

The ubiquity of Gaussian measures and their perturbation

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Description. Perturbations of Gaussian measures (i.e. normal distributed random variables) arise in a wide range of fields of mathematics and theoretical physics. In the context of probability and more precisely statistical mechanics, (discrete) Gaussian measures often take the form

$$d\mu_C(\phi) = \frac{e^{-\frac{1}{2}(\phi, C^{-1}\phi)}}{Z_\Lambda} d\phi^\Lambda$$

where $\phi : \Lambda \rightarrow \mathbb{R}^n$, $\Lambda \subset \subset \mathbb{Z}^d$ is a finite set, $C \in \mathbb{R}_{sym}^{\Lambda \times \Lambda}$ satisfies $C > 0$ as a quadratic form and $(\phi, C^{-1}\phi) := \sum_{ij \in \Lambda} C_{ij}^{-1}(\phi_i, \phi_j)$. The constant $Z_\Lambda > 0$ is a normalization and is often called partition function. A Gaussian measure with local perturbation takes the form

$$d\mu_{V,C}(\varphi) = \prod_{j \in \Lambda} e^{-V(\varphi_j)} d\mu_C(\phi)$$

for some potential $V : \mathbb{R}^n \rightarrow \mathbb{R}$. Depending on the parameters in the problem this measure may be

- almost Gaussian $d\mu_{V,C}(\varphi) \sim d\mu_C(\phi)$
- almost diagonal $d\mu_{V,C}(\varphi) \sim \prod_{j \in \Lambda} e^{-V(\varphi_j)} d\varphi_j$

in the limit of large volume. Two famous examples are

$$V(\varphi) = \lambda|\varphi|^4, \quad V(\varphi) = \lambda(|\varphi|^2 - 1)^2, \quad \lambda > 0.$$

In the second example we converge as $\lambda \rightarrow \infty$ to random variables concentrated on the nonlinear manifold $\mathcal{S}^{n-1}$. The corresponding model is called $O(n)$ model and is one of the most studied problems in statistical mechanics.

The technical challenge is to extract detailed enough information on these measures as the number of variables grows. The goal of this lecture is to introduce some of these tools (such as cluster expansions, contour expansions, convexity methods) in some concrete examples.

Prerequisites. Basic notions in probability and integration. Most of the tools we will consider are essentially analytical and/or combinatorial in nature.

Some literature. Most of the references below are available online.

- S. Friedli, Y. Velenik: *Statistical Mechanics of Lattice Systems: a Concrete Mathematical Introduction*, book; [esp. Chapter 8 (Gaussian Free Field) and 9 ($O(n)$ models)]
- D. Brydges: *A short course on cluster expansions*, Les Houches, Session XLIII, 1984
- D. Brydges *Statistical mechanics and the renormalization group* Lecture notes for the 2009 summer school in probability; [lecture 4 and 5 (lattice Gaussian fields)]
- D. Brydges, J. Imbrie, G. Slade: *Functional integral representation for self-avoiding walks* Probability Surveys (2009)
- R. Bauerschmidt: *A short course on spin systems* Lecture notes Darmstadt 2018;