

[21: 18.12.2023]

[22: 21.12.2021]

Proof of Schauder I, Theorem 4.3. Let X be a real Banach space, $K \subset X$ convex and compact, $F\colon K \to K$ a continuous function. Our goal is to show that $\exists x_0 \in K$ such that $F(x_0) = x_0$.

Claim. $\forall \varepsilon > 0$ $\exists K_\epsilon \subset K$ and $F_\varepsilon\colon K \to K$ such that

- K_ε convex, bounded and closed, and $K_\varepsilon \subset V_\varepsilon$ where V_ε is a finite dimensional linear subspace of X,
- F_ε is continuous, $F_\varepsilon(K_\varepsilon) \subset K_\varepsilon$ and $\sup_{x\in K} \|F_\varepsilon(x) - F(x)\| \le \varepsilon$.

Consequence. By Brouwer's fixed point theorem, there exists a point $x_\varepsilon \in K_\varepsilon$ such that $F_\varepsilon(x_\varepsilon) = x_\varepsilon$. Since K is compact, we can find $x \in K$ and a sequence $n \to \varepsilon_n$ such that $\varepsilon_n \to 0$ and $x_{\varepsilon_n} \to x$. We claim that x is a fixed point for F. Indeed we argue

$$\|x - F(x)\| \le \|x - x_{\varepsilon_n}\| + \|x_{\varepsilon_n} - F(x_{\varepsilon_n})\| + \|F(x_{\varepsilon_n}) - F(x)\|.$$

Since $x_{\varepsilon_n} \to x$ and F is continuous we have $\|x - x_{\varepsilon_n}\| \to 0$ and $\|F(x_{\varepsilon_n}) - F(x)\| \to 0$. Finally, using $F_{\varepsilon_n}(x_{\varepsilon_n}) = x_{\varepsilon_n}$ we argue

$$\|x_{\varepsilon_n} - F(x_{\varepsilon_n})\| = \|F_{\varepsilon_n}(x_{\varepsilon_n}) - F(x_{\varepsilon_n})\| \le \sup_{x\in K} \|F_{\varepsilon_n}(x) - F(x)\| \le \varepsilon_n \to 0,$$

and hence $F(x) = x$.

Proof of the Claim.
• We construct K_ε. Since K is compact $\forall \varepsilon > 0$ $\exists N_\varepsilon \ge 1$, $x_1^\varepsilon, \dots x_{N_\varepsilon}^\varepsilon \in K$ such that

$$K \subset \bigcup_{j=1}^{N_\varepsilon} B_\varepsilon(x_j^\varepsilon).$$

We define $V_\varepsilon := \operatorname{span}\{x_1^\varepsilon, \dots x_{N_\varepsilon}^\varepsilon\}$ and

$$K_\varepsilon := \operatorname{conv}\{x_1^\varepsilon, \dots x_{N_\varepsilon}^\varepsilon\} = \{y = \sum_{j=1}^{N_\varepsilon} \lambda_j x_j^\varepsilon |\ \lambda_j \ge 0, \sum_{j=1}^{N_\varepsilon} \lambda_j = 1\}.$$

It holds $\dim V_\varepsilon \le N_\varepsilon < \infty$. Moreover $K_\varepsilon \subset K$, since K is convex, $K_\varepsilon \subset V_\varepsilon$ by construction, K_ε is closed (exercise) and bounded since $|y| \le \sum_{j=1}^{N_\varepsilon} \lambda_j |x_j^\varepsilon| \le \max_j |x_j^\varepsilon|$ $\forall y \in K_\varepsilon$.

• We construct F_ε. The easiest choice $F_\varepsilon := F$ does not work since $F(K_\varepsilon) \not\subset K_\varepsilon$ in general.

It is enough to find a continuous function $J_\varepsilon\colon K \to K_\varepsilon$ such that

$$\sup_{x\in K} \|J_\varepsilon(x) - x\| \le \varepsilon.$$

Indeed, given such a function, we define $F_\varepsilon(x) := J_\varepsilon(F(x))$. Then F_ε is continuous, $F_\varepsilon(K_\varepsilon) \subset K_\varepsilon$ and

$$\sup_{x\in K} \|F_\varepsilon(x) - F(x)\| = \sup_{x\in K} \|J_\varepsilon(F(x)) - F(x)\| \le \sup_{x\in K} \|J_\varepsilon(x) - x\| \le \varepsilon.$$

To contruct J_ε it is enough to find N_ε continuous functions $\lambda_j\colon K \to [0, 1]$ such that:
$\sum_{j=1}^{N_\varepsilon} \lambda_j(x) = 1$ $\forall x \in K$ and $\lambda_j(x) = 0$ $\forall x \notin B_\varepsilon(x_j^\varepsilon)$.

Indeed, set $J_\varepsilon(x) := \sum_{j=1}^{N_\varepsilon} \lambda_j(x) x_j^\varepsilon$.

By construction, $J_\varepsilon(x) \in K_\varepsilon \ \forall x \in K$. Moreover J_ε is continous and

$$\|J_\varepsilon(x) - x\| = \|\sum_{j=1}^{N_\varepsilon} \lambda_j(x) \ (x_j^\varepsilon - x)\| \leq \sum_{j=1}^{N_\varepsilon} \lambda_j(x) \|x_j^\varepsilon - x\| \leq \varepsilon \sum_{j=1}^{N_\varepsilon} \lambda_j(x) = \varepsilon,$$

where we used $\lambda_j(x) = 0$ if $\|x_j^\varepsilon - x\| > \varepsilon$.

To construct the functions λ_j we argue as follows. Consider $a_j(x) := \operatorname{dist}(x, B_\varepsilon(x_j^\varepsilon)^c)$. This function is continuos and satisfies

$$a_j(x) = 0 \quad \forall x \notin B_\varepsilon(x_j^\varepsilon), \qquad 0 < a_j(x) \leq \varepsilon \quad \forall x \in B_\varepsilon(x_j^\varepsilon), \qquad a_j(x_j^\varepsilon) = \varepsilon.$$

We define now

$$\lambda_j(x) := \frac{a_j(x)}{\sum_{k=1}^{N_\varepsilon} a_k(x)}.$$

This function is well defined, since $\forall x \in K \ \exists j$ such that $x \in B_\varepsilon(x_j^\varepsilon)$, and hence $\sum_{k=1}^{N_\varepsilon} a_k(x) > 0$. Moreover λ_j has all the required properties (exercise). This concludes the proof of the claim and of the theorem.

$\square$

To prove the second version of Schauder's theorem, we need some preliminary definitions and results.

Definition 4.7. *Let X be a Banach space, $F \colon X \to X$ a map.*

F is compact if $F(B)$ is precompact $\forall B \subset X$ bounded set.

Reminders from Functional Analysis. Let X be a Banach space, $A \subset X$ a subset.

- A is precompact if $\forall \varepsilon > 0 \ \exists N \geq 1$ and $x_1, \ldots, x_N \in A$ such that

$$A \subset \bigcup_{j=1}^{N} B_\varepsilon(x_j) = \{x_1, \ldots, x_N\} + B_\varepsilon(0),$$

  i.e. $\forall x \in A \ \exists j \in \{1, \ldots, N\}$ and $y \in B_\varepsilon(0)$ such that $x = x_j + y$.

- A is precompact $\Leftrightarrow \overline{A}$ is compact.

- The convex hull of A is the set

$$\operatorname{conv} A := \left\{ y = \sum_{j=1}^{n} \lambda_j y_j | \ n \geq 1, \ \lambda_j \geq 0, \sum_{j=1}^{n} \lambda_j = 1, \ y_1, \ldots, y_n \in A \right\}.$$

Lemma 4.8. *Let X be a Banach space, $A \subset X$ a subset.*

A is precompact $\Rightarrow$ $\operatorname{conv} A$ is precompact.

Proof.
• A is precompact, then $\forall \varepsilon > 0 \ \exists N \geq 1$ and $x_1, \ldots, x_N \in A$ such that $A \subset \{x_1, \ldots, x_N\} + B_\varepsilon(0)$. We show that $\operatorname{conv} A \subset \operatorname{conv}\{x_1, \ldots, x_N\} + B_\varepsilon(0)$ holds.

Indeed, set $z \in \operatorname{conv} A$. By definition $\exists k \geq 1$, $z_1, \dots, z_k \in A$, $\lambda_1, \dots, \lambda_k \geq 0$ with $\sum_{l=1}^k \lambda_l = 1$ such that $z = \sum_{l=1}^k \lambda_l z_l$.
Since $z_l \in A$ we can write $z_l = x_{j_l} + y_l$, for some $j_l \in \{x_1, \dots, x_N\}$ and $y_l \in B_\varepsilon(0)$. It follows

$$z = \sum_{l=1}^k \lambda_l z_l = \sum_{l=1}^k \lambda_l x_{j_l} + \sum_{l=1}^k \lambda_l y_l \in \operatorname{conv}\{x_1, \dots, x_N\} + B_\varepsilon(0),$$

and hence $\operatorname{conv} A \subset \operatorname{conv}\{x_1, \dots, x_N\} + B_\varepsilon(0)$.

• We show that $\operatorname{conv} A$ is precompact.
Indeed $\operatorname{conv}\{x_1, \dots, x_N\}$ is compact, since it is bounded, closed and

$$\operatorname{conv}\{x_1, \dots, x_N\} \subset \operatorname{span}\{x_1, \dots, x_N\}, \qquad \text{with } \dim \operatorname{span}\{x_1, \dots, x_N\} \leq N.$$

Therefore $\exists m \geq 1$, $\tilde{x}_1, \dots, \tilde{x}_m \in \operatorname{conv}\{x_1, \dots, x_N\}$, such that

$$\operatorname{conv}\{x_1, \dots, x_N\} \subset \{\tilde{x}_1, \dots, \tilde{x}_N\} + B_\varepsilon(0).$$

It follows

$$\operatorname{conv} A \subset \operatorname{conv}\{x_1, \dots, x_N\} + B_\varepsilon(0) \subset \{\tilde{x}_1, \dots, \tilde{x}_N\} + B_{2\varepsilon}(0).$$

This conclusion the proof of the lemma.

□

Proof of Schauder II, Theorem 4.4. Let X be a real Banach space, $A \subset X$ convex, closed and bounded, $F \colon X \to X$ satisfying

- F is continous and compact.
- $F(A) \subset A$.

Our goal is to show that F admits a fixed point in A.

The idea is to reduce to Schauder I. Indeed it is enough to find $K \subset A$ such that K is convex and compact, and $F(K) \subset K$. Then, by Schauder's theorem I, $\exists x \in K \subset A$ such that $F(x) = x$.

To construct K we proceed as follows. Since F is compact and A is bounded, $F(A)$ is precompact and hence, by Lemma 4.8, $\operatorname{conv} F(A)$ is precompact and convex. We define

$$K := \overline{\operatorname{conv} F(A)}.$$

Then K is convex and compact. We show now: $K \subset A$ and $F(K) \subset K$.

Since $F(A) \subset A$ and A is convex it follows that $\operatorname{conv} F(A) \subset A$. Since A is closed we get

$$K = \overline{\operatorname{conv} F(A)} \subset \overline{A} = A,$$

and hence $K \subset A$. Finally, since $K \subset A$, we have $F(K) \subset F(A) \subset \operatorname{conv} F(A) \subset K$. □

Proof of Schaefer's fixed point theorem 4.5. Let X be a real Banach space, $F\colon X \to X$ satisfy

- F is continous and compact,
- the set $\mathcal{A} := \{x \in X \,|\, x = \lambda F(x) \;$ for some $0 \le \lambda \le 1\}$ is bounded.

Our goal is to prove that F admits a fixed point. For this purpose we reduce to Schauder II. A natural candidate for the set K is $\overline{B_M(0)}$, since this set is convex, bounded and closed. But in general $F(\overline{B_M(0)}) \not\subset \overline{B_M(0)}$. We consider instead a regularized version of F constructed as follows.

Since the family $\mathcal{A}$ is bounded, we have $M_0 := \sup_{x\in\mathcal{A}} \|x\| < \infty$. Set now $M > M_0$. We define the function $F_M\colon X \to X$ as $F_M := T_M(F(x))$ where $T_M\colon X \to X$ is given by

$$T_M(x) := \begin{cases} x & \text{if } \|x\| \le M \\ \frac{M}{\|x\|}x & \text{if } \|x\| > M. \end{cases}$$

i.e.

$$F_M(x) := \begin{cases} F(x) & \text{if } \|F(x)\| \le M \\ \lambda_x F(x) & \text{if } \|x\| > M. \end{cases} \qquad \text{where } \lambda_x := \frac{M}{\|F(x)\|} \in (0,1).$$

• SInce T is continuous (exercise) and F is continuous by assumption, F_M is continuous too.

• $F_M(X) \subset \overline{B_M(0)}$ by construction, and hence $F_M(\overline{B_M(0)}) \subset \overline{B_M(0)}$.

• F_M is compact. Indeed let $n \mapsto x_n \in X$ be a bounded sequence in X. Since the function F is compact there exists a subsequence $j \mapsto x_{n_j}$ and a point $y \in X$ such that $F(x_{n_j}) \to y$. Since T is continuous we have $F_M(x_{n_j}) = T(F(x_{n_j})) \to T(y)$.

Therefore, by Shauder's theorem 4.4, $\exists x_0 \in \overline{B_M(0)}$ such that $F_M(x_0) = x_0$.
We distinguish now two cases

If $\|F(x_0)\| \le M$ then $F(x_0) = F_M(x_0) = x_0$.

If $\|F(x_0)\| > M$ then $\lambda_{x_0} F(x_0) = F_M(x_0) = x_0$, where $\lambda_{x_0} = M/\|F(x_0)\| < 1$. Therefore $x_0 \in \mathcal{A}$ and hence $\|x_0\| \le M_0 < M$. Using again $F(x_0) = \lambda_{x_0}^{-1} x_0$, we argue

$$\|F(x_0)\| = \frac{1}{\lambda_{x_0}}\|x_0\| = \|F(x_0)\| \frac{\|x_0\|}{M} < \|F(x_0)\|$$

which gives a contradiction. Therefore this case cannot occur.

□

The fixed point theorems above can be used to prove existence (thought not uniqueness) of several nonlinear PDEs. Three examples (see Sheet 11) are

- $-\Delta u(x) = f(x, u(x))$,
- $-\mathrm{div}\,(a(u)Du) = f$ where $a : \mathbb{R} \to \mathbb{R}$ is a scalar function,
- $-\Delta u(x) + \mu u(x) = f(x, Du(x))$.

In all these cases the main trick is to construct a function $F : X \to X$, where $X = L^2(\Omega)$ or $H_0^1(\Omega)$ where $v = Fu$ is defined as the unique weak solution of the linear PDE $-\Delta v(x) = f(x, u(x))$, (resp. $-\mathrm{div}\,(a(u)Dv) = f$, $-\Delta v(x) + \mu v(x) = f(x, Du(x))$).
To prove continuity of the function F we need some continuity of the operator $u \to f(x, u(x))$. This is the content of the next theorem.

Lemma 4.9 (Nemitski composition theorem)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Set $1 \le p, q < \infty$, and $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ a function with the following properties.*

- *f is a Carathéodory function i.e. $x \to f(x,s)$ is measurable $\forall s \in \mathbb{R}$ and $s \to f(x,s)$ continuous for a.e. $x \in \Omega$.*
- *$\exists b \ge 0$, a function $g \in L^q(\Omega; [0,\infty))$ and $0 < \beta \le \frac{p}{q}$ such that*

$$|f(x,s)| \le g(x) + b|s|^\beta \qquad \forall s \in \mathbb{R},\ \textit{for a.e. } x \in \Omega.$$

Then the Nemitski operator

$$F\colon \quad \begin{aligned} &L^p(\Omega) \to L^q(\Omega) \\ &u \mapsto F(u)(x) := f(x, u(x)) \end{aligned}$$

is well defined and continuous.

Proof. Sheet 11. □

[22: 21.12.2021]

[23: 08.01.2024]

5 Quasilinear elliptic PDEs

Let $\Omega \subset \mathbb{R}^d$ be open and bounded. A quasilinear PDE of second order has the (non-divergence) form

$$-\operatorname{Tr}\left[M(x,u,Du)D^2u\right] = f(x,u,Du) \tag{5.1}$$

where

$$M\colon \begin{array}{ll} \Omega \times \mathbb{R} \times \mathbb{R}^d & \to \mathbb{R}^{d\times d}_{sym} \\ (x,s,p) & \mapsto M(x,s,p) \end{array}, \qquad f\colon \begin{array}{ll} \Omega \times \mathbb{R} \times \mathbb{R}^d & \to \mathbb{R} \\ (x,s,p) & \mapsto f(x,s,p). \end{array}$$

To write the corresponding divergence formulation, which is more practical to study weak solutions, we need first some notation.

Consider a function $F\colon \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^n$. We will denote by $\partial/\partial x_j F$ the derivative of the function $x \mapsto F(x,s,p)$ and by $d/dx_j F$ the derivative of the function $x \mapsto F(x, u(x), Du(x))$.

For $a\colon \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, differentiable we define

$$\begin{aligned}
\operatorname{div} a(x,u,Du) :&= \sum_{j=1}^d \frac{d}{dx_j} a_j(x,u,Du) \\
&= \sum_{j=1}^d \left[\partial_{x_j} a_j(x,u,Du) + \partial_s a_j(x,u,Du)\partial_{x_j} u + \sum_{l=1}^d \partial_{p_l} a_j(x,u,Du)\partial^2_{x_l x_j} u\right] \\
&= \operatorname{Tr}\left[A(x,u,Du)D^2u\right] + B(x,u,Du) = \operatorname{Tr}\left[(\operatorname{Re}A)D^2u\right] + B,
\end{aligned}$$

where

$$\begin{aligned}
A\colon \quad & \begin{array}{ll} \Omega \times \mathbb{R} \times \mathbb{R}^d & \to \mathbb{R}^{d\times d} \\ (x,s,p) & \mapsto A_{ij}(x,s,p) := \partial_{p_i} a_j(x,s,p) \end{array}, \\
B\colon \quad & \begin{array}{ll} \Omega \times \mathbb{R} \times \mathbb{R}^d & \to \mathbb{R} \\ (x,s,p) & \mapsto B(x,s,p) := \sum_j \left[\partial_{x_j} a_j(x,u,Du) + \partial_s a_j(x,u,Du)\partial_{x_j} u\right], \end{array}
\end{aligned}$$

and we used $\mathrm{Re}A = \frac{1}{2}(A + A^*) = \frac{1}{2}(A + A^t)$ and $(D^2u)^t = (D^2u)$.
Hence, u is a solution of $-\mathrm{div}\, a(x, u, Du) = f(x, u, Du)$ iff u is a solution of $-\mathrm{Tr}\,[M(x, u, Du)D^2u] = \tilde{f}(x, u, Du)$ with $M := \mathrm{Re}A$ and $\tilde{f}(x, u, Du) := f(x, u, Du) + B(x, u, Du)$.

Note that, performing integration by parts directly in (5.1) we do not obtain an expression of the form $\mathrm{div}\, a(x, u, Du) + B(x, u, Du)$ in general:

$$\begin{aligned} -\mathrm{Tr}\,[M(x, u, Du)D^2u] &= -\mathrm{div}\,(M(x, u, Du)Du) + \sum_{k=1}^{d} \partial_k u \;\mathrm{div}\, M_{\cdot,k} \\ &= -\mathrm{div}\,(M(x, u, Du)Du) + \sum_{k=1}^{d} \partial_k u \sum_{jl=1}^{d} \partial_{p_l} M(x, u, Du)_{jk}(D^2u)_{jl} + \tilde{B}(x, u, Du). \end{aligned}$$

In the following we consider the boundary value problem:

$$\begin{cases} -\mathrm{div}\, a(x, u, Du) = f(x, u, Du) & \text{in } \Omega \\ u_{|\partial\Omega} = 0 \end{cases} \tag{5.2}$$

where $a\colon \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ and $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ are nonlinear functions. If a is differentiable in the third variable we define $A_{jl}(x, s, p) := \partial_{p_l} a_j(x, s, p)$.

5.1 Ellipticity and weak formulation

Definition 5.1 (ellipticity version I)**.** *Assume the function $a\colon \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is C^1 in the third variable. In particular $A\colon \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is well defined. We say that the (formal) differential operator $u \mapsto -\mathrm{div}\, a(x, u, Du)$ is*

- *elliptic, if $ReA(x, s, p) \geq 0$ as a quadratic form $\forall (s, p) \in \mathbb{R} \times \mathbb{R}^d$ and a.e. $x \in \Omega$,*
- *uniformly elliptic, if $ReA(x, s, p) \geq \theta\mathrm{Id}$, for some $\theta > 0$, $\forall (s, p) \in \mathbb{R} \times \mathbb{R}^d$ and a.e. $x \in \Omega$.*

The next lemma summarizes some important properties of vector fields. Since A is constructed from the derivative of a in p, we can neglect the x, s dependence for the moment.

Lemma 5.2 (properties of vector fields)**.** *Fix $n \in \mathbb{N}$, $n \geq 1$.*

(i) Consider a function $F\colon \mathbb{R}^n \to \mathbb{R}$, C^2 and convex. Set $a(p) := DF(p)$ and $A_{ij} := \partial_i a_j$.
Then A is well defined, $A^t = A$ and $A \geq 0$.

(ii) Assume the vector field $a\colon \mathbb{R}^n \to \mathbb{R}^n$ is differentiable and monotone i.e.

$$[a(p) - a(q)] \cdot (p - q) \geq 0 \qquad \forall p, q \in \mathbb{R}^d.$$

Set $A_{ij} := \partial_i a_j$.
Then A is well defined and $ReA \geq 0$.

(iii) (zeros of a vector field) Let $v\colon \mathbb{R}^n \to \mathbb{R}^n$ be a vector field satisfying

(a) v is continous and
(b) $\exists R > 0$ such that $v(x) \cdot x \geq 0$ $\forall x \in \partial B_R(0)$.

Then $\exists x_0 \in \overline{B_R(0)}$ such that $v(x_0) = 0$.

Proof.
(i) Since $p \mapsto F(x,s,p)$ is C^2, the hessian matrix $H_{ij} := \partial_{p_i}\partial_{p_j} F = \partial_{p_j}\partial_{p_i} F$ is well defined and symmetric. Hence $A_{ij} = \partial_{p_i} a_j = H_{ij}$ is well defined and symmetric.
Since $p \mapsto F(p)$ is convex we have $H = A \geq 0$.

(ii) Set $p := q + h\xi$, with $h \in \mathbb{R} \setminus 0$ and $\xi \in \mathbb{R}^d$. Then $p - q = h\xi$ and monotonicity gives

$$0 \leq [a(p) - a(q)] \cdot (p - q) = h[a(q + h\xi) - a(q)] \cdot \xi,$$

hence, since $h^2 > 0$,

$$\frac{1}{h^2} h[a(x,s,q + h\xi) - a(x,s,q)] \cdot \xi \ = \ \frac{1}{h}[a(x,s,q + h\xi) - a(x,s,q)] \cdot \xi \geq 0 \qquad \forall h \in \mathbb{R}.$$

Taking the limit $h \to 0$ we get

$$0 \leq \sum_{jl} \partial_{p_l} a_j(x,s,q) \xi_l \xi_j = \xi \cdot A\xi = \xi \cdot \mathrm{Re} A\xi.$$

The result follows since ξ is arbitrary.

(iii) By contradiction assume $v(x) \neq 0$ $\forall x \in \overline{B_R(0)}$. We define

$$\begin{aligned} w\colon \quad \overline{B_R(0)} \quad & \to \partial B_R(0) \subset \overline{B_R(0)} \\ x \quad & \mapsto w(x) := -\tfrac{R}{|v(x)|} v(x). \end{aligned}$$

Since v is continous and $v \neq 0$ the function w is also continous. Moreover $\overline{B_R(0)} \subset \mathbb{R}^n$ is convex, bounded and closed. By Brouwer's fixed point theorem, it follows $\exists x_0 \in \overline{B_R(0)}$ such that

$$x_0 = w(x_0) = -\frac{R}{|v(x_0)|} v(x_0), \qquad \text{hence} \quad v(x_0) = -\frac{|v(x_0)|}{R} x_0.$$

Since $w(\overline{B_R(0)}) \subset \partial B_R(0)$, we have $x_0 \in \partial B_R(0)$, and hence, since $v(x_0) \neq 0$,

$$0 \leq x_0 \cdot v(x_0) = -\frac{|v(x_0)|}{R} |x_0|^2 = -|v(x_0)| R < 0,$$

which gives a contradiction. $\square$

We can use monotonicity to define ellipticity also when a is not differentiable

Definition 5.3 (monotone vector field)**.** *A vector field $v\colon \mathbb{R}^n \to \mathbb{R}^n$ is*

(i) monotone, if $[v(p) - v(q)] \cdot (p - q) \geq 0$ $\forall p, q \in \mathbb{R}^n$,

(ii) strictly monotone if $[v(p) - v(q)] \cdot (p - q) > 0$ $\forall p \neq q \in \mathbb{R}^n$,

(iii) uniformly monotone if $\exists \theta > 0$ such that $[v(p) - v(q)] \cdot (p - q) \geq \theta |p - q|^2$ $\forall p, q \in \mathbb{R}^n$.

Definition 5.4 (ellipticity version II)**.** *Let $a\colon \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ be a given function. The (formal) differential operator $u \mapsto -\operatorname{div} a(x, u, Du)$ is*

- *elliptic if $p \mapsto a(x,s,p)$ is a monotone vector field $\forall s \in \mathbb{R}$ and a.e. $x \in \Omega$.*
- *strictly elliptic if $p \mapsto a(x,s,p)$ is a striclty monotone vector field $\forall s \in \mathbb{R}$ and a.e. $x \in \Omega$.*
- *uniformly elliptic if $p \mapsto a(x,s,p)$ is a uniformly monotone vector field $\forall s \in \mathbb{R}$ and a.e. $x \in \Omega$, with constant θ independent of (x,s).*

Example Assume $\exists F\colon \Omega\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$, such that F is differentiable and convex in the third variable and $a = \partial_p F$. Then $p \mapsto a(x,s,p)$ is a monotone vector field and the (formal) differential operator $u \mapsto -\mathrm{div}\, a(x,u,Du)$ is elliptic (exercise).

Definition 5.5 (weak formulation)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and bounded, $a\colon \Omega\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$ and $f\colon \Omega\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ two functions. We consider the (formal) PDE*

$$\begin{cases} -\mathrm{div}\, a(x,u,Du) = f(x,u,Du) & \text{in } \Omega \\ u_{|\partial\Omega} = 0 \end{cases} \tag{5.3}$$

A function $u \in H_0^1(\Omega)$ is a weak solution of (5.3) if

$$\int_\Omega Dv\cdot a(x,u,Du)\,dx = \int_\Omega v f(x,u,Du)\,dx \qquad \forall v \in H_0^1(\Omega),$$

provided the integrals above are well defined and finite $\forall u, v \in H_0^1(\Omega)$.

Regularity requirements.

• Consider first $\int_\Omega Dv\cdot a(x,u,Du)\,dx$. Since $Dv \in L^2(\Omega)$ we need $x \mapsto a(x,u(x),Du(x)) \in L^2(\Omega)$ $\forall u \in H_0^1(\Omega)$. Assume the function is measurable in all variables and $\exists C, \alpha, \beta \ge 0$ such that

$$|a(x,s,p)| \le C[1 + |s|^\alpha + |p|^\beta] \qquad \forall (x,s,p).$$

Therefore we need $u^{2\alpha} \in L^1(\Omega)$ and $Du^{2\beta} \in L^1(\Omega)$ $\forall u \in H_0^1(\Omega)$. Since $Du \in L^2(\Omega)$ we must have $\beta \le 1$. On the contrary, any $\alpha < \infty$ works for $d = 1,2$ while for $d \ge 3$ we need $2\alpha \le 2^* = \frac{2d}{d-2}$.

• Consider now $\int_\Omega v f(x,u,Du)\,dx$. Since $v \in H_0^1(\Omega)$ we have $v \in L^q(\Omega)$ $\forall 1 \le q < \infty$ in $d = 1,2$ and $v \in L^{2^*}(\Omega)$ in $d \ge 3$. Therefore we need $x \mapsto a(x,u(x),Du(x)) \in L^m(\Omega)$ with $m > 1$ for $d = 1,2$ and $m = (2^*)' = \frac{2d}{d+2}$ for $d \ge 3$.
Assume f is measurable in all variables and $\exists C', \alpha', \beta' \ge 0$ such that

$$|f(x,s,p)| \le C'[1 + |s|^{\alpha'} + |p|^{\beta'}] \qquad \forall (x,s,p).$$

Arguing as above we need: $\beta' \le \frac{2}{m} < 2$ for $d \ge 1$, $0 < \alpha' < \infty$ for $d = 1,2$ and $\alpha' \le \frac{2^*}{(2^*)'} = \frac{d+2}{d-2}$ for $d \ge 3$ (exercise).

• In the following we will assume $|a(x,s,p)| \le C[1 + |s| + |p|]$ and $|f(x,s,p)| \le |g(x)|$ where $g \in L^2(\Omega)$. These assumptions garantee the weak formulation is well defined in any dimension.

5.2 Monotonicity and existence of weak solutions

Theorem 5.6 (Leray-Lions)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and bounded.*
Let $a\colon \Omega\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$ and $f\colon \Omega\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ satisfy the following properties.

(i) Both a and f are Carathéodory functions

[i.e. they are continuous in (s,p) for a.e. $x \in \Omega$ and they are measurable in x $\forall (s,p) \in \mathbb{R}\times\mathbb{R}^d$.]

(ii) $\exists C > 0$ and $g \in L^2(\Omega)$ such that

$$|a(x,s,p)| \le C\,[1+|s|+|p|], \qquad |f(x,s,p)| \le |g(x)| \qquad \forall (s,p) \in \mathbb{R}\times\mathbb{R}^d \qquad \text{for a.e. } x \in \Omega.$$

(iii) a is strictly monotone in the third variable i.e

$$[a(x,s,p)-a(x,s,q)]\cdot(p-q)>0 \qquad \forall s\in\mathbb{R}, p\neq q\in\mathbb{R}^d, \text{ for a.e. } x\in\Omega.$$

(iv) (coercivity) $\exists\beta>0$ *such that*

$$a(x,s,p)\cdot p\geq\beta|p|^2 \qquad \forall s\in\mathbb{R}, p\in\mathbb{R}^d, \text{ for a.e. } x\in\Omega.$$

Then there exists a function $u\in H_0^1(\Omega)$ *weak solution of* (5.3).

Remarks.

- Condition *(ii)* garantees the weak formulation is well defined.
- The weak solution is not unique in general but we will see that if we replace strict monotonicity with uniform monotonicity, the solution is unique.

We will not see the general proof of Leray-Lions theorem (too long). Instead we will see the proof in the simpler case, stated in the theorem below.

Theorem 5.7. *Let* $\Omega\subset\mathbb{R}^d$ *be open and bounded.*
Let $a\colon\mathbb{R}^d\to\mathbb{R}^d$ *and* $f\colon\Omega\to\mathbb{R}$ *satisfy the following properties.*

(i) a is continuous and f is measurable.

(ii) $f\in L^2(\Omega)$ *and* $\exists C>0$ *such that* $|a(p)|\leq C\,[1+|p|]$, $\forall p\in\mathbb{R}^d$.

(iii) a is a monotone vector field i.e $[a(p)-a(q)]\cdot(p-q)\geq 0\ \forall p,q\in\mathbb{R}^d$.

(iv) $\exists\beta>0,\gamma\geq 0$ *such that* $a(p)\cdot p\geq\beta|p|^2-\gamma\ \forall p\in\mathbb{R}^d$.

Then there exists a function $u\in H_0^1(\Omega)$ *weak solution of*

$$\begin{cases} -\operatorname{div} a(Du)(x)=f(x) & \text{in } \Omega \\ u_{|\partial\Omega}=0. \end{cases} \tag{5.4}$$

Moreover, if a is uniformly monotone, then the weak solution is also unique.

Remark. Note that in *(iii)* and *(iv)* we need weaker conditions than in Leray-Lions.

Proof of Thm 5.7: unicity. Assume a is uniformly monotone, i.e $[a(p)-a(q)]\cdot(p-q)\geq\theta|p-q|^2$ $\forall p,q\in\mathbb{R}^d$, with $\theta>0$. Our goal is to show that the weak solution, in case it exists, is unique. By contradiction, let $u_1,u_2\in H_0^1(\Omega)$ be two weak solutions. Since $f=f(x)$ is independent of s,p, we have

$$\int_\Omega Dv\cdot a(Du_1)\,dx=\int_\Omega v\ f(x)\,dx=\int_\Omega Dv\cdot a(Du_2)\,dx \qquad \forall v\in H_0^1(\Omega),$$

and hence $\int_\Omega[a(Du_1)-a(Du_2)]\cdot Dv\ dx=0\ \ \forall v\in H_0^1(\Omega)$. Setting $v=u_1-u_2$ we obtain, by uniform monotonicity,

$$0=\int_\Omega[a(Du_1)-a(Du_2)]\cdot(Du_1-Du_2)\,dx\geq\theta\|D(u_1-u_2)\|^2_{L^2(\Omega)}\geq 0.$$

Therefore $D(u_1-u_2)=0$ and hence, since $u_1-u_2\in H_0^1(\Omega)$, it holds $u_1=u_2$ a.e. in Ω. □

Strategy for the existence proof: Galerkin's method. Let X be a reflexive and separable Banach space. Consider $A\colon X \to X^*$ a given map and $F \in X^*$ a given element in the dual space. We look for a solution $u \in X$ of the equation

$$A(u) = F, \qquad \text{i.e.} \qquad A(u)(v) = F(v)\ \forall v \in X.$$

In our specific case, $X = H_0^1(\Omega)$,

$$A(u)(v) := \int_\Omega Dv \cdot a(Du)\, dx, \qquad F(v) := \int_\Omega v f(x)\, dx.$$

Galerkin's method can be organized in three steps.

Step 1: restriction to a finite dimensional problem.

Since X is separable, there exists $\{w_k\}_{k=1}^\infty$ dense subset. We define $V_n := \operatorname{span}\{w_1, \dots, w_n\}$. Then $d_n := \dim V_n \le n < \infty$. We say that $u \in V_n$ is a solution of the restricted problem (in V_n) if

$$A(u)_{|V_n} = F_{|V_n} \qquad \text{i.e.} \qquad A(u)(v) = F(v)\ \forall v \in V_n.$$

Step 2: solving the restricted problem.

We have d_n equations and d_n unknowns. In some cases one can prove that for each n $\exists u_n \in V_n$ solution of the restricted problem.

Step 3: convergence to a solution of the starting problem.

The idea is to prove that $n \mapsto u_n$ is a bounded sequence in X, hence, since X is reflexive, there exists $u \in X$ and a subsequence $j \to u_{n_j}$ such that $u_{n_j} \rightharpoonup u$ weakly in X.
The hard part is to show that the limit function is a solution of $A(u) = F$. Indeed, since $u \mapsto A(u)$ is nonlinear, $u_n \rightharpoonup u \not\Rightarrow A(u_n) \to A(u)$ in any sense. Here we will use monotoniticy.

Proof of Thm 5.7: existence. Set $X := H_0^1(\Omega)$, $A\colon X \to X^*$, and $F \in X^*$ defined as

$$A(u)(v) := \int_\Omega Dv \cdot a(Du)\, dx, \qquad F(v) := \int_\Omega v f(x)\, dx.$$

Our goal is to find $u \in X$ solution of $A(u) = F$. We use Galerkin's method.

Step 1: restriction to a finite dimensional problem. $H_0^1(\Omega)$ is a separable Hilbert space so there exists an o.n. countable basis. To construct such a basis consider the differential operator $L := -\Delta$.
By Theorem 2.13, the real spectrum of L has the form

$$\Sigma(L) = \{\lambda_1, \lambda_2, \dots\} \qquad \text{with} \qquad 0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \qquad \text{and} \qquad \lim_{k\to\infty} \lambda_k = \infty.$$

Moreover there exists an o.n. basis of $L^2(\Omega)$ $\{w_n\}_{n\ge1}$ such that $-\Delta u_n = \lambda_n u_n$ $\forall n$. It follows (exercise) that $\{e_n\}_{n\ge1}$ with $e_n := \frac{1}{\sqrt{\lambda_n}} w_n$ is an o.n. basis of $\left(H_0^1(\Omega), B_L[\cdot,\cdot]\right)$ where $B_L[u,v] := \int_\Omega Du \cdot Dv\, dx$. Hence $\{e_n\}_{n\ge1}$ is an orthogonal basis of $\left(H_0^1(\Omega), (\cdot,\cdot)_{H_0^1}\right)$ with

$$\|e_n\|_{H_0^1}^2 = \|e_n\|_{L^2}^2 + \|De_n\|_{L^2}^2 = \frac{1}{\lambda_n} + 1.$$

We define now $V_n := \operatorname{span}\{e_1, \dots, e_n\}$. Then $\dim V_n = n$.

Step 2: solving the restricted problem. We show that $\forall n \geq 1$ $\exists u_n \in V_n$ solution of the restricted problem $A(u)_{|V_n} = F_{|V_n}$.

Fix $n \geq 1$. For $\alpha \in \mathbb{R}^n$ set $u^\alpha := \sum_{j=1}^n \alpha_j e_j$. We look for $\alpha \in \mathbb{R}^n$ solution of

$$\int_\Omega Dv \cdot a\big(u^\alpha\big)\, dx \;=\; A(u^\alpha)(v) = F(v) \;=\; \int_\Omega v f(x)\, dx \qquad \forall v \in V_n.$$

We have $A(u)_{|V_n} = F_{|V_n}$ iff $A(u^\alpha)(e_k) = F(e_k)$ $\forall k = 1, \dots, n$, iff $A(u^\alpha)(e_k) - F(e_k) = 0$ $\forall k = 1, \dots, n$, iff $v(\alpha) = 0$ where

$$\begin{aligned} v\colon \quad & \mathbb{R}^n && \to \mathbb{R}^n \\ & \alpha && \mapsto v(\alpha)_k := A(u^\alpha)(e_k) - F(e_k),\ k = 1, \dots n. \end{aligned}$$

Therefore the problem is reduced to finding a zero for the vector field v. By Lemma 5.2 (*iii*) above, it is sufficient to check that v is continous and $\exists R > 0$ such that $v(\alpha) \cdot \alpha \geq 0$ $\forall \alpha \in \partial B_R(0)$.

Continuity. The map $p \mapsto a(p)$ is continous and $|a(p)| \leq C\,[1 + |p|]$, therefore the function

$$\begin{aligned} \Phi\colon \quad & L^2(\Omega)^d && \to L^2(\Omega)^d \\ & U && \mapsto \Phi(U) := a(U) \end{aligned}$$

is well defined and continous (proof: use Nemitski composition Lemma 4.9). It follows (exercise) that $\alpha \mapsto A(u^\alpha)(e_k)$ is continuous $\forall k$ and hence v is continous.

Positivity on the boundary of a sphere. We compute

$$v(\alpha) \cdot \alpha = \sum_{k=1}^n \alpha_k v(\alpha)_k = \sum_{k=1}^n \alpha_k [A(u^\alpha)(e_k) - F(e_k)] = A(u^\alpha)(u^\alpha) - F(u^\alpha).$$

Since $a(p) \cdot p \geq \beta |p|^2 - \gamma$ we have

$$A(u^\alpha)(u^\alpha) = \int_\Omega Du^\alpha \cdot a(Du^\alpha)\, dx \geq \ \beta \|Du^\alpha\|^2_{L^2(\Omega)} - \gamma|\Omega|,$$

and hence

$$v(\alpha) \cdot \alpha \geq \|Du^\alpha\|^2_{L^2(\Omega)} - \gamma|\Omega| - F(u^\alpha) \geq \|Du^\alpha\|^2_{L^2(\Omega)} - \|u^\alpha\|_{L^2(\Omega)} \|f\|_{L^2(\Omega)} - \gamma|\Omega|.$$

Since $\{e_n\}_{n \geq 1}$ is o.n. with respect to $B_L[\cdot, \cdot]$ and orthogonal with respect to $(\cdot, \cdot)_{L^2(\Omega)}$ we obtain

$$\begin{aligned} \|Du^\alpha\|^2_{L^2(\Omega)} =& B_L[u^\alpha, u^\alpha] = \sum_{j=1}^n \alpha_j^2 = |\alpha|^2 \\ \|u^\alpha\|^2_{L^2(\Omega)} =& (u^\alpha, u^\alpha)_{L^2(\Omega)} = \sum_{j=1}^n \alpha_j^2 \|w_j\|^2_{L^2(\Omega)} = \sum_{j=1}^n \alpha_j^2 \frac{1}{\lambda_j} \leq \frac{1}{\lambda_1} |\alpha|^2. \end{aligned}$$

Therefore, using $|\alpha| = R$,

$$v(\alpha) \cdot \alpha \geq |\alpha|^2 - \|f\|_{L^2(\Omega)} \frac{1}{\sqrt{\lambda_1}} |\alpha| - \gamma|\Omega| = R^2 - \|f\|_{L^2(\Omega)} \frac{1}{\sqrt{\lambda_1}} R - \gamma|\Omega| \geq 0$$

for R large enough.

By Lemma 5.2 (iii) it follows that v admits a zero and hence there exists a solution $u_n \in V_n$ of the restricted problem $A(u)_{|V_n} = F_{|V_n}$.

Step 3: convergence to a solution of the starting problem.

Let $n \mapsto u_n \in V_n$ the family of solutions we constructed in Step 2. Since $A(u)_{|V_n} = F_{|V_n}$, we have

$$A(u_n)(u_n) = F(u_n) \qquad \forall n \geq 1. \tag{5.5}$$

- We show that the sequence $n \mapsto u_n$ is bounded in $H_0^1(\Omega)$. Indeed

$$A(u_n)(u_n) = \int_\Omega Du_n \cdot a(Du_n)\ dx = \int_\Omega u_n f\ dx = F(u_n) \qquad \forall n.$$

Using (iv) and Poincaré inequality we have

$$\int_\Omega Du_n \cdot a(Du_n)\ dx \geq \beta \|Du_n\|^2_{L^2(\Omega)} - \gamma|\Omega| \geq \tilde{\beta}\|u_n\|^2_{H_0^1(\Omega)} - \gamma|\Omega|,$$

with $\tilde{\beta} > 0$. By Cauchy-Schwarz and then Young inequality we have

$$|\int_\Omega u_n f\ dx| \ \leq\ \|u_n\|_{L^2(\Omega)}\|f\|_{L^2(\Omega)} \leq \|u_n\|_{H_0^1(\Omega)}\|f\|_{L^2(\Omega)} \leq \frac{\varepsilon}{2}\|u_n\|^2_{H_0^1(\Omega)} + \frac{1}{2\varepsilon}\|f\|^2_{L^2(\Omega)}.$$

Putting all this together and setting $\varepsilon \leq \tilde{\beta}$ we obtain

$$\sup_{n\geq 1}\|u_n\|^2_{H_0^1(\Omega)} \leq \frac{2}{\tilde{\beta}}\left[\gamma|\Omega| + \frac{1}{2\varepsilon}\|f\|^2_{L^2(\Omega)}\right] =: C_1^2.$$

Therefore the sequence $n \mapsto u_n$ is bounded.

- We show that the sequence $n \mapsto A(u_n)$ is bounded in $H_0^1(\Omega)^*$. Indeed using (ii),

$$\begin{aligned}\|A(u_n)\|_{op} &= \sup_{\|v\|_{H_0^1}=1} |A(u_n)(v)| \leq \sup_{\|v\|_{H_0^1}=1} \int_\Omega |Dv|\ |a(Du_n)|\ dx \\ &\leq \|a(Du_n)\|_{L^2(\Omega)} \leq C\,\|1 + |Du_n|\|_{L^2(\Omega)} \leq C\left(\sqrt{|\Omega|} + \|Du_n\|_{L^2(\Omega)}\right) \leq C\Big(\sqrt{|\Omega|} + C_1\Big)\end{aligned}$$

where in the last step we used $\sup_{n\geq 1}\|u_n\|_{H_0^1(\Omega)} \leq C_1$.

- We have proved that the sequence $n \mapsto u_n$ (resp. $n \mapsto A(u_n)$) is bounded in $H_0^1(\Omega)$ (resp in $H_0^1(\Omega)^*$). Therefore, since X is reflexive, $\exists\ j \mapsto n_j$ subsequence, $u \in H_0^1(\Omega)$ and $T \in H_0^1(\Omega)^*$ such that

$$u_{n_j} \rightharpoonup u, \quad \text{in } H_0^1(\Omega), \qquad A(u_{n_j}) \rightharpoonup T, \quad \text{in } H_0^1(\Omega)^*.$$

It remains to show that $T = F$ and $T = A(u)$, which then implies $A(u) = F$.

- We show that $T = F$. Since $A(u_{n_j}) \rightharpoonup T$ we have

$$\lim_{j\to\infty} A(u_{n_j})(v) = T(v) \qquad \forall v \in X.$$

By construction we have $A(u_{n_j})(e_k) = F(e_k)$ $\forall n_j \geq k$, therefore

$$T(e_k) = \lim_{j\to\infty} A(u_{n_j})(e_k) = F(e_k) \qquad \forall k \geq 1,$$

and hence $T(e_k) = F(e_k)$ $\forall k$. The assertion now follows since $\{e_k\}_{k\geq 1}$ is a basis for X.

• We show that $T = A(u)$. By monotonicity of a we get

$$[A(u_{n_j}) - A(v)](u_{n_j} - v) = \int_\Omega [a(Du_{n_j}) - a(Dv)] \cdot (Du_{n_j} - Dv)\, dx \geq 0 \qquad \forall v \in X, j \geq 1.$$

We compute

$$[A(u_{n_j}) - A(v)](u_{n_j} - v) = A(u_{n_j})(u_{n_j}) - A(u_{n_j})(v) - A(v)(u_{n_j} - v).$$

By (5.5) we have

$$A(u_{n_j})(u_{n_j}) = F(u_{n_j}) = T(u_{n_j}) \to T(u)$$

where we used $T = F$ and $u_{n_j} \rightharpoonup u$.
$u_{n_j} \rightharpoonup u$ also implies $A(v)(u_{n_j} - v) \to A(v)(u - v)$.
Finally, $A(u_{n_j}) \rightharpoonup T$ implies $A(u_{n_j})(v) \to A(u)(v)$ and hence

$$[T - A(v)](u - v) \geq 0 \qquad \forall v \in X.$$

We take v 'near' u as follows. Set $v := u - \lambda w$, with $\lambda > 0$ and $w \in X$. The inequality above becomes

$$[T - A(u - \lambda w)](w) \geq 0 \qquad \forall w \in X.$$

Since $\lambda \mapsto A(u - \lambda w)(w)$ is continuous, we can take the limit $\lambda \to 0$:

$$\lim_{\lambda\to 0}[T - A(u - \lambda w)](w) = [T - A(u)](w) \geq 0 \qquad \forall w \in X,$$

which implies $T = A(u)$. □

6 Calculus of variations

A powerful tool to solve nonlinear PDEs is calculus of variations. The idea is to replace the problem of solving a PDE with the problem of minimizing some functional. The latter is sometimes easier.

Example. Let $\Omega \subset \mathbb{R}^d$ open and bounded, with Lipschitz boundary.
We look for solutions of the PDE

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u_{|\partial\Omega} = g \end{cases} \tag{6.1}$$

where $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ are given functions. A function $u \in H^1(\Omega)$ is a weak solution if $\operatorname{Tr} u = g$ and

$$\int_\Omega [Dv \cdot Du - vf]\, dx = 0 \qquad \forall v \in H^1_0(\Omega).$$

Assume $\exists u_g \in H^1(\Omega)$ such that $\operatorname{Tr} u_g = g$ (otherwise there can be no solution). We have seen (cf. Lemma 2.4) that $u \in H^1(\Omega)$ is a weak solution of (6.1) iff $w := u - u_g \in H_0^1(\Omega)$ is a weak solution of

$$\begin{cases} -\Delta w = f - \sum_{j=1}^d f_j & \text{in } \Omega \\ w_{|\partial\Omega} = 0 \end{cases}$$

where $f_j := -\partial_j f \in L^2(\Omega)$. By Lax-Milgram the weak solution exists and is unique.

We reformulate now the problem as a functional minimization. Set $X := H^1(\Omega)$ and $Y := \{u \in X|\ \operatorname{Tr} u = g\}$. We have $Y \neq \varnothing$ and $Y = u_g + H_0^1(\Omega)$. Define

$$\begin{aligned} I\colon \quad & X \to \mathbb{R} \\ u \quad & \mapsto I(u) := \textstyle\int_\Omega \left[\frac{|Du|^2}{2} - uf\right]\, dx. \end{aligned}$$

This map is well defined since $u, \partial_j u, f \in L^2(\Omega)$.

Claim. Assume $u_0 \in Y$ is a minimizer for I on Y, i.e. $I(v) \geq I(u_0)$ $\forall v \in Y$. Then u_0 is a weak solution for (6.1).

Proof. $u_0 \in Y$ hence $u_0 + \tau w \in Y$ $\forall \tau \in \mathbb{R}$ and $\forall w \in H_0^1(\Omega)$. Therefore, since u_0 is a minimizer, we have $I(u_0 + \tau w) \geq I(u_0)$ $\forall \tau \in \mathbb{R}$ and $\forall w \in H_0^1(\Omega)$. For a fixed $w \in H_0^1(\Omega)$ define

$$\begin{aligned} i_w\colon \quad & \mathbb{R} \to \mathbb{R} \\ \tau \quad & \mapsto\ i_w(\tau) := I(u_0 + \tau w). \end{aligned}$$

By direct computation, this function is a polynome is τ :

$$i_w(\tau) = \frac{\tau^2}{2}\|Dw\|^2_{L^2(\Omega)} + \tau \int_\Omega [Dw \cdot Du_0 - wf]\ dx + I(u_0).$$

Since u_0 is a minimizer, $\tau = 0$ is minimizer for $i_w(\tau)$ and hence

$$\int_\Omega [Dw \cdot Du_0 - wf]\ dx = i'_w(0) = 0 \qquad \forall w \in H_0^1(\Omega).$$

Therefore u_0 is a weak solution of (6.1). □

The general strategy is to construct, if possible, a functional associated to the PDE we consider and instead of solving the PDE to look for a minimizer of the functional.

[24: 11.01.2024]

[25: 15.01.2024]

6.1 Characterization of minimizers: Euler-Lagrange equation

In this section we look for the PDE associated to the mininimizer of a functional.

Definition 6.1 (minimizer)**.** *Let X be a real Banach space, $I\colon X \to \mathbb{R}$ a given map.*

(i) $u_0 \in X$ is a minimizer for I over X (or I attains its minimum at u_0, or u_0 is a global minimizer) if

$$I(u) \geq I(u_0) \qquad \forall u \in X.$$

(ii) Let $Y \subset X$ a subset. $u_0 \in Y$ is a minimizer for I over Y (or restricted to Y) if

$$I(u) \geq I(u_0) \qquad \forall u \in Y.$$

Remark. In the Example above, $X = H^1_0(\Omega)$ and $Y = u_g + H^1_0(\Omega)$, with $u_g \in H^1(\Omega)$, and $I(u) := \int_\Omega \left[\frac{|Du|^2}{2} - uf\right] dx$. Note that u_0 is a minimizer for I over Y iff $u_0 - u_g \in H^1_0(\Omega)$ is a global minimizer of $\tilde{I}\colon H^1_0(\Omega) \to \mathbb{R}$ defined via $\tilde{I}(v) := I(u_g + v)$.

Lemma 6.2 (characterization of minimizers)**.** *Let X be a real Banach space, $I\colon X \to \mathbb{R}$ a map, and $u_0 \in X$ fixed. For each $w \in X$ we define the map*

$$\begin{aligned} i_w\colon \quad & \mathbb{R} \to \mathbb{R} \\ & \tau \mapsto i_w(\tau) := I(u_0 + \tau w). \end{aligned}$$

Then the following hold.

(i) u_0 is a minimizer $\Leftrightarrow$ i_w attains its minimum at $\tau = 0$ $\forall w \in X$, i.e. $i_w(\tau) \geq i_w(0)$ $\forall \tau \in \mathbb{R}$ and $\forall w \in X$.

(ii) Assume i_w is differentiable in $\tau = 0$. Then if u_0 is a minimizer we have $i'_w(0) = 0$.

(iii) Assume i_w is C^2. Then if u_0 is a minimizer we have $i'_w(0) = 0$ and $i''_w(0) \geq 0$.

Proof. Exercise □

Notation. $i'_w(0)$ is called "first variation of I at u_0 in the direction w".
$i''_w(0)$ is called "second variation of I at u_0 in the direction w".

Definition 6.3. *Let X be a real Banach space, $I\colon X \to \mathbb{R}$ a map.*

(i) I is Gateaux differentiable at $u \in X$ in the direction $w \in X$ if the map $\tau \mapsto I(u + \tau w)$ is differentiable at $\tau = 0$.

I is Gateaux differentiable at $u \in X$ if I is Gateaux differentiable at u in all direction $w \in X$. In this case we denote by

$$I'(u)(w) := \lim_{\tau \to 0} \frac{I(u + \tau w) - I(u)}{\tau}$$

the Gateaux derivative at u in the direction w. In this case the function $I'(u)\colon X \to \mathbb{R}$ is called the Gateaux derivative of I at u.

(ii) I is Fréchet differentiable at $u \in X$ if $\exists A_u \in X^*$ such that

$$\lim_{\|w\|_X \to 0} \frac{|I(u + w) - I(u) - A_u(w)|}{\|w\|_X} = 0.$$

I is Fréchet differentiable if it is Fréchet differentiable at each $u \in X$. In this case $I'(u) = A_u$ $\forall u$ and $I'\colon X \to X^*$.

(iii) I is in $C^1(X)$ if I is Fréchet differentiable and the map I' is continuous.

Remarks.

- I Fréchet differentiable at $u \Rightarrow I$ Gateaux differentiable at u and $I'(u) = A_u$.

  The inverse implication does not hold.

- The Gateaux derivative is not necessarily additive or continuous. As an axample consider $F\colon \mathbb{R}^2 \to \mathbb{R}$ defined via $F(x,y) = \frac{x^3}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $F(0,0) = 0$. Then $F'(0,0)(a,b) = F(a,b)$ and is not linear in (a,b).
- Below we mostly need Gateaux differentiability, but our Gateaux derivatives $I'(u)\colon X \to \mathbb{R}$ will satify $I'(u) \in X^*$.
- If $u \mapsto I'(u)$ is well defined and continuous in a neighborhood U of u_0, and $I'(u) \in X^*$ for all $u \in U$, then I is Fréchet differentiable at u_0 with $A_{u_0} = I'(u_0)$.

Lemma 6.4 (Euler-Lagrange equation)**.** *Let X be a real Banach space, $I\colon X \to \mathbb{R}$ a map. Assume $u_0 \in X$ is a minimizer for I on X and I is Gateaux differentiable at u_0.*
Then u_0 is a solution of

$$I'(u_0)(w) = 0 \qquad \forall w \in X.$$

This is called the Euler-Lagrange equation associated to I.

Proof. Since I is Gateaux differentiable at u_0 the map $\tau \mapsto i_w(\tau) := I(u_0 + \tau w)$ is differentiable at $\tau = 0$. Since u_0 is a minimizer, it follows $0 = i_w'(0) = I'(u_0)(w)$ $\forall w \in X$. □

Remark. In some cases the Euler-Lagrange equation corresponds to a PDE in weak formulation. In the example above we can set $X := H_0^1(\Omega)$,

$$I(u) := \int_\Omega \left[\frac{|D(u_g + u)|^2}{2} - (u_g + u)f \right] \, dx,$$

with $u_g \in H^1(\Omega)$ fixed. This functional is Gateaux differentiable everywhere and the Euler-Lagrange equation is

$$0 = I'(u_0)(w) = \int_\Omega \left[Dw \cdot D(u_g + u_0) - wf \right] \, dx = 0 \qquad \forall w \in H_0^1(\Omega).$$

u_0 is a solution iff $u := u_g + u_0$ is a weak solution of the PDE

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u_{|\partial\Omega} = g. \end{cases}$$

Note that I is even Fréchet differentiable since

$$\limsup_{\|w\|_{H_0^1} \to 0} \frac{|I(u+w) - I(u) - I'(u)(w)|}{\|w\|_{H_0^1}} = \limsup_{\|w\|_{H_0^1} \to 0} \frac{\|Dw\|_{L^2}^2}{2\|w\|_{H_0^1}} \leq \limsup_{\|w\|_{H_0^1} \to 0} \frac{\|w\|_{H_0^1}}{2} = 0.$$

Moreover

$$I'(u_1)(w) - I'(u_2)(w) = \int_\Omega \left[Dw \cdot D(u_1 - u_2) \right] \, dx,$$

hence

$$\|I'(u_1) - I'(u_2)\|_{op} \leq \|Du_1 - Du_2\|_{L^2} \leq \|u_1 - u_2\|_{H_0^1}$$

and therefore $I \in C^1(X)$.

PDE associated to the Euler-Lagrange equation We will consider functionals of the form

$$\begin{aligned} I\colon \quad & X \to \mathbb{R} \\ & u \mapsto I(u) := \int_\Omega L(x, u(x), Du(x))\ dx, \end{aligned}$$

where $\Omega \subset \mathbb{R}^d$ is open and bounded, $X = W^{1,q}(\Omega)$, with $1 \leq q < \infty$ and

$$\begin{aligned} L\colon \quad & \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \\ & (x, s, p) \mapsto L(x, s, p). \end{aligned}$$

Remark 1. In the example above $L(x, s, p) = \frac{|p|^2}{2} - f(x)s$.

Remark 2. We perform a non rigorous computation, to derive the expected expression for the Euler-Lagrange equation. We assume L is differentiable in (s, p) for all x, and we can exchange limits and integrals. We compute

$$\begin{aligned} I'(u)(w) &= \lim_{\tau \to 0} \frac{I(u + \tau w) - I(u)}{\tau} = \lim_{\tau \to 0} \int_\Omega \frac{L(x, u + \tau w, Du + \tau Dw) - L(x, u, Du)}{\tau} dx \\ &= \int_\Omega \left[w \partial_s L(x, u, Du) + Dw \cdot \partial_p L(x, u, Du) \right]\ dx \qquad \forall w \in X \end{aligned}$$

where we used the abbreviated notation

$$\partial_s L(x, u, Du) := \partial_s L(x, s, p)_{|s=u(x), p=Du(x)}, \qquad \partial_p L(x, u, Du) := \partial_p L(x, s, p)_{|s=u(x), p=Du(x)}. \tag{6.2}$$

The Euler-Lagrange equation is then

$$\int_\Omega \left[w \partial_s L(x, u, Du) + Dw \cdot \partial_p L(x, u, Du) \right]\ dx = 0 \qquad \forall w \in X. \tag{6.3}$$

Remark 3.(PDE in weak formulation) Assume L and u above are smooth functions. Then we obtain the quasi-linear second order PDE

$$-\operatorname{div}\left[a(x, u, Du) \right] = -\partial_s L(x, u, Du) \tag{6.4}$$

with $a_j(x, s, p) := \partial_{p_j} L(x, s, p)$ and $\partial_s L(x, u, Du) = \partial_s L(x, u, Du) = \partial_s L(x, s, p)_{|s=u(x), p=Du(x)}$. $u \in W^{1,q}(\Omega)$ is a weak solution of (6.4) if (6.3) holds $\forall w \in W_0^{1,p}(\Omega)$.

Note that the weak formulation above is well defined if $\forall u \in W^{1,q}(\Omega)$ it holds

- $x \mapsto \partial_{p_j} L(x, u(x), Du(x)) \in L^{q'}(\Omega)$ with $\frac{1}{q'} = 1 - \frac{1}{q}$,
- $x \mapsto \partial_s L(x, u(x), Du(x)) \in L^\alpha(\Omega)$ with $1 \leq \alpha < \infty$ in the case $d \leq q$ and $\alpha \geq (q^*)'$ if $d > q$, where q^* is the Sobolev number $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{d}$ and $\frac{1}{(q^*)'} = 1 - \frac{1}{q^*}$.

We look now for sufficient regularity conditions on L such that

- $I\colon X \to \mathbb{R}$ is well defined,
- I is Gateaux differentiable with $I'(u) \in X^*$ and the EL equation is of the form (6.3).

Theorem 6.5. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded.*
Set $X := W^{1,q}(\Omega)$, with $1 \leq q < \infty$ and let

$$L\colon \quad \begin{aligned} &\Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \\ &(x,s,p) \mapsto L(x,s,p) \end{aligned}$$

be a map satisfying the following two properties.

C1) L is Carathéodory, i.e. $x \mapsto L(x,s,p)$ is measurable $\forall (s,p)$ and $(s,p) \mapsto L(x,s,p)$ is continuous for a.e. x.

Moreover $\exists f_1 \in L^1(\Omega;[0,\infty))$ and $C_1 > 0$ such that

$$|L(x,s,p)| \ \leq \ f_1(x) + C_1\left[|s|^q + |p|^q\right]$$

C2) L is differentiable in (s,p) for a.e. $x \in \Omega$, $\partial_s L$ and $\partial_p L$ are Carathéodory functions, and $\exists f_2, f_3 \in L^{q'}(\Omega;[0,\infty))$ and $C_2, C_3 > 0$, with $\frac{1}{q'} = 1 - \frac{1}{q}$, such that

$$\begin{aligned} |\partial_s L(x,s,p)| \ &\leq \ f_2(x) + C_2\left[|s|^{\frac{q}{q'}} + |p|^{\frac{q}{q'}}\right] = f_2(x) + C_2\left[|s|^{q-1} + |p|^{q-1}\right] \\ |\partial_p L(x,s,p)| \ &\leq \ f_3(x) + C_3\left[|s|^{\frac{q}{q'}} + |p|^{\frac{q}{q'}}\right] = f_3(x) + C_3\left[|s|^{q-1} + |p|^{q-1}\right]. \end{aligned}$$

Then the following holds.

(i) The map $I\colon X \to \mathbb{R}$ defined by $I(u) := \int_\Omega L(x,u(x),Du(x))\,dx$ is Gateaux differentiable everywhere in X and

$$I'(u)(w) = \int_\Omega \left[w\partial_s L(x,u,Du) + Dw \cdot \partial_p L(x,u,Du)\right]\,dx, \qquad \forall u,w \in X. \tag{6.5}$$

where we used the notation introduced in (6.2). Moreover $I'(u) \in X^*$ with

$$\|I'(u)\|_{op} \leq \|\partial_s L(x,u,Du)\|_{L^{q'}(\Omega)} + \|\partial_p L(x,u,Du)\|_{L^{q'}(\Omega)}.$$

(ii) Assume $\partial\Omega$ is Lipschitz continuous and $g \in T(W^{1,q}(\Omega)) \subset L^q(\partial\Omega, \mathcal{H}^{d-1})$.

Define $X_g := \{u \in X \mid \operatorname{Tr} u = g\}$. In particular $X_0 = W_0^{1,q}(\Omega)$.

Then, if $u_0 \in X_g$ is a minimizer for I on X_g, u_0 is a weak solution of

$$\begin{cases} -\operatorname{div}\partial_p L(x,u,Du) = -\partial_s L(x,u,Du) & \text{in } \Omega \\ u_{|\partial\Omega} = g \end{cases}$$

i.e.

$$\int_\Omega \left[w\partial_s L(x,u_0(x),Du_0(x)) + Dw \cdot \partial_p L(x,u_0(x),Du_0(x))\right]\,dx = 0 \qquad \forall w \in W_0^{1,q}(\Omega).$$

Proof.
(i)

$C1) \Rightarrow x \mapsto L(x,u(x),Du(x)) \in L^1(\Omega)$ $\forall u \in W^{1,q}(\Omega)$ and hence I is well defined.

$C2) \Rightarrow x \mapsto \partial_s L(x,u(x),Du(x)), \partial_p L(x,u(x),Du(x)) \in L^{q'}(\Omega)\ \forall u \in W^{1,q}(\Omega)$ and hence the integral in (6.5) is well defined.

We study

$$I'(u)(w) = \lim_{\tau\to 0} \frac{I(u+\tau w) - I(u)}{\tau} = \lim_{\tau\to 0} \int_\Omega L^\tau(x)\, dx,$$

where

$$L^\tau(x) \ = \frac{L(x, u+\tau w, Du+\tau Dw) - L(x,u,Du)}{\tau}.$$

Since L is differentiable in (s,p) for a.e. x we have

$$\lim_{\tau\to 0} L^\tau(x) = w\partial_s L(x,u,Du) + Dw\cdot \partial_p L(x,u,Du)$$

pointwise a.e. in Ω. Moreover, using $C2)$ again, we obtain $\forall |\tau| \le 1$

$$\begin{aligned}
|L^\tau(x)| &= \Big| \int_0^1 [(w\partial_s + Dw\cdot\partial_p)L(x, u+t\tau w, Du+t\tau Dw)]\ dt \Big| \\
&\le \big[|w|f_2 + |Dw|f_3\big] \ + \ \big(C_2|w| + C_3|Dw|\big) \int_0^1 \Big[|u+t\tau w|^{\frac{q}{q'}} + |Du+t\tau Dw|^{\frac{q}{q'}}\Big]\ dt \\
&\le \big[|w|f_2 + |Dw|f_3\big] \ + \ \big(C_2|w| + C_3|Dw|\big)\ [(|u|+|w|)^{\frac{q}{q'}} + (|Du|+|Dw|)^{\frac{q}{q'}}] =: F(x),
\end{aligned}$$

where $F \in L^1(\Omega)$. The result now follows by dominated convergence.

Finally $I'(u)$ is linear and

$$\begin{aligned}
|I'(u)(w)| &\le \int_\Omega [|w|\ |\partial_s L(x,u,Du)| + |Dw|\ |\partial_p L(x,u,Du)|]\, dx \\
&\le \|w\|_{L^q} \|\partial_s L(x,u,Du)\|_{L^{q'}} + \|Dw\|_{L^q}\|\partial_p L(x,u,Du)\|_{L^{q'}} \\
&\le \|w\|_{W^{1,q}} \left[\|\partial_s L(x,u,Du)\|_{L^{q'}} + \|\partial_p L(x,u,Du)\|_{L^{q'}}\right].
\end{aligned}$$

The result follows.

(ii) exercise. □

Remark. We can always restrict ourselves to the case $g=0$. Indeed $u \in X_g$ is a mininizer for I on $X_g \Leftrightarrow u - u_g$ is a minimizer for $\tilde{I}\colon X_0 \to \mathbb{R}$ defined by $\tilde{I}(u) := I(u_g + u)$.

[25: 15.01.2024]

[26: 18.01.2024]

6.2 Existence of minimizers: direct method of calculus of variations

As a preparation, consider the following result from Analysis 1.

Lemma 6.6. *Assume $f \in C(\mathbb{R})$ satisfies*

(i) f is bounded below, i.e. $\exists M \in \mathbb{R}$ such that $f(x) \ge M\ \forall x \in \mathbb{R}$.

(ii) f is coercive, i.e. $f(x) \to \infty$ as $|x| \to \infty$.

Then f admits a minimizer, i.e. $\exists x_0 \in \mathbb{R}$ such that $f(x_0) = \inf_{x\in\mathbb{R}} f(x)$.

The proof is elementary, but very instructive.

Proof.
Step 1. $(i) \Rightarrow m := \inf_x f \in \mathbb{R}$ and hence $\exists$ a sequence $n \mapsto x_n \in \mathbb{R}$ such that $f(x_n) \to m$.

Step 2. $(ii) \Rightarrow$ the sequence $n \mapsto x_n$ is bounded. Indeed otherwise there would be a subsequence $j \mapsto x_{n_j}$ with $|x_{n_j}| \to \infty$ and hence $m = \lim_{j\to\infty} f(x_{n_j}) = +\infty$ which gives a contradiction.

Step 3. Since $n \mapsto x_n$ is bounded, there exists a convergent subsequence $x_{n_j} \to x$. By continuity of f it follows $m = \lim_{j\to\infty} f(x_{n_j}) = f(x)$.

□

We must extend now this strategy to $I \colon X \to \mathbb{R}$.

Definition 6.7. *Let X be a real Banach space, $I \colon X \to \mathbb{R}$ a map.*

- *I is bounded below if $\exists M \in \mathbb{R}$ such that $I(u) \geq M$ $\forall u \in X$.*
- *I is coercive if $I(u) \to \infty$ as $\|u\|_X \to \infty$.*

Assume now $I \colon X \to \mathbb{R}$ is bounded below and coercive. We try repeating the steps in the proof of Theorem 6.6.

Since I is bounded below it holds $m := \inf_{u\in X} I(u) \in \mathbb{R}$ and hence $\exists$ a sequence $n \mapsto u_n \in X$ such that $I(u_n) \to m$.

Since I is coercive, the sequence $n \mapsto u_n \in X$ is bounded.

Problem 1: $n \mapsto u_n$ bounded $\not\Rightarrow$ there is a convergent subsequence. If X is reflexive, then there exists a subsequence $j \mapsto u_{n_j}$, and $u \in X$ such that $u_{n_j} \rightharpoonup u$ weakly.

Problem 2: I is not weakly continuous in general, i.e. $u_n \rightharpoonup u \not\Rightarrow I(u_n) \to I(u)$. We will see that we only need weak lower semicontinuity.

Definition 6.8. *Let X be a real Banach space, $I \colon X \to \mathbb{R}$ a map.*
I is weak lower semicontinuous (w.l.s.c. in short) if

$$u_n \rightharpoonup u \quad \Rightarrow \quad \liminf_{n\to\infty} I(u_n) \geq I(u).$$

Theorem 6.9 (Weierstrass)**.** *Let X be a real reflexive Banach space.*
Let $I \colon X \to \mathbb{R}$ be bounded below, coercive and weakly lower semicontinuous.
Then I admits a minimizer i.e. $\exists u_0 \in X$ such that $I(u_0) \leq I(u)$ $\forall u \in X$.

Proof.
Step 1. I bounded below $\Rightarrow m := \inf_{u\in X} I(u) \in \mathbb{R}$ and hence $\exists$ a sequence $n \mapsto u_n \in X$ such that $I(u_n) \to m$.

Step 2. I is coercive $\Rightarrow$ the sequence $n \mapsto u_n$ is bounded in X. It follows, since X is reflexive, there exists a subsequence $j \mapsto u_{n_j}$, and $u \in X$ such that $u_{n_j} \rightharpoonup u$ in X.

Step 3. I is weakly lower semicontinuous $\Rightarrow$

$$m = \lim_{j\to\infty} I(u_{n_j}) = \liminf_{j\to\infty} I(u_{n_j}) \geq I(u) \geq m.$$

It follows $I(u) = m$.

□

The hard part is to prove weak lower semicontinuity. The next result shows that it is sufficient that $p \mapsto L(x, s, p)$ is convex.

Theorem 6.10 (convex L)**.** *Let $\Omega \subset \mathbb{R}^d$ open and bounded with Lipschitz boundary.*
Set $X := W^{1,q}(\Omega)$ with $1 \leq q < \infty$.
We consider the map

$$I\colon \quad X \to \mathbb{R}$$
$$u \mapsto I(u) := \textstyle\int_\Omega L(x, u(x), Du(x))\ dx,$$

where $L\colon \Omega \times \mathbb{R} \times \mathbb{R}^d$ is such that

(i) $(C1)$ and $(C2)$ from Theorem 6.5 hold,

(ii) $p \mapsto L(x, s, p)$ is convex $\forall s \in \mathbb{R}$ and a.e. $x \in \Omega$.

Then I is weakly lower semicontinuous.

Remark. $p \mapsto L(x, s, p)$ differentiable and convex $\Rightarrow$ the Euler-Lagrange equation is an elliptic second order PDE.

Proof. Let $n \mapsto u_n \in X$ be a sequence with $u_n \rightharpoonup u \in X$.
Our goal is to show that $\liminf_{n\to\infty} I(u_n) \geq I(u)$ holds.

• Since $u_n \rightharpoonup u$, it follows that the sequence is bounded.
Moreover $W^{1,q}(\Omega) \subset\subset L^q(\Omega)\ \forall 1 \leq q < \infty\ d \geq 1$.
Therefore, since $n \mapsto u_n$ is bounded and $u_n \rightharpoonup u$ in $W^{1,q}(\Omega)$, it follows $u_n \to u$ in $L^q(\Omega)$.

• Assume first $L = L(x, s)$ is independent of p. By $(C1)$ and Nemitski composition theorem it follows that the function

$$\Phi\colon \quad L^q(\Omega) \to L^1(\Omega)$$
$$u \mapsto \Phi(u) := L(x, u(x))$$

is well defined and continuous. Hence $u_n \to u$ in $L^q(\Omega)$ implies $I(u_n) \to I(u)$, i.e. I is weakly continuous.

• Assume L is linear in p, i.e $I(u) = \int_\Omega f(x) \cdot Du(x)\ dx$. In this case

$$Du_n \rightharpoonup Du \quad \Rightarrow \quad I(u_n) = \int_\Omega f(x) \cdot Du_n(x)\ dx \to \int_\Omega f(x) \cdot Du(x)\ dx = I(u),$$

i.e. I is weakly continuous.

• Consider now the general case $L = L(x, s, p)$. We only know $Du_n \rightharpoonup Du$, hence we cannot use continuity. We will use convexity to compare $I(u_n)$ with an expression linear in Du_n.
Ineed, since L is convex and differentiable in the p variable, it holds

$$L(x, s, p + p') \geq L(x, s, p) + p' \cdot \partial_p L(x, s, p) \qquad \forall p, p' \in \mathbb{R}^d, s \in \mathbb{R} \text{ and a.e. } x \in \Omega.$$

It follows

$$L(x, u_n, Du_n) \geq L(x, u_n, Du) + (Du_n - Du) \cdot \partial_p L(x, u_n, Du)$$

and hence

$$\begin{aligned} \liminf_{n\to\infty} I(u_n) &= \liminf_{n\to\infty} \int_\Omega L(x, u_n, Du_n)\, dx \\ &\geq \liminf_{n\to\infty} \left[\int_\Omega L(x, u_n, Du)\, dx + \int_\Omega (Du_n - Du) \cdot \partial_p L(x, u_n, Du)\, dx \right]. \end{aligned}$$

Fix $u \in W^{1,q}(\Omega)$ and consider the function $\tilde{L}(x,s) := L(x,s,Du(x))$. By $(C1)$ and Nemitski composition theorem again it follows that the function

$$\begin{aligned} \Phi\colon \quad & L^q(\Omega) \to L^1(\Omega) \\ & v \mapsto \Phi(v) := \tilde{L}(x,v(x)) \end{aligned}$$

is well defined and continuous. Hence $u_n \to u$ in $L^q(\Omega)$ implies

$$\lim_{n\to\infty} \int_\Omega L(x,u_n,Du)\,dx = \lim_{n\to\infty} \int_\Omega \tilde{L}(x,u_n)\,dx = \int_\Omega \tilde{L}(x,u)\,dx = I(u).$$

Finally we show $\lim_{n\to\infty} \int_\Omega (Du_n - Du)\cdot \partial_p L(x,u_n,Du)\,dx = 0$, which concludes the proof. Indeed

$$\begin{aligned} \int_\Omega (Du_n - Du)\cdot \partial_p L(x,u_n,Du)\,dx = \int_\Omega & (Du_n - Du)\cdot [\partial_p L(x,u_n,Du) - \partial_p L(x,u,Du)]\,dx \\ & + \int_\Omega (Du_n - Du)\cdot \partial_p L(x,u,Du)\,dx. \end{aligned}$$

Since $u_n \rightharpoonup u$ we have $\lim_{n\to\infty} \int_\Omega (Du_n - Du)\cdot \partial_p L(x,u,Du)\,dx = 0$. Since $n \to Du_n - Du$ is a bounded sequence in $L^q(\Omega)$ and $\partial_p L(x,u_n,Du) \to \partial_p L(x,u,Du)$ in $L^{q'}(\Omega)$ (which holds again by Nemitski), it follows

$$\lim_{n\to\infty} \int_\Omega (Du_n - Du)\cdot [\partial_p L(x,u_n,Du) - \partial_p L(x,u,Du)]\,dx = 0.$$

□

Example 1. Set $X := W_0^{1,q}(\Omega)$, with $1 < q < \infty$ and consider $I(u) := \|Du\|_{L^q}^q = \int_\Omega |Du|^q dx$. This functional is weakly lower semicontinuous (exercise).

Example 2. Set $X := W_0^{1,q}(\Omega)$, with $1 < q < \infty$. We consider the map

$$\begin{aligned} I\colon \quad & X \to \mathbb{R} \\ & u \mapsto I(u) := \textstyle\int_\Omega L(x,u(x),Du(x))\;dx. \end{aligned}$$

Assume L satisfies the assumptions of Theorem 6.10 and in addition $\exists \alpha > 0, \beta \in L^{q'}(\Omega;[0,\infty))$, with $\frac{1}{q} + \frac{1}{q'} = 1$, such that

$$L(x,s,p) \geq \alpha|p|^q - \beta(x)|s|.$$

This functional admits a minimizer.

Proof. By Theorem 6.10, I is weakly lower semicontinuous. It remains to prove that I is bounded below and coercive. We compute

$$\begin{aligned} I(u) = \int_\Omega L(x,u(x),Du(x))\;dx &\geq \alpha \int_\Omega |Du|^q\;dx - \int_\Omega \beta(x)|u(x)|\;dx \\ &\geq \alpha\|Du\|_{L^q}^q - \|\beta\|_{L^{q'}}\|u\|_{L^q} \geq \tilde{\alpha}\|u\|_{W_0^{1,q}}^q - \|\beta\|_{L^{q'}}\|u\|_{W_0^{1,q}}, \end{aligned}$$

for some constant $\tilde{\alpha} > 0$. In the last line we used Poincaré inequality. It follows by Young's inequality

$$I(u) \geq \left[\tilde{\alpha} - \frac{\varepsilon}{q}\right]\|u\|_{W_0^{1,q}}^q - \frac{1}{q'\varepsilon}\|\beta\|_{L^{q'}}^{q'},$$

and hence $\exists \gamma_1, \gamma_2 > 0$ such that $\forall u \in W_0^{1,q}(\Omega)$

$$I(u) \geq \gamma_1\|u\|_{W_0^{1,q}}^q - \gamma_2.$$

Therefore I is bounded below and coercive. □

6.3 Regularity of minimizers

Let $\Omega \subset \mathbb{R}^d$ be open and bounded and consider the map

$$I: \quad \begin{array}{l} H_0^1(\Omega) \to \mathbb{R} \\ u \mapsto I(u) := \int_\Omega F(Du(x))\, dx, \end{array}$$

with $F \in C^\infty(\mathbb{R}^d)$ and such that $|F(p)| \le \alpha |p|^2$ for some $\alpha > 0$ and $\forall p \in \mathbb{R}^d$.
Define $a := DF \in C^\infty(\mathbb{R}^d; \mathbb{R}^d)$ and assume $\exists \beta, \theta > 0$ such that $|a(p)| \le \beta |p|$ and $Da(p) \ge \theta \mathrm{Id}$ $\forall p$. The following hold.

- I is Gateaux differentiable and $I'(u)(w) = \int_\Omega Dw \cdot a(Du(x))\, dx$ $\forall u, w \in H_0^1(\Omega)$.
- If $u \in H_0^1(\Omega)$ is a minimizer for I it follows that u is a weak solution of

$$\begin{cases} -\mathrm{div}\, a(Du) = 0 & \text{in } \Omega \\ u_{|\partial\Omega} = 0 \end{cases}$$

Since $A := Da \ge \theta \mathrm{Id}$ this is a quasi-linear second order uniformly elliptic PDE.

We investigate now regularity of the solution.

[26: 18.01.2024]

[27: 22.01.2024]

In the case of a linear PDE $-\mathrm{div}\, M(x) Du(x) = 0$ with $M \in L^\infty(\Omega; \mathbb{R}^{d\times d}_{sym})$ and $M \ge \theta \mathbb{I}$ a.e. we have seen the following results: assume $u \in H^1(\Omega)$ is a weak solution.

- By the lemma of de Giorgi we have $u \in C^{0,\sigma}_{loc}(\Omega)$ for some $0 < \sigma < 1$.
- If in addition $M \in C^{1+k}(\Omega; \mathbb{R}^{d\times d}_{sym})$ we have $u \in H^{k+2}_{loc}(\Omega)$.

We will need the following generalization of de Giorgi (see Giaquinta-Martinazzi Sec. 5.4):

$$M \in C^{k,\sigma}_{loc}(\Omega; \mathbb{R}^{d\times d}_{sym}) \Rightarrow u \in C^{k+1,\sigma}_{loc}(\Omega) \qquad \forall k \ge 0. \tag{6.6}$$

Theorem 6.11. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded.*
Let $a \in C^\infty(\mathbb{R}^d; \mathbb{R}^d)$ be a function satisfying $\forall p \in \mathbb{R}^d$

(i) $|a(p)| \le C_1 |p|$,

(ii) $Da(p) = Da(p)^t$ *and* $Da(p) \ge \theta \mathrm{Id}$, *where* $Da(p)_{ij} = \partial_{p_j} a_i$,

(iii) $|Da(p)| \le C_2$,

for some constants $C_1, C_2, \theta > 0$. Assume $u \in H^1(\Omega)$ is a weak solution of

$$-\mathrm{div}\, a(Du) = 0, \qquad in \qquad \Omega. \tag{6.7}$$

Then $u \in C^\infty(\Omega)$ and $\forall j = 1, \dots, d$ the function $v_j := \partial_j u$ is a solution of the linear elliptic PDE

$$-\mathrm{div}\,(A Dv_j) = 0 \tag{6.8}$$

where the matrix-valued function $A \in C^\infty(\Omega; \mathbb{R}^{d\times d})$ is defined via $A(x) := Da(Du(x))$.

Proof.
Step 1. We show that $u \in H^2_{loc}(\Omega)$. Since $u \in H^1(\Omega)$ we need to show that $Du \in H^1(V)$ holds for all V open $V \subset\subset \Omega$.
Let $V \subset\subset \Omega$ be fixed. We can construct U open such that $V \subset\subset U \subset\subset \Omega$ and $\zeta \in C_c^\infty(U;[0,1])$ such that $\zeta_{|V} = 1$. Set

$$h_0 := \frac{\operatorname{dist}(U, \partial\Omega)}{2}.$$

To prove $Du \in H^1(V)$ it is sufficient to find a constant $C = C_{U,V,\zeta} > 0$ such that (cf. Thm 2.17 and proof of Thm 2.18)

$$\sup_{0<|h|\le h_0} \|\zeta D_j^h Du\|_{L^2(\Omega)} \le C \qquad \forall j = 1, \dots, d. \tag{6.9}$$

Since u is a weak solution of (6.7), it holds

$$\int_\Omega Dw \cdot a(Du) dx \qquad \forall w \in H_0^1(\Omega).$$

For $w \in H_0^1(U)$ we have $D_j^{-h} w \in H_0^1(\Omega)$ $\forall 0 < |h| \le h_0$ and hence, using $DD_j^h w = D_j^h Dw$ and partial integration, we get

$$0 = \int_\Omega -D_j^h Dw \cdot a(Du) dx = \int_\Omega Dw \cdot D_j^h(a(Du))\ dx \qquad \forall w \in H_0^1(U), 0 < |h| \le h_0. \tag{6.10}$$

We compute

$$D_j^h(a(Du))(x)) = \frac{1}{h}\Big[a(Du(x+he_j)) - a(Du(x))\Big] = \int_0^1 Da(U_{h,t}(x))\ D_j^h(Du)(x)\, dt\ =\ A_h(x) D_j^h Du(x),$$

where

$$U_{h,t}(x) := Du(x) + thD_j^h Du(x) \in \mathbb{R}^d, \qquad A_h(x) := \int_0^1 Da(U_{h,t}(x)) dt \in \mathbb{R}^{d\times d}.$$

Since $Da(p) \ge \theta \mathrm{Id}$ and $|Da(p)| \le C_2$ we have

$$A_h(x) \ge \theta \mathrm{Id} \quad \text{ and } \quad |A_h(x)| \le C_2 \qquad \text{for a.e } x \in V. \tag{6.11}$$

Inserting these formulas in (6.10) we obtain

$$\int_\Omega Dw \cdot A_h D_j^h Du\ dx\ = 0 \qquad \forall w \in H_0^1(U), 0 < |h| \le h_0. \tag{6.12}$$

We consider $w := \zeta^2 D_j^h u$. This function is well defined and satisfies $w \in H_0^1(U)$ $\forall 0 < |h| \le h_0$ and hence (6.12) holds. Using

$$Dw = \zeta^2 D_j^h Du + 2\zeta D_j^h u D\zeta,$$

(6.11) and (6.12) we argue

$$\begin{aligned}\theta \|\zeta D_j^h(Du)\|^2_{L^2(\Omega)} &\le \int_\Omega \zeta^2\ D_j^h Du \cdot A_h D_j^h Du\, dx = -2\int_\Omega \zeta\ D_j^h u\ D\zeta \cdot\ A_h D_j^h Du\, dx \\ &\le\ 2\|A_h D\zeta\|_{L^\infty(\Omega)}\, \|D_j^h u\|_{L^2(U)}\, \|\zeta D_j^h(Du)\|_{L^2(\Omega)}.\end{aligned}$$

Since $u \in H^1(\Omega)$, by Thm 2.17 there is a constant $C_3 > 0$ such that

$$\|D_j^h u\|_{L^2(U)} \leq C_3 \|Du\|_{L^2(\Omega)} \qquad \forall 0 < |h| \leq h_0.$$

Inserting this bound above we obtain (6.9). Therefore $u \in H^2_{loc}(\Omega)$ and $v_j := \partial_j u \in H^1_{loc}(\Omega)$.

Step 2. We show that $v_j := \partial_j u$ is a local weak solution for $-\mathrm{div}\,(ADv_j) = 0$, where $A(x) = Da(Du(x)) = A_{h=0}(x)$. We need to show

$$\int_\Omega Dw \cdot A_0 D\partial_j u \, dx = 0 \qquad \forall w \in H^1_0(V).$$

We claim there is a sequence $n \mapsto h_n$ with $h_n \to 0$ and

$$\lim_{n\to\infty} \int_\Omega Dw \cdot A_{h_n} D_j^{h_n} Du \; dx = \int_\Omega Dw \cdot A_0 D\partial_j u \, dx,$$

so that the result follows from (6.12). We proceed now to prove the claim.

• By Step 1 we have $\|D_j^h Du\|_{L^2(V)} \leq \|\zeta D_j^h Du\|_{L^2(\Omega)} \leq C\|Du\|_{L^2(\Omega)}$ for all $0 \leq |h| \leq h_0$ and hence ($L^2(V)$ is reflexive) there is a sequence $n \mapsto h_n$ with $h_n \to 0$ and $v \in L^2(V)$ such that $D_j^{h_n} Du \rightharpoonup v$ in $L^2(V)$. Using the definition of weak derivative we obtain $v = D\partial_j u$.

• Since $n \mapsto D_j^{h_n} Du$ is a bounded sequence we have $\lim_{n\to\infty} hD_j^{h_n} Du = 0$ and hence $U_{h_n,t} \to Du$ strongly in $L^2(V)$. Since a is continuous it follows, by Nemitski composition theorem, $A_{h_n} Dw \to A_0 Dw$ strongly in $L^2(V)$.

• Putting these results together we argue

$$\int_\Omega Dw \cdot A_{h_n} D_j^{h_n} Du \; dx = \int_\Omega Dw \cdot A_0 D_j^{h_n} Du \; dx + \int_\Omega Dw \cdot (A_{h_n} - A_0) D_j^{h_n} Du \; dx = I_{1,n} + I_{2,n}.$$

Since $D_j^{h_n} Du \rightharpoonup \partial_j Du$ in $L^2(V)$ we have

$$\lim_{n\to\infty} I_{1,n} = \int_\Omega Dw \cdot A_0 \partial_j Du \; dx.$$

Since $n \mapsto D_j^{h_n} Du$ is bounded and $A_{h_n} Dw - A_0 Dw \to 0$ strongly in $L^2(V)$ we have $\lim_{n\to\infty} I_{2,n} = 0$, which concludes the proof of the claim.

Step 3. We show that v_j is smooth.
By Step 2, $v_j = \partial_j u \in H^1_{loc}(\Omega)$ is a local weak solution of $-\mathrm{div}\, ADv_j = 0$, with A uniformly elliptic and $x \mapsto A(x) \in L^\infty(\Omega)$. Hence, by Corollary 2.38 v_j is locally Hölder continuous, i.e $v_j \in C^{0,\sigma}_{loc}(\Omega)$ for some $0 < \sigma < 1$ and for all $j = 1, \ldots, d$.
Since $Du \in C^{0,\sigma}_{loc}(\Omega)$ and $a \in C^\infty$ we have $A_{ij} \in C^{0,\sigma}_{loc}(\Omega)$. Applying (6.6) we obtain $v_j \in C^{1,\sigma}_{loc}(\Omega)$, which implies $A_{ij} \in C^{1,\sigma}_{loc}(\Omega)$. Repeating this argument we obtain smoothness.

□

Remark. Note that even if $a \in C^\infty$, the map $x \mapsto A(x) := Da(Du(x))$ is only as regular as the function Du. Therefore we cannot argue as in the case of linear PDEs with smooth coefficients.

6.4 Constrained minimizers

Let $X = W^{1,q}(\Omega)$, $I\colon X \to \mathbb{R}$ be a given map, $\mathcal{A}$ a subset of X. We say that $u_0 \in \mathcal{A}$ is a constrained minimizer for I on $\mathcal{A}$ if

$$I(u_0) = \inf_{u\in\mathcal{A}} I(u).$$

We have already seen the case when $\mathcal{A} = \{u \in X \,|\, \operatorname{Tr} u = g\}$ for some $g \in L^q(\partial\Omega)$. We will consider now two important types of constraint:

1. integral contraint: in this case $\mathcal{A} = \{u \in X \,|\, J(u) = 0\}$ where $J\colon X \to \mathbb{R}$ is of the form $J(u) = \int_\Omega G(x, u(x))\, dx$;

2. unilateral constraint: in this case $\mathcal{A} = \{u \in X \,|\, u \geq h \text{ a.e.}\}$ where $h \in C^\infty(\overline{\Omega})$ is the 'obstacle'.

Example 1. Set $d = 3$, $\Omega \subset \mathbb{R}^3$ open and bounded. We consider

$$\begin{aligned} I\colon \quad & H_0^1(\Omega) \to \mathbb{R} \\ & u \mapsto I(u) := \tfrac{1}{2}\textstyle\int_\Omega |Du|^2\, dx, \end{aligned} \qquad \begin{aligned} J\colon \quad & H_0^1(\Omega) \to \mathbb{R} \\ & u \mapsto I(u) := \textstyle\int_\Omega \left[1 + \frac{u^3}{3}\right] dx, \end{aligned}$$

Note that J is well defined since $d = 3$ and hence $H_0^1(\Omega) \subset L^3(\Omega)$.
The unique global minimizer for I on X is $u = 0$. But $0 \notin \mathcal{A}$, hence it cannot be a constrained minimizer on $\mathcal{A}$.

Example 2. Set $\Omega = B_2(0) \subset \mathbb{R}^d$ and $X = H_0^1(\Omega)$. Consider $h \in C_c^\infty(\Omega; [0,1])$ with $h_{|B_1(0)} = 1$, and $I(u) := \frac{1}{2}\int_\Omega |Du|^2\, dx$. Once again the unique global minimizer for I on X is $u = 0$. But $0 \notin \mathcal{A}$, hence it cannot be a constrained minimizer on $\mathcal{A}$.

[27: 22.01.2024]

[28: 25.01.2024]

Theorem 6.12 (existence of constrained minimizers)**.**
Let $\Omega \subset \mathbb{R}^d$ be open and bounded, $X = W_0^{1,q}(\Omega)$ with $1 < q < \infty$.
We consider

$$\begin{aligned} I\colon \quad & X \to \mathbb{R} \\ & u \mapsto I(u) := \textstyle\int_\Omega L(x, u(x), Du(x))\, dx, \end{aligned}$$

and we assume I is bounded below, coercive and weakly lower semicontinuous.

(i) (integral constraint) Let $1 \leq p < \infty$ such that $W_0^{1,q}(\Omega) \subset\subset L^p(\Omega)$ and consider

$$\begin{aligned} J\colon \quad & L^p(\Omega) \to \mathbb{R} \\ & u \mapsto J(u) := \textstyle\int_\Omega G(x, u(x))\, dx, \end{aligned}$$

such that J is continuous. Let $\mathcal{A} := \{u \in X \,|\, J(u) = 0\}$ and assume $\mathcal{A} \neq \varnothing$.

Then $\exists u_0 \in \mathcal{A}$ such that $I(u_0) = \inf_{u\in\mathcal{A}} I(u)$.

(ii) (unilateral constraint) Define $\mathcal{A} := \{u \in X \,|\, u \geq h$ a.e. in $\Omega\}$, where $h \in C^\infty(\overline{\Omega})$ is a given function and assume $\mathcal{A} \neq \varnothing$.

Then $\exists u_0 \in \mathcal{A}$ such that $I(u_0) = \inf_{u \in \mathcal{A}} I(u)$.

Moreover, if $(s,p) \to L(x,s,p)$ is strictly convex for a.e. $x \in \Omega$, the minimizer u_0 is also unique.

Proof.
(i) Set $m_{\mathcal{A}} := \inf_{u \in \mathcal{A}} I(u)$.
I is bounded below and $\mathcal{A} \neq \varnothing$, hence $m_{\mathcal{A}} \in \mathbb{R}$ and $\exists n \to u_n \in \mathcal{A}$ such that $I(u_n) \to m_{\mathcal{A}}$. Since I is coercive, the sequence $u \mapsto u_n$ is bounded, and therefore, since X is reflexive, there exists a weakly convergent subsequence $u_{n_j} \rightharpoonup u \in X$. Since I is weakly lower semicontinuous it follows $m_{\mathcal{A}} = \liminf_{j \to \infty} I(u_{n_j}) \geq I(u)$.

We show now $u \in \mathcal{A}$, and hence $m_{\mathcal{A}} = I(u)$.
Indeed, $u_{n_j} \rightharpoonup u$ and $W_0^{1,q}(\Omega) \subset\subset L^p(\Omega)$, imply that $u_{n_j} \to u$ in $L^p(\Omega)$ and hence, since J is continuous, $\lim_{j \to \infty} J(u_{n_j}) = J(u)$.
Since $u_{n_j} \in \mathcal{A}$ we have $J(u_{n_j}) = 0 \ \forall j$, and hence

$$J(u) = \lim_{j \to \infty} J(u_{n_j}) = 0.$$

Therefore $u \in \mathcal{A}$.

(ii) (existence) Set $m_{\mathcal{A}} := \inf_{u \in \mathcal{A}} I(u)$. Arguing as in (i), $\exists n \to u_n \in \mathcal{A}$ and a function $u \in X$ such that $u_n \rightharpoonup u$ in X and $m_{\mathcal{A}} \geq I(u)$.

It remains to prove that $u \in \mathcal{A}$. We argue as follows.
Since $u_n \rightharpoonup u \in X$ and $W_0^{1,q} \subset\subset L^q(\Omega)$ it follows $u_n \to u$ in $L^q(\Omega)$, and therefore there is a subsequence $j \to u_{n_j}$ such that $u_{n_j} \to u$ pointwise a.e. in Ω, i.e $\exists \tilde{\Omega} \subset \Omega$ with $|\Omega \setminus \tilde{\Omega}| = 0$ and $u_{n_j} \to u$ pointwise in $\tilde{\Omega}$.

$u_{n_j} \in \mathcal{A}$ for all j, hence for each j $\exists \Omega_j \subset \Omega$ such that $|\Omega \setminus \Omega_j| = 0$ and $u_{n_j}(x) \geq h(x) \ \forall x \in \Omega_j$.

We define now $\overline{\Omega} := \tilde{\Omega} \cap \bigcap_j \Omega_j$ We have $|\Omega \setminus \overline{\Omega}| = 0$, $u_{n_j}(x) \geq h(x) \ \forall x \in \overline{\Omega}$ and $\forall j$, and $u_{n_j} \to u$ pointwise in $\overline{\Omega}$. Therefore $u(x) \geq h(x) \ \forall x \in \overline{\Omega}$ and hence $u \in \mathcal{A}$. This completes the proof of existence.

(ii) (unicity) By contradiction assume $u_1, u_2 \in \mathcal{A}$ are two different minimizers over $\mathcal{A}$, i.e $u_1 \neq u_2$ and $I(u_1) = I(u_2) = m_{\mathcal{A}}$. The set $\mathcal{A}$ is convex (exercise), therefore $\lambda u_1 + (1-\lambda)u_2 \in \mathcal{A}$ and hence $I(\lambda u_1 + (1-\lambda)u_2) \geq m_{\mathcal{A}} \ \forall \lambda \in (0,1)$. Since $(s,p) \to L(x,s,p)$ is strictly convex we have

$$L(x, \lambda u_1 + (1-\lambda)u_2, \lambda Du_1 + (1-\lambda)Du_2) < \lambda L(x, u_1, Du_1) + (1-\lambda)L(x, u_2, Du_2)$$

and hence

$$m_{\mathcal{A}} \leq I(\lambda u_1 + (1-\lambda)u_2) < \lambda I(u_1) + (1-\lambda)I(u_2) = m_{\mathcal{A}},$$

which gives a contradiction.

□

Remark. Note that the set $\mathcal{A} := \{u \in X \,|\, J(u) = 0\}$ is not convex in general (unless J is linear) hence the argument above does not apply for integral constraints.

We investigate now the PDE associated to a constrained minimizer.

Theorem 6.13 (Euler-Lagrange equation for constrained minimizers)**.** *Let $\Omega \subset \mathbb{R}^d$ be open and bounded, $X = W_0^{1,q}(\Omega)$ with $1 < q < \infty$.*
We consider

$$I\colon \quad X \to \mathbb{R}$$
$$u \mapsto I(u) := \int_\Omega L(x, u(x), Du(x))\, dx,$$

and we assume I is bounded below, coercive, weakly lower semicontinuous and $C^1(X)$ i.e. I is Fréchet differentiable and the map $I'\colon X \to X^*$ is continuous (cf. Def. 6.3)

(i) (integral constraint) Let $1 \le p < \infty$ such that $W_0^{1,q}(\Omega) \subset\subset L^p(\Omega)$ and consider

$$J\colon \quad L^p(\Omega) \to \mathbb{R}$$
$$u \mapsto J(u) := \int_\Omega G(x, u(x))\, dx,$$

such that J is $C^1(X)$. Let $\mathcal{A} := \{u \in X \,|\, J(u) = 0\}$ and assume $\mathcal{A} \neq \varnothing$. Let $u_0 \in \mathcal{A}$ be a constrained minimizer (whose existence is ensured by Thm. 6.12).

Assume in addition $\exists v = v_{u_0} \in X$, $v \neq 0$ such that $J'(u_0)(v) \neq 0$.

Then u_0 is a solution of

$$I'(u_0)(w) = \lambda(u_0) J'(u_0)(w) \qquad \forall w \in X. \tag{6.13}$$

where

$$\lambda(u_0) := \frac{I'(u_0)(v_{u_0})}{J'(u_0)(v_{u_0})} \in \mathbb{R}.$$

is called the "Lagrange multiplier for the integral constraint J".

(ii) (unilateral constraint) Define $\mathcal{A} := \{u \in X \,|\, u \ge h$ a.e. in $\Omega\}$, where $h \in C^\infty(\overline{\Omega})$ is a given function and assume $\mathcal{A} \neq \varnothing$. Let $u_0 \in \mathcal{A}$ be a constrained minimizer (whose existence is ensured by Thm. 6.12).

Then u_0 is a solution of

$$I'(u_0)(w) \ge I'(u_0)(u_0) \qquad \forall w \in \mathcal{A}. \tag{6.14}$$

Proof.
(i) Without constraint we would study the function $\tau \mapsto I(u_0 + \tau w)$. The problem is that $u, w \in \mathcal{A}$ does not imply $u_0 + \tau w \in \mathcal{A}$. The solution is to "shift" τw back onto $\mathcal{A}$. Precisely, $\forall w \in X$, $\exists \delta = \delta_{w,v} > 0$ and $\phi = \phi_{w,v}\colon \mathbb{R} \to \mathbb{R}$ such that

- $\phi \in C^1(\mathbb{R})$ and $\phi(0) = 0$,
- $u_0 + \tau w + \phi(\tau) v \in \mathcal{A}$ $\forall |\tau| \le \delta$.

To prove this consider

$$j\colon \quad \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$
$$(\tau, \sigma) \mapsto j(\tau, \sigma) := J(u_0 + \tau w + \sigma v).$$

Since $J \in C^1$ we have $j \in C^1(\mathbb{R} \times \mathbb{R})$ and

$$\begin{cases} \partial_\tau j(\tau, \sigma) = J'(u_0 + \tau w + \sigma v)(w) \\ \partial_\sigma j(\tau, \sigma) = J'(u_0 + \tau w + \sigma v)(v). \end{cases}$$

In particular $\partial_\sigma j(0,0) = J'(u_0)(v) \neq 0$. Therefore, by the implicit function theorem, there exists a function $\phi \in C^1(\mathbb{R})$ and a parameter $\delta > 0$ such that $\phi(0) = 0$ and $j(\tau, \phi(\tau)) = 0$ $\forall |\tau| \leq \delta$.
We define now

$$\begin{aligned} i = i_w \colon \quad & \mathbb{R} \to \mathbb{R} \\ & \tau \mapsto i(\tau) := I(u_0 + w(\tau)), \end{aligned}$$

where $w(\tau) := \tau w + \phi(\tau) v$.

Since $I \in C^1(X)$ and $\tau \to w(\tau) \in C^1(\mathbb{R}; X)$ we have $i \in C^1(\mathbb{R})$. Moreover, since $u_0 + w(\tau) \in \mathcal{A}$ $\forall |\tau| \leq \delta$, and u_0 is a constrained minimizer on $\mathcal{A}$, we have $i(\tau) \geq i(0)$ $\forall |\tau| \leq \delta$ and hence $i'(0) = 0$. We compute

$$\begin{aligned} i'(0) &= \lim_{\tau \to 0} \frac{I(u_0 + w(\tau)) - I(u_0)}{\tau} \\ &= \lim_{\tau \to 0} \frac{I(u_0 + w(\tau)) - I(u_0) - I'(u_0)(w(\tau))}{\|w(\tau)\|_X} \frac{\|w(\tau)\|_X}{\tau} + \lim_{\tau \to 0} \frac{I'(u_0)(w(\tau))}{\tau}. \end{aligned}$$

Note that $\lim_{\tau \to 0} w(\tau) = 0$ and

$$\frac{\|w(\tau)\|_X}{\tau} = w + \frac{\phi(\tau)}{\tau} v = w + \frac{\phi(\tau) - \phi(0)}{\tau} v \to_{\tau \to 0} w + \phi'(0) v.$$

Inserting these results above and using that I is Fréchet differentiable we get

$$0 = i'(0) = \lim_{\tau \to 0} \frac{I'(u_0)(w(\tau))}{\tau} = I'(u_0)(w) + \phi'(0) I'(u_0)(v). \tag{6.15}$$

We use now the relation $J(u) = 0$ $\forall u \in \mathcal{A}$ to derive a formula for $\phi'(0)$.
Indeed $u_0 + w(\tau) \in \mathcal{A}$ $\forall |\tau| \leq \delta$, hence $J(u_0 + w(\tau)) = 0$ $\forall |\tau| \leq \delta$. In particular, arguing as for $\tau \mapsto I(u_0 + w(\tau))$, we obtain

$$0 = \frac{d}{d\tau} J(u_0 + w(\tau))_{|\tau=0} = \lim_{\tau \to 0} \frac{J(u_0 + w(\tau)) - J(u_0)}{\tau} = J'(u_0)(w) + \phi'(0) J'(u_0)(v),$$

and hence, since $J'(u_0)(v) \neq 0$

$$\phi'(0) = -\frac{J'(u_0)(w)}{J'(u_0)(v)}.$$

Inserting in (6.15) we obtain the result.

(ii) As in (i), the main problem is that $u_0, w \in \mathcal{A} \not\Rightarrow u_0 + \tau w \in \mathcal{A}$.
Since $\mathcal{A}$ is convex it holds

$$(1-\tau) u_0 + \tau w = u_0 + \tau(w - u_0) \in \mathcal{A} \qquad \forall w \in \mathcal{A}, \forall \tau \in [0,1].$$

Fix now $w \in \mathcal{A}$ and let $i \colon [0,1] \to \mathbb{R}$ be the map defined via $i(\tau) := I(u_0 + \tau(w - u_0))$. Then $i(\tau) \geq i(0)$ $\forall 0 < \tau \leq 1$ and therefore

$$0 \leq \lim_{\tau \downarrow 0} \frac{i(\tau) - i(0)}{\tau} = I'(u_0)(w - u_0).$$

This concludes the proof of the theorem. $\square$

[28: 25.01.2024]
[29: 29.01.2024]

Example 1. Set $d = 3$, $\Omega \subset \mathbb{R}^3$ open and bounded, $X = X_0 = H_0^1(\Omega)$. We consider again

$$I(u) := \frac{1}{2}\int_\Omega |Du|^2\,dx, \quad J(u) := \int_\Omega \left[1 + \frac{u^3}{3}\right]dx.$$

- It holds $H_0^1(\Omega) \subset\subset L^3(\Omega)$.
- $I \geq 0$ hence I is bounded below.
- By Poincaré, $I(u) = \frac{1}{2}\|Du\|_{L^2(\Omega)} \geq C\|Du\|^2_{H_0^1(\Omega)}$, hence I is coercive. Moreover $p \mapsto \frac{|p|^2}{2}$ is convex and hence I is weakly lower semicontinuous (cf. Thm 6.10)
- $J\colon L^3(\Omega) \to \mathbb{R}$ is continous. Moreover $\mathcal{A} = \{u \in H_0^1(\Omega) \,|\, J(u) = 0\} \neq \varnothing$. Indeed take $u \in H_0^1(\Omega; [0,\infty))$, $u \neq 0$ and consider $u_\alpha := \alpha u$, for $\alpha \in \mathbb{R}$ to choose. We compute

$$J(u_\alpha) = J(\alpha u) = |\Omega| + \alpha^3 \frac{1}{3}\int_\Omega u^3 = |\Omega| + \alpha^3 \frac{\|u\|^3_{L^3}}{3},$$

where in the last step we used $u \geq 0$. We obtain $J(u_\alpha) = 0$ for $\alpha := -(3|\Omega|)^{\frac{1}{3}}/\|u\|_{L^3}$.

Hence, by Theorem 6.12, there exists $u_0 \in \mathcal{A}$ such that $I(u_0) = \inf_{\mathcal{A}} I$. We look now for the corresponding Euler-Lagrange equation.

- $I \in C^1(X)$. Indeed we have $I(u + w) - I(u) = (Du, Dw)_{L^2(\Omega)} + \frac{1}{2}\|Dw\|^2_{L^2(\Omega)}$. Therefore (exercise) I is Fréchet differentiable with

$$I'(u)(w) = (Du, Dw)_{L^2(\Omega)} = \int_\Omega Du \cdot Dw\ dx.$$

Finally

$$\begin{aligned}
\|I'(u) - I'(v)\|_{op} &= \sup_{\|w\|_{H_0^1}=1} \left|\int_\Omega (Du - Dv)\cdot Dw\ dx\right| \\
&\leq \sup_{\|w\|_{H_0^1}=1} \|Du - Dv\|_{L^2}\|Dw\|_{L^2} \leq \|Du - Dv\|_{L^2} \leq \|u - v\|_{H_0^1}.
\end{aligned}$$

Therefore $I'\colon X \to X^*$ is continuous.

- $J \in C^1(X)$. Indeed we have $J(u + w) - J(u) = \int_\Omega u^2 w\,dx + \frac{1}{3}\int_\Omega w^3 dx$. Therefore (exercise) J is Fréchet differentiable with

$$J'(u)(w) = \int_\Omega u^2 w\,dx.$$

Finally

$$\begin{aligned}
\|J'(u) - J'(v)\|_{op} &= \sup_{\|w\|_{H_0^1}=1} \left|\int_\Omega (u^2 - v^2)w\,dx\right| \\
&\leq \|u^2 - v^2\|_{L^2} = \|(u - v)(u + v)\|_{L^2} \leq \|u - v\|_{L^4}\|u + v\|_{L^4} \\
&\leq C\|u - v\|_{H_0^1}\|u + v\|_{H_0^1} \leq C\|u - v\|_{H_0^1}(\|u\|_{H_0^1} + \|v\|_{H_0^1}).
\end{aligned}$$

Therefore $J'\colon X \to X^*$ is continuous.

- $\exists v \neq 0$ such that $J'(u_0)(v) \neq 0$. Indeed, by contradiction, assume $J'(u_0)(v) = 0$ $\forall v \in X$. Then

$$\int u_0^2 v = 0 \quad \forall v \in C_c^\infty(\Omega)$$

and hence $u_0 = 0$ a.e. in Ω. But this is impossible since $0 \notin \mathcal{A}$.

By Theorem 6.13 it follows that u_0 is a weak solution of the nonlinear eigenvalue equation

$$-\Delta u_0 = \lambda(u_0)u_0^2, \qquad \text{where} \qquad \lambda(u_0) = \frac{\int_\Omega Du_0 \cdot Dv\,dx}{\int_\Omega u_0^2 v\,dx}.$$

Example 2. Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Set $X = H_0^1(\Omega)$. We consider again

$$I(u) := \frac{1}{2}\int_\Omega |Du|^2\,dx.$$

Let $h \in C_c^\infty(\Omega)$ be a given function and set $\mathcal{A} := \{u \in X \,|\, u \geq h \text{ a.e. in } \Omega\}$.
With this choice $\mathcal{A} \neq \varnothing$ since $h \in \mathcal{A}$.
I is bounded, coercive and w.l.s.c. and the map $p \mapsto L(p) = |p|^2/2$ is strictly convex.
Then, by Theorem 6.12 $\exists! u_0 \in \mathcal{A}$ such that $I(u_0) \leq I(u)$ $\forall u \in \mathcal{A}$.
In addition $I \in C^1(X)$, hence, by Theorem 6.13, u_0 satisfies

$$I'(u_0)(w - u_0) \geq 0 \qquad \forall w \in \mathcal{A}.$$

In particular we have $w = u_0 + \tau v \in \mathcal{A}$ $\forall \tau \geq 0, v \in H_0^1(\Omega; [0,\infty))$, since $w \geq u_0 \geq h$ a.e. in Ω. Hence

$$I'(u_0)(w - u_0) = \tau I'(u_0)(v) \geq 0 \qquad \forall \tau > 0, v \in H_0^1(\Omega; [0,\infty)).$$

Dividing by τ we obtain

$$\int_\Omega Du_0 \cdot Dv\,dx = I'(u_0)(v) \geq 0 \qquad \forall v \in H_0^1(\Omega; [0,\infty))$$

and hence u_0 is a weak sub-solution for $-\Delta u = 0$.

6.5 Critical points

As an illustrative example let $d = 3$, $X = H_0^1(\Omega)$, $f \in L^2(\Omega)$ and consider the functional

$$I(u) := \int_\Omega \left(\frac{|Du|^2}{2} - \frac{u^3}{3} + fu \right) dx. \tag{6.16}$$

This functional is well defined and in $C^1(X)$ (exercise) with

$$I'(u)(w) = \int_\Omega \left(Du \cdot Dw - u^2 w + fw \right)\,dx.$$

If $I'(u) = 0$ then u is a weak solution of $-\Delta u = u^2 - f$.
Note that the functional I is <u>not</u> coercive. Indeed take $v \in H_0^1(\Omega; [0,\infty))$ and consider $u_\lambda := \lambda v$ with $\lambda > 0$. Then $I(u_\lambda) \to -\infty$ as $\lambda \to \infty$. This shows also that I admits no global minimizer. We will see that we can still find u_0 solution of $I'(u_0) = 0$. This will be a critical point, but not necessarily a local minimizer.

Definition 6.14 (critical points). *Let X be a real Banach space and $I\colon X \to \mathbb{R}$ a functional satisfying $I \in C^1(X)$.*

(i) The point $u_0 \in X$ is a critical point *for I if $I'(u_0) = 0$.*

In this case $c_0 := I(u_0)$ is called a critical value *for I.*

(ii) A real number $c \in \mathbb{R}$ is called a regular value *for I if c is not a critical value, i.e $I'(u) \neq 0$ for all $u \in X$ such that $I(u) = c$.*

Remark 1. If c_0 is a critical value then $\exists u_0 \in X$ solution of the E-L equation $I'(u_0) = 0$ i.e. u_0 is a weak solution for the corresponding PDE.

Remark 2. When proving existence of minimizers we start with a minimizing sequence $I(u_n) \to \inf_X I$. Coercivity implies that the sequence $n \mapsto u_n$ is bounded. If in addition X is reflexive, there exists then a weakly convergent subsequence.
When proving existence of critical points we replace coercivity+reflexivity with the so-called Palais-Smale condition.

Definition 6.15 (Palais-Smale condition)**.** *Let X be a Banach space and $I \in C^1(X)$. We say that*

(i) I satisfies Palais-Smale condition at level $c \in \mathbb{R}$ if $\forall n \mapsto u_n \in X$ such that

$$I(u_n) \to c \quad \text{in } \mathbb{R}, \qquad \text{and} \quad I'(u_n) \to 0 \quad \text{in } X^*,$$

there exists a strongly convergent subsequence $u_{n_j} \to u \in X$,

(ii) I satisfies Palais-Smale condition, if I satisfies Palais-Smale condition at all level $c \in \mathbb{R}$.

Example. The function $f\colon \mathbb{R} \to \mathbb{R}$ defined via $f(x) := e^x$ does not satisfy PS.
Indeed take $x_n := -n$. It holds $f(x_n) \to c = 0$ and $f'(x_n) \to 0$. But the sequence admits no convergent subsequence.

Topological characterization of the critical points.

We will often use the notation

$$\{I \leq c\} := \{u \in X \,|\, I(u) \leq c\} = I^{-1}((-\infty, c]).$$

In the same way we define $\{I = c\}$, $\{a \leq I \leq b\}$, ecc.

Informal statement: $c \in \mathbb{R}$ is a critical point for I if there exists no continuous deformation $\Phi\colon \{I(u) \leq c + \delta\} \to \{I(u) \leq c - \delta\}$, where $\delta > 0$. This means the two sets $\{I(u) \leq c + \delta\}$ and $\{I(u) \leq c - \delta\}$ are topologically different. Lemma 6.16 below makes this statement precise.

Example. Let $X = \mathbb{R}$ and $I(x) := x^3 - 3x$. This function has two critical points $x_1 = -1$, $x_2 = 1$ with critical values $c_1 = I(x_1) = 2$, and $c_2 = I(x_2) = -2$. We investigate now the set $\{I \leq c\}$ for different values of c. We have

$$\begin{aligned}
&c < -2 \Rightarrow && \{I \leq c\} = (-\infty, \alpha_1] \\
&-2 < c < 2 \Rightarrow && \{I \leq c\} = (-\infty, \alpha_1'] \cup [\alpha_2, \alpha_3] \\
&2 < c \Rightarrow && \{I \leq c\} = (-\infty, \alpha_1''].
\end{aligned}$$

Therefore passing through a critical value the number of connected components changes.

Note that this topological characterization is a sufficient but not necessary condition for existence of a critical point. Consider for example the function $I(x) := x^3$. Then $x = 0$ is a critical point but both $\{I(u) \leq \delta\}$ and $\{I(u) \leq -\delta\}$ have only one connected component.

Lemma 6.16 (Deformation lemma)**.** *Let X be a Banach space.*
Assume

- *$I \in C^1(X)$ and satisfies Palais-Smale condition,*
- *$c \in \mathbb{R}$ is a regular value for I.*

Then the following hold.

(i) *$\exists 0 < \delta_0 \leq 1$ and $0 < \sigma \leq 1$ such that*

$$\|I'(u)\|_{X^*} \geq \sigma > 0, \quad \forall u \in B_\delta := \{c - \delta \leq I(u) \leq c + \delta\}, \ \forall 0 < \delta \leq \delta_0.$$

(ii) *Let σ and δ_0 be the constants introduced above. Then $\forall 0 < \varepsilon$ and $0 < \delta < \min\{\delta_0, \frac{\sigma^2}{2}, \varepsilon\}$ $\exists \eta = \eta_{\varepsilon,\delta} \in C([0,1] \times X; X)$ such that*

(a) *$\eta(0, u) = u \ \forall u \in X$,*
(b) *$\forall t \in [0,1]$ the function $\eta(t, \cdot) \colon X \to X$ is a homeomorphism (i.e continuous and invertible with continuous inverse),*
(c) *$u \in A_\varepsilon := \{c - \varepsilon < I < c + \varepsilon\}^c \Rightarrow \eta(t, u) = u \ \forall t \in [0,1]$,*
(d) *the map $t \to I(\eta(t, u))$ is non-increasing $\forall u \in X$,*
(e) *$\eta(0, u) = u \in \{I \leq c + \delta\} \Rightarrow \eta(1, u) \in \{I \leq c - \delta\}$.*

Proof.
(i) By contradiction assume there are three sequences $\delta_n \to 0$, $\sigma_n \to 0$ and $n \to u_n$ such that $u_n \in B_{\delta_n} = \{c - \delta_n \leq I \leq c + \delta_n\}$ and $\|I'(u_n)\|_H \leq \sigma_n \ \forall n$.
From $\delta_n \to 0$ and $c - \delta_n \leq I(u_n) \leq c + \delta_n$, it follows $I(u_n) \to c$. Since $\|I'(u_n)\|_H \leq \sigma_n$ and $\sigma_n \to 0$ we also have $I'(u_n) \to 0$.
By Palais-Smale, there exists a convergent subsequence $u_{n_j} \to u$ in X. Since I and I' are continuous we have

$$I(u) = \lim_{j \to \infty} I(u_{n_j}) = c, \quad I'(u) = \lim_{j \to \infty} I'(u_{n_j}) = 0,$$

which is impossible since c is a regular value for I.

(ii) We will see the proof only in the special case $X = H$ is a Hilbert space and hence we can identify X and X^* via $I'(u)(w) = (v_{I'(u)}, w)_H$. By abuse of notation we will sometimes write $I'(u)$ instead of $v_{I'(u)}$.
We will assume in addition the map $\hat{I}' \colon H \to H$ defined via $\hat{I}'(u) := v_{I'(u)}$ is locally Lipschitz-continuous, i.e. Lipschitz-continuous on bounded sets.

In the general case of a Banach space, we need to replace the map $\hat{I}' \colon H \to H$ by a pseudo-gradient vector field, i.e. a map $W \colon X_r \to X$, with $X_r := \{u \in X \,|\, I'(u) \neq 0\}$ satisfying

- W is locally Lipschitz-continuous,
- W "approximates" $I'(u)$.

One can prove that such a function always exists. See the book by S. Kesavan ("Nonlinear functional analysis") for more details.

Assume now $X = H$ is a Hilbert space and $u \mapsto v_{I'(u)}$ is locally Lipschitz-continuous. To construct η we look for a vector field $V\colon H \to H$ such that the ODE

$$\begin{cases} \partial_t \eta(t,u) = V(\eta(t,u)) \\ \eta(0,u) = u \end{cases} \tag{6.17}$$

has a unique solution $\eta \in C([0,1] \times X; X)$ satisfying $(i) - (v)$.
Note that, if the potential V is bounded and locally Lipschitz, then for each $u \in H$ (6.17) has a unique solution $\eta(\cdot, u) \in C^1(\mathbb{R}; H)$ (by adapting the proof of Picard-Lindelöf). Moreover, the solution satisfies $\eta \in C([0,1] \times H; H)$.

The rest of the proof was not discussed in class

Part 1: construction of V. Assume η is the unique solution of (6.17), and satisfies $(i) - (v)$. We will use now $(i) - (v)$ to impose restrictions on the choice of V.

Step 1. By (iv), the function $t \to I(\eta(t,u))$ must be non-increasing $\forall u \in X$, i.e

$$0 \geq \frac{d}{dt} I(\eta(t,u) = I'(\eta(t,u))(\partial_t \eta(t,u)) = \Big(I'(\eta(t,u)), \partial_t \eta(t,u)\Big)_H = \Big(I'(\eta(t,u)), V(\eta(t,u))\Big)_H$$

where in the last identity we used (6.17). Therefore we define

$$V(u) := -\Phi(u) I'(u),$$

where $\Phi\colon H \to [0, \infty)$ will be chosen later. With this choice

$$\Big(I'(u), V(u)\Big)_H = -\Phi(u) \|I'(u)\|_H^2 \leq 0 \qquad \forall u \in H. \tag{6.18}$$

Step 2. We want V to be bounded i.e.

$$\sup_{u \in H} \|V(u)\|_H = \sup_{u \in H} \Phi(u) \|I'(u)\|_H < \infty.$$

The map $u \mapsto \|I'(u)\|_H$ is not bounded in general hence we neet Φ to compensate. We introduce the cut-off function

$$h\colon \quad \begin{array}{l} [0,\infty) \to \mathbb{R} \\ \tau \to h(\tau) \end{array}, \qquad h(\tau) := \begin{cases} 1 & 0 \leq \tau \leq 1 \\ \frac{1}{\tau} & 1 < \tau. \end{cases}$$

This function is Lipschitz continuous and satisfies $\sup_{\tau \geq 0} \tau h(\tau) \leq 1$. We define now

$$\Phi(u) := \tilde{\Phi}(u)\, h(\|I'(u)\|_H)$$

where $\tilde{\Phi}\colon H \to [0, \infty)$ is a bounded function to be chosen later. With this choice

$$\sup_{u \in H} \Phi(u) \|I'(u)\|_H = \sup_{u \in H} \tilde{\Phi}(u)\, h(\|I'(u)\|_H) \|I'(u)\|_H \leq \sup_{u \in H} \tilde{\Phi}(u) < \infty.$$

Step 3. We have now $V(u) = -\tilde{\Phi}(u)\, h(\|I'(u)\|_H) I'(u)$.

Let $u \in A_\varepsilon = \{c - \varepsilon < I < c + \varepsilon\}^c$. By (iii), we have $\eta(t, u) = u$ $\forall t \in [0, 1]$, hence

$$0 = \partial_t \eta(t, u) = V(\eta(t, u)) = V(u) \qquad \forall t \in [0, 1].$$

Therefore we need $V(u) = 0$ $\forall u \in A_\varepsilon$.

Assume now $c - \delta < I(u) \leq c + \delta$. By (v) we have $I(\eta(1, t)) \leq c - \delta < I(u) = I(\eta(0, u))$. Therefore $\eta(1, u) \neq \eta(0, u)$ and hence $V(\eta(t, u)) = \partial_t \eta(t, u) \neq 0$ is some time interval.

Putting these two conditions together we look for $\tilde{\Phi}\colon H \to [0, \infty)$ such that

- $\tilde{\Phi}$ is bounded and locally Lipschitz,
- $\tilde{\Phi} = 0$ on $A_\varepsilon := \{c - \varepsilon < I < c + \varepsilon\}^c$,
- $\tilde{\Phi} = 1$ on $B_\delta := \{c - \delta \leq I \leq c + \delta\}$.

Since I is continuous, both A_ε and B_δ are closed sets. Since $\delta < \varepsilon$ we have $A_\varepsilon \cap B_\delta = \varnothing$ and hence $\operatorname{dist}(A_\varepsilon, B_\delta) > 0$. We define then

$$\tilde{\Phi}(u) := \frac{\operatorname{dist}(u, A_\varepsilon)}{\operatorname{dist}(u, A_\varepsilon) + \operatorname{dist}(u, B_\delta)}.$$

With this choice $0 \leq \tilde{\Phi} \leq 1$, $\tilde{\Phi}(u)$ is locally Lipschitz, $\tilde{\Phi} = 0$ on A_ε and $\tilde{\Phi} = 1$ on B_δ. This concludes Part 1.

Part 2: construction of η. Let us define V as above. Then V is bounded and locally Lipschitz, and hence for each $u \in H$ (6.17) has a unique solution $\eta(\cdot, u) \in C^1(\mathbb{R}; H)$. Moreover, the solution satisfies $\eta \in C([0, 1] \times H; H)$. We check now that the solution satisfies $(i) - (v)$.

• (i) holds since $\eta(0, u) = u$ by construction.

• We have $\eta(t+s, u) = \eta(t, \eta(s, u))$ $\forall t, s, u$. In particular $\eta(-t, \eta(t, u)) = \eta(t-t, u) = \eta(0, u) = u$. Therefore $\eta(t, \cdot)\colon X \to X$ is invertible with $\eta(t, \cdot)^{-1} = \eta(-t, \cdot)$, and hence (ii) holds.

• To check (iii) take $u \in A_\varepsilon$ and consider the constant function $\tilde{\eta}(t) = u$. Since $V(u) = 0$ this function is a solution of (6.17). By unicity of the solution it follows $\tilde{\eta} = \eta(\cdot, u)$.

• (iv) follows directly from (6.18).

• It remains to check (v). Note that until now $\delta < \varepsilon$ are free parameters. O ur goal is to choose them such that $u \in \{I \leq c + \delta\} \Rightarrow \eta(1, u) \in \{I \leq c - \delta\}$. We distinguish two cases.

Case 1. If $I(u) \leq c - \delta$, we also have $I(\eta(1, u)) \leq c - \delta$, since $t \to I(\eta(t, u))$ is non-increasing.

Case 2: Let $c - \delta < I(u) \leq c + \delta$, i.e $u \in B_\delta$. By contradiction assume $c - \delta < I(\eta(1, u))$. Then $\eta(t, u) \in B_\delta$ $\forall t \in [0, 1]$ and hence, using $\tilde{\Phi} = 1$ on B_δ

$$V(\eta(t, u)) = -h(\|I'(\eta(t, u))\|_H) I'(\eta(t, u)) \qquad \forall t \in [0, 1].$$

Therefore

$$\frac{d}{dt} I(\eta(t, u)) = -h(\|I'(\eta(t, u))\|_H) \|I'(\eta(t, u))\|_H^2 = \begin{cases} -\|I'(\eta(t, u))\|_H^2 & \text{if } \|I'(\eta(t, u))\| \leq 1 \\ -\|I'(\eta(t, u))\|_H & \text{if } \|I'(\eta(t, u))\| > 1. \end{cases}$$

By (i) $\exists 0 < \delta_0 < 1$ and $0 < \sigma \leq 1$ such that $\|I'(u)\| \geq \sigma$ $\forall u \in B_\delta$ with $0 < \delta \leq \delta_0$. We have

$$\frac{d}{dt} I(\eta(t, u)) \leq \left. \begin{cases} -\sigma^2 & \text{if } \|I'(\eta(t, u))\| \leq 1 \\ -1 & \text{if } \|I'(\eta(t, u))\| > 1, \end{cases} \right\} \leq -\sigma^2 \qquad \forall t \in [0, 1].$$

Therefore

$$I(\eta(1,u)) = I(\eta(0,u)) + \int_0^1 \frac{d}{dt} I(\eta(s,u))ds \le c + \delta - \sigma^2 < c - \delta,$$

if we take $0 < \delta < \frac{\sigma^2}{2}$. This contradicts $c - \delta < I(\eta(1,u))$, hence the result holds.

□

[29: 29.01.2024]

[30: 01.02.2024]

Existence of critical points.

Example. Consider $f \in C^1(\mathbb{R}^d;\mathbb{R})$ satisfying

- $f(0) = 0$,
- $\exists r, a > 0$ such that $f(x) \ge a \ \forall |x| = r$,
- $\exists x_0 \in \mathbb{R}^d$ such that $|x_0| > r$ and $f(x_0) \le 0$.

Then $\exists \bar{x} \in \mathbb{R}^d$ such that $0 < |\bar{x}| < |x_0|$, $f(\bar{x}) \ge a$ and $Df(\bar{x}) = 0$.
The next theorem extends this result to general Banach spaces.

Theorem 6.17 (Mountain Pass Theorem)**.** *Let X be a Banach space, and $I \in C^1(X)$ a functional satisfying Palais-Smale condition. Assume in addition*

- $I(0) = 0$,
- $\exists r, a > 0$ *such that* $I(v) \ge a \ \forall \|v\|_X = r$,
- $\exists u_0 \in X$ *such that* $\|u_0\|_X > r$ *and* $I(u_0) \le 0$.

Let $\Gamma := \{\gamma \in C([0,1];X) |\, \gamma(0) = 0, \gamma(1) = u_0\}$ be the set of continuous paths starting in 0 and ending in u_0. Then

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))$$

is a critical value for I, i.e. $\exists \bar{u} \in X$ such that $I(\bar{u}) = c$ and $I'(\bar{u}) = 0$.

Remark 1. We are looking for the path from the valley containing 0 to the valley contining u_0 with the lowest altitude (the mountain pass). The point $\bar{u}$ may be a saddle, a local max or a local min.

Remark 2. γ must cross the mountain range $\|v\| = r$, therefore $\max_{t\in[0,1]} I(\gamma(t)) \ge a \ \forall \gamma \in \Gamma$ and hence $c \ge a$.

Remark 3. Since $I(\bar{u}) = c \ge a > 0$ it holds $\bar{u} \ne 0$ and $\bar{u} \ne u_0$. In particular, if we already know that 0 or u_0 is a critical point, the theorem implies there are at least two solutions for the Euler-Lagrange equation of I.

Proof.
By Remark 2, $c \geq a > 0$. By contradiction, assume c is a regular value. We will show that in this case $\exists \tilde{\gamma} \in \Gamma$ such that $\max_{t\in[0,1]} I(\gamma(t)) < c$ which contradicts the definition of c.

• We construct a candidate for $\tilde{\gamma}$.
c is a regular point, then, by the Deformation Lemma, $\forall \varepsilon > 0\ \exists 0 < \delta < \varepsilon$ and $\tilde{\eta}\colon X \to X$ ($\tilde{\eta} = \eta(1, \cdot)$) such that

- $\tilde{\eta}$ is a homeomorphism,
- $\tilde{\eta}(u) = u\ \forall u \in A_\varepsilon = \{c - \varepsilon < I < c + \varepsilon\}^c$,
- $\tilde{\eta}(\{I \leq c + \delta\}) \subset \{I \leq c - \delta\}$.

In particular we can choose $\varepsilon < 1$ small enough s.t.

$$c - \varepsilon > 0 = I(0) \geq I(u_0)$$

and hence $\tilde{\eta}(0) = 0$ and $\tilde{\eta}(u_0) = u_0$. Since $c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t))$, $\exists \gamma_0 \in \Gamma$ such that

$$\max_{t\in[0,1]} I(\gamma_0(t)) \leq c + \delta. \tag{6.19}$$

We define $\tilde{\gamma} := \tilde{\eta} \circ \gamma_0$.

• We check $\tilde{\gamma}$ is the correct choice.
Indeed, $\tilde{\gamma} \in C([0, 1]; X)$ since both $\tilde{\eta}$ and γ_0 are continuous. Moreover

$$\tilde{\gamma}(0) = \tilde{\eta}(\gamma_0(0)) = \tilde{\eta}(0) = 0, \quad \tilde{\gamma}(1) = \tilde{\eta}(\gamma_0(1)) = \tilde{\eta}(u_0) = u_0,$$

hence $\tilde{\gamma} \in \Gamma$.
By construction $\gamma_0(t) \in \{I \leq c + \delta\}\ \forall t \in [0, 1]$ hence $\tilde{\gamma}(t) = \tilde{\eta}(\gamma_0(t)) \in \{I \leq c - \delta\}\ \forall t \in [0, 1]$, and therefore

$$c \leq \max_{t\in[0,1]} I(\tilde{\gamma}(t)) \leq c - \delta,$$

which gives a contradiction.

□

Example. Set $d = 3$, $\Omega \subset \mathbb{R}^3$ open and bounded, $X := H_0^1(\Omega)$, $f \in L^2(\Omega)$ and consider the functional (6.16), i.e.

$$I(u) := \int_\Omega \Big[\frac{|Du|^2}{2} - \frac{u^3}{3} + fu\Big]\, dx.$$

This functional is well defined and is in $C^1(X)$ with

$$I'(u)(w) = \int_\Omega \big(Du \cdot Dw - u^2 w + fw\big)\, dx.$$

We show now that I satisfies Palais-Smale condition. Let $n \to u_n \in H_0^1(\Omega)$ be a sequence such that

$$I(u_n) \to c \in \mathbb{R}, \quad I'(u_0) \to 0.$$

Our goal is to find a strongly convergent subsequence. For this purpose we show first that the sequence is bounded, and hence, since X is reflexive, there is a weakly convergent subsequence

$u_{n_j} \rightharpoonup u$. Since $H^1_0(\Omega) \subset\subset L^2(\Omega)$ this implies $u_{n_j} \to u$ strongly in $L^2(\Omega)$. The last step is to prove that also $Du_{n_j} \to Du$ strongly in $L^2(\Omega)$.

• We prove that the sequence $n \to u_n$ is bounded. Since I is not coercive, we will use instead the exact expressions for $I(u_n)$ and $I'(u_n)$. We have, by direct computation

$$I(u_n) = \frac{1}{2}\|Du_n\|^2_{L^2} - \frac{1}{3}\int_\Omega u_n^3 dx + (f, u_n)_{L^2} \tag{6.20}$$

$$I'(u_n)(u_n) = \|Du_n\|^2_{L^2} - \int_\Omega u_n^3 dx + (f, u_n)_{L^2}. \tag{6.21}$$

From the first equation we get

$$-\int_\Omega u_n^3 dx = 3\left[I(u_n) - \frac{1}{2}\|Du_n\|^2_{L^2} - (f, u_n)_{L^2}\right].$$

Inserting this into (6.21) we get

$$I'(u_n)(u_n) = -\frac{1}{2}\|Du_n\|^2_{L^2} + 3I(u_n) - 2(f, u_n)_{L^2}$$

and hence, using Poincaré,

$$\begin{aligned}
C\|u_n\|^2_{H^1_0} \leq \frac{1}{2}\|Du_n\|^2_{L^2} &= 3I(u_n) - I'(u_n)(u_n) - 2(f, u_n)_{L^2} \\
&\leq 3I(u_n) + \|I'(u_n)\|_{op}\|u_n\|_{H^1_0} + 2\|f\|_{L^2}\|u_n\|_{L^2} \\
&\leq 3|I(u_n)| + \|u_n\|_{H^1_0}\left[\|I'(u_n)\|_{op} + 2\|f\|_{L^2}\right]
\end{aligned}$$

Since $I(u_n) \to c$ and $I'(u_n) \to 0$ the two sequences $n \mapsto |I(u_n)|$ and $n \mapsto \|I'(u_n)\|_{op}$ are bounded and therefore

$$C\|u_n\|^2_{H^1_0} \leq \alpha + \beta\|u_n\|_{H^1_0},$$

where $\alpha = 3\sup_n |I(u_n)|$ and $\beta = 2\|f\|_{L^2} + \sup_n \|I'(u_n)\|_{op}$. It follows that the sequence $n \to u_n$ is bounded ad hence, since X is reflexive, there is a weakly convergent subsequence $u_{n_j} \rightharpoonup u$. Since $H^1_0(\Omega) \subset\subset L^p(\Omega)$ $\forall p < 6$ this implies $u_{n_j} \to u$ strongly in $L^p(\Omega)$ $\forall p < 6$. In particular $u_{n_j} \to u$ in $L^2(\Omega)$

• We prove $Du_{n_j} \to Du$ strongly in $L^2(\Omega)$.
Remember that

$$I'(u_{n_j})(w) = (Du_{n_j}, Dw)_{L^2} - \int_\Omega u^2_{n_j} w\, dx + (f, w)_{L^2}.$$

Since $I'(u_{n_j}) \to 0$ we have $I'(u_{n_j})(w) \to 0$. Since $Du_{n_j} \rightharpoonup Du$ we have $(Du_{n_j}, Dw)_{L^2} \to (Du, Dw)_{L^2}$. Finally since $u_{n_j} \to u$ in $L^4(\Omega)$ we have

$$\int_\Omega |u^2_{n_j} - u^2|\,|w|\, dx \leq \|u_n - u\|_{L^4}\|u_n + u\|_{L^4}\|w\|_{L^2} \to 0.$$

Putting all this together we obtain

$$0 = (Du, Dw)_{L^2} - \int_\Omega u^2 w\, dx + (f, w)_{L^2} \qquad \forall w \in H^1_0(\Omega). \tag{6.22}$$

In particular, setting $w = u$,

$$0 = \|Du\|^2_{L^2} - \int_\Omega u^3\, dx + (f, u)_{L^2}. \tag{6.23}$$

From (6.21) we obtain, for all j,

$$\|Du_{n_j}\|_{L^2}^2 = I'(u_{n_j})(u_{n_j}) + \int_\Omega u_{n_j}^3\,dx - (f, u_{n_j})_{L^2}.$$

Since $I'(u_{n_j}) \to 0$ and the sequence $j \mapsto u_{n_j}$ is bounded we have $I'(u_{n_j})(u_{n_j}) \to 0$. Since $u_{n_j} \to u$ in $L^3(\Omega)$ we have $\int_\Omega u_{n_j}^3\,dx \to \int_\Omega u^3\,dx$. Since $u_{n_j} \to u$ in $L^2(\Omega)$ we have $(f, u_{n_j})_{L^2} \to (f, u)$. Putting all this together,and using (6.23) we obtain

$$\lim_{j\to\infty} \|Du_{n_j}\|_{L^2}^2 = 0 + \int_\Omega u^3\,dx - (f, u) = \|Du\|_{L^2}^2. \tag{6.24}$$

Since X is a Hilbert space $Du_{n_j} \rightharpoonup Du$ and $\|Du_{n_j}\|_{L^2} \to \|Du\|_{L^2}$ implies $Du_{n_j} \to Du$ in $L^2(\Omega)$, which concludes the proof.

As an example of application consider the equation

$$\begin{cases} -\Delta u = u^2 - f & \text{in } \Omega \subset \mathbb{R}^3 \\ u_{|\partial\Omega} = 0 \end{cases} \tag{6.25}$$

with $f \in L^2(\Omega)$. In weak formulation this PDE becomes

$$\int_\Omega [Du \cdot Dw - u^2 w + fw]\,dx = 0 \qquad \forall w \in H_0^1(\Omega).$$

Therefore a weak solution of (6.25) is a critical point for the functional I defined in (6.16). Assume now, Ω is connected, $f \in C^\infty(\overline{\Omega}; [0, \infty))$ and $\exists x_0 \in \Omega$ such that $f(x_0) > 0$. By Lemma 3.9 (sub-supersolution method) there exists at least one non-positive weak solution $u_0 \in H_0^1(\Omega; (-\infty, 0])$. We will use now the Mountain Pass theorem to show that there is at least a second weak solution.

In order to apply the Mountain Pass theorem, we replace $u = 0$ with u_0 (point in the first valley) and $I(0) = 0$ with the value of $I(u_0)$.

Claim. $\exists r, a > 0$ and $u_1 \in H_0^1(\Omega)$ such that

(i) $\forall \|u - u_0\|_{H_0^1} = r$ it holds $I(u) \geq I(u_0) + a$,

(ii) $\|u_1 - u_0\|_{H_0^1} > r$ and $I(u_1) \leq I(u_0)$.

Consequence. By the Mountain Pass theorem, there exists $\bar{u} \in H_0^1$ critical point for I such that $I(\bar{u}) \geq I(u_0) + a$. Therefore $\bar{u} \neq u_0$ and hence there are at least two weak solutions for (6.25).

Proof of the Claim.
• If $\|u - u_0\|_{H_0^1} = r$ then $u = u_0 + rw$ with $\|w\|_{H_0^1} = 1$. We have

$$\begin{aligned} I(u) - I(u_0) = I(u_0 + rw) - I(u_0) &= r\left[(Du_0, Dw)_{L^2} - \int_\Omega u_0^2 w\,dx + (f, w)_{L^2}\right] \\ &+ r^2\left[\frac{1}{2}\|Dw\|_{L^2}^2 - \int_\Omega u_0 w^2\,dx\right] - \frac{1}{3}r^3 \int_\Omega w^3\,dx. \end{aligned}$$

Since u_0 is a critical point we have $(Du_0, Dw)_{L^2} - \int_\Omega u_0^2 w\,dx + (f, w)_{L^2} = 0$ hence

$$I(u) - I(u_0) = r^2\left[\frac{1}{2}\|Dw\|_{L^2}^2 - \int_\Omega u_0 w^2\,dx\right] - \frac{1}{3}r^3 \int_\Omega w^3\,dx.$$

Since $u_0 \le 0$ a.e we have $-\int_\Omega u_0 w^2\,dx \ge 0$ and hence

$$I(u) - I(u_0) \ge r^2 \frac{1}{2}\|Dw\|_{L^2}^2 - \frac{1}{3} r^3 \|w\|_{L^3}^3$$

By Poincaré and Sobolev inequality we have

$$\|Dw\|_{L^2}^2 \ge C_1 \|w\|_{H_0^1}^2 = C_1, \qquad \|w\|_{L^3}^3 \le C_2 \|w\|_{H_0^1}^3 = C_2,$$

where $C_1, C_2 > 0$, and we used $\|w\|_{H_0^1} = 1$. Therefore

$$I(u) - I(u_0) \ge r^2 \frac{1}{2} C_1 - \frac{1}{3} r^3 C_2 \ge r^2 \frac{1}{4} C_1 > 0$$

for r small enough. So (i) holds.

• We construct u_1. Set $v := u_0 + t\varphi$ where $\varphi \in H_0^1(\Omega; [0,\infty))$ is a given function satisfying $\varphi > 0$ on a set of positive measure, and $t > 0$ is a parameter to choose. We have

$$\|u_1 - u_0\|_{H_0^1} = t\|\varphi\|_{H_0^1} > r$$

if $t > r/\|\varphi\|_{H_0^1}$. Finally

$$I(u_1) - I(u_0) = t^2 \left[\frac{1}{2}\|D\varphi\|_{L^2}^2 - \int_\Omega u_0 \varphi^2\,dx\right] - \frac{1}{3} r^3 \int_\Omega \varphi^3\,dx = t^2\alpha_1 - t^3\alpha_2,$$

where $\alpha_1 \in \mathbb{R}$ and $\alpha_2 > 0$. It follows $I(u_1) - I(u_0) < 0$ if t is large enough. Hence (ii) holds.