Topics in percolation (SS2025)

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Description Percolation is a probabilistic model which has many applications, including in physics, statistical mechanics and network theory. Generally, one is interested in what happens to the connectivity properties of a (infinite) graph, after randomly removing some of the edges or vertices. Another physical motivation is a porous medium, where one wants to understand if a liquid can pass through from top to bottom.

In this seminar we will restrict ourselves to *Bernoulli bond percolation* on the lattice $\mathbb{Z}^d$, where the setting is the following: Let $p \in [0,1]$. We consider $\mathbb{Z}^d$ as an infinite graph with vertex set V given by the lattice points and edges (or bonds)

$$E := \Big\{ \{i,j\} \subset \mathbb{Z}^d \times \mathbb{Z}^d \colon \|i-j\|_1 = 1 \Big\}.$$

Here $\|x\|_1 := \sum_{k=1}^d |x_k|$ denotes the 1-norm. We consider a collection $\{\sigma_e\}_{e\in E}$ of independent Bernoulli random variables taking the value 1 with probability p and 0 with probability $1-p$. We will say that an edge e is *open* if $\sigma_e = 1$, and *closed* otherwise. The corresponding probability space will be $(\{0,1\}^E, \mathcal{F}, \mathbb{P}_p)$, where $\mathcal{F}$ is the sigma algebra of events generated by finitely many edges, and $\mathbb{P}_p$ is the corresponding product measure with each coordinate being a Bernoulli random variable. The associated expectation will be denoted by $\mathbb{E}_p$.

In percolation one is typically interested in the behavior and geometry of *open clusters*, i.e. the connected components of the graph after deleting all closed edges (which is a probabilistic object). Two very important quantities we will study throughout the seminar are: the probability that zero belongs to an infinite cluster $\Theta(p) := \mathbb{P}_p(0 \text{ is in an infinite cluster})$ and the average size of the open cluster containing zero $\chi(p) := \mathbb{E}_p[|C|]$, where $|C|$ is the size of the open cluster at zero.

The goal of this seminar will be to answer some of the most classical questions in percolation, including the following:

- It is clear that $\Theta(0) = 0$ and $\Theta(1) = 1$. Does there exist a unique *critical probability* $0 < p_c < 0$ such that $\Theta(p) = 0$ for $p < p_c$ and $\Theta(p) > 0$ for $p > p_c$?
- What is the behavior of $\chi(p)$ when $p < p_c$? If the open cluster at 0 is finite, can we say more about its size?
- When $p > p_c$, there exists an infinite open cluster. But is it unique?
- Are the function Θ and χ continuous or even differentiable?
- What happens **at** p_c?

While the questions are easy to state, we will discover some of the very rich and clever mathematics needed to answer them, which will result in some very intriguing proof ideas.

While out of scope of this seminar, we remark that there are several alternative models one can study: instead of declaring edges as open or closed, one can attach a random variable to each vertex, which in the i.i.d. case leads to *Bernoulli site percolation*, a model that is in fact very closely related to bond percolation. Furthermore, one can consider other graphs and models with long range interactions, and ask which properties are *universal*. Moreover, there are many ways one could introduce dependence between the random variables, with one prominent example being the so called *random cluster model* (or *FK percolation*).

Prerequisites basic notions on probability measures

Literature Selected parts of

- G. Grimmet: "Percolation", 1999.
- H. Duminil-Copin: "Introduction to Bernoulli percolation", 2017. `https://www.ihes.fr/~duminil/publi/2017percolation.pdf`