Summary of the course

Functional integrals involving commuting and anticommuting variables

Margherita Disertori
Bonn University
Winter term 2024/25

These notes are for the students in the class V5B7 at Bonn University, Winter term 2024/2025.

Please send typos and corrections to `disertori@iam.uni-bonn.de`.

Here is a list of books (to be updated/corrected), papers and lecture notes that contain some parts of the material covered in class or may be useful to learn more on the topic. Most of the references given are easily available online.

Gaussian measures in statistical mechanics

- S. Friedli, Y. Velenik: *Statistical Mechanics of Lattice Systems: a Concrete Mathematical Introduction* , book; [Chapter 8 (Gaussian Free Field)]
- D. Brydges *Statistical mechanics and the renormalization group* Lecture notes for the 2009 summer school in probability; [lecture 4 and 5 (lattice Gaussian fields)]
- D. Brydges, J. Imbrie, G. Slade: *Functional integral representation for self-avoiding walks* Probability Surveys (2009)
  [review on (perturbations of) gaussian integrals and their application to probability,]
  [contains also an introduction to Grassman variables]
- R. Bauerschmidt: *A short course on spin systems* Lecture notes Darmstadt 2028;
  [spin systems and Gaussian integrals]

Gaussian measures in quantum mechanics

- Gurau, R., V. Rivasseau, and A. Sfondrini: *Renormalization: an advanced overview.* 2014.
  [Chapter 3.1–3.3 Quantum Field Theory and Gaussian measure]
- Salmhofer, M. *Renormalization, an introduction* 1999, book
  [Chapter 1 Field Theory]
- Mehta, M. L.: *Random matrices*, book
  [the most complete book on random matrix models]
- B. Eynard, Kimura, Ribault: *Random matrices*
  [review on certain types of matrix models and matrix integrals not covered on class]
- M. Aizenman and S. Warzel: *Random operators*, book
  [the most complete book on random operators]

Grassmann variables

- R. Bauerschmidt: *Supersymmetry for probabilists* lecture notes 2018
- F. Wegner: *Supermathematics and its applications in statistical physics* book
- J. Feldman, H. Knörrer, E. Trubowitz: *Fermionic Functional Integrals and the Renormalization Group* lecture notes 1999

Some papers/reviews on random matrix models using supersymmetric representation

- G. Antinucci, L. Fresta and M. Porta: *A supersymmetric hierarchical model for weakly disordered 3d semimetals.* Ann. Henri Poincaré 21.11 (2020), 3499–3574.
- M. Campanino and A. Klein: *A supersymmetric transfer matrix and differentiability of the den- sity of states in the one-dimensional Anderson model* Comm. Math. Phys. 104.2 (1986), 227–241.
- A. Bovier: *The density of states in the Anderson model at weak disorder: a renormalization group analysis of the hierarchical model* J. Statist. Phys. 59.3-4 (1990),
- M. Shcherbina and T. Shcherbina: *Universality for 1d random band matrices.* Comm. Math. Phys. 385.2 (2021), 667–716.
- M. Disertori, H. Pinson and T. Spencer: *Density of states for random band matrices.* Comm. Math. Phys. 232.1 (2002), 83–124.
- M. Disertori: *Density of states for GUE through supersymmetric approach.* Reviews in Mathematical Physics, 16(09):1191–1225, 2004
- T. Spencer: *Random banded and sparse matrices* in: The Oxford handbook of random matrix theory. Oxford Univ. Press, Oxford, 2011, pp. 471–488.

Some papers on random walks with memory

- G. Kozma: *Reinforced random walk*: https://arxiv.org/abs/1208.0364
- F Merkl, S. Rolles: *Linearly edge-reinforced random walks* IMS Lecture Notes–Monograph Series Dynamics & Stochastics Vol. 48 (2006) 66–77
- C. Sabot, P. Tarrés: *Edge-reinforced random walk, vertex-reinforced jump process and the supersymmetric hyperbolic sigma model* J. Eur. Math. Soc. 17, 2353–2378

Contents

1 Preliminary definitions and results

1.1 Introduction

Our goal in this lecture is to study integrals in $N \gg 1$ variables. These can be represented as

$$I_N = \int_{\mathcal{M}^N} d\rho_N(\varphi) f(\varphi)$$

where $\mathcal{M}$ is a manifold, $\varphi\colon \{1,\dots,N\} \to \mathcal{M}$ is the *spin configuration* or *field configuration*, the measure $d\rho_N(\varphi)$ has the form

$$d\rho_N(\varphi) = \prod_{j=1}^{N} d\varphi_j \; e^{-H(\varphi)},$$

and f is the function we wish to average (*the observable*). In the following we often replace $j \in \{1,\dots,N\}$ with $j \in \Lambda \subset\subset \mathbb{Z}^d$ or $j \in$ finite graph.

Examples

• Ising model: $\mathcal{M} = \{-1, 1\}$ with $d\varphi = \frac{1}{2}(\delta_{-1} + \delta_{+1})$;

• $O(n)$ model: $\mathcal{M} = \mathcal{S}^{n-1}$ $n \geq 2$ with $d\varphi = d\mathcal{H}^{n-1}$ the surface measure. For $n = 1$ we recover the Ising model. For $n = 2$ this is also called *XY* or *rotator* model. For $n = 3$ this is called *Heisenberg model*.

• unbounded spin: $\mathcal{M} = \mathbb{R}$ or $\mathbb{C}$. In the first case we use $d\varphi$ the real Lebesgue measure. In the second case we use

$$d\overline{\varphi} d\varphi := 2 \; d\mathrm{Re}\varphi \; d\mathrm{Im}\varphi \tag{1.1}$$

Note that there are other possible conventions in the literature (no 2 factor, an additional i factors). The choice above is motivated by the computation $(dx - idy) \wedge (dx + idy) = 2idxdy$ The above construction can be generalized to $\mathcal{M} = \mathbb{R}^m$ or $\mathbb{C}^m$ with $m \geq 1$.

• more generally we will use $\varphi \in \mathbb{R}^{n\times n}$ or $\varphi \in \mathbb{C}^{n\times n}$, and φ may have in addition anticommuting components.

Anticommuting variables arise naturally in physics (in the context of the quantum description of particles satisfying Fermi statistics) and in mathematics (in the context ot differential forms and graded algebras). Here we will use them principally as a tool to reformulate some integral as a new integral that is hopefully easier to study (duality).

Two easy examples of duality

1. mean field $O(n)$. This example is motivated by statistical mechanics.
Consider $S = (S_1, \dots, S_N)$ with $S_j \in \mathcal{S}^{n-1}$, $n \geq 1$. We write

$$dS := \prod_{j=1}^{N} dS_j$$

where dS_j is the normalized Hausdorff measure $d\mathcal{H}^{n-1}$ i.e.

$$\int_{\mathcal{S}^{n-1}} dS_j = 1.$$

We consider the measure

$$d\rho_N(S) := dS \; e^{\frac{\beta}{N}\sum_{j,k=1}^{N} S_j\cdot S_k} e^{h\sum_{j=1}^{N} S_j\cdot\hat{e}},$$

where $\beta, h > 0$ are parameters and $\hat{e} \in \mathcal{S}^{n-1}$ is a fixed direction. In the case $h > 0$, this measure favours the (unique) configuration $S_j = \hat{e}\ \forall j$. For $h = 0$ the measure favours configurations of the form $S_j = S_k\ \forall j, k$. Our goal is to study properties of this measure as $N \to \infty$. We have the following dual representation (that will be proved later)

$$\int_{(\mathcal{S}^{n-1})^N} d\rho_N(S) = \left(\tfrac{N}{2\pi\beta}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} d\varphi \; e^{-N\left(\frac{|\varphi - h\hat{e}|^2}{2\beta} - \ln I(\varphi)\right)} \tag{1.2}$$

where

$$I(\varphi) = \int_{\mathcal{S}^{n-1}} dS \; e^{S\cdot\varphi} > 0.$$

Note that, while in the first integral we have $O(Nn)$ variables, in the second we only have n variables. The large number N remains only as a large parameter in the exponent and can be used to perform rigorous Laplace method. This dual representation is obtained using real Gaussian integrals.

Example 2 large random matrix. This example is motivated by quantum mechanics (self-adjoint operator describing the energy levels of a large nucleus). Our variable is now a matrix $H \in \mathbb{C}^{N\times N}_{herm}$. By self-adjointness we have $H_{jj} \in \mathbb{R}\ \forall i = 1, \dots, N$ and $H_{ji} = \overline{H}_{ij}\ \forall i < j$. We consider the measure

$$d\rho_N(H) := \prod_{j=1}^{N} dH_{jj} e^{-\frac{N}{2}H_{jj}^2} \prod_{i<j=1}^{N} d\overline{H}_{ij} dH_{ij} e^{-N\overline{H}_{ij}H_{ij}} = dH \; e^{-\frac{N}{2}\operatorname{Tr} H^2}$$

where $dH := \prod_{j=1}^{N} dH_{jj} \prod_{i<j=1}^{N} d\overline{H}_{ij} dH_{ij}$. This measure is real and finite hence becomes a probability measure after normalization. Moreover it is invariant under unitary rotations $H \mapsto U^*HU$ with $U^*U = 1$. The corresponding set of random matrices is called *Gaussian Unitary Ensemble (GUE).* We will use the notation

$$\langle f\rangle_N := \frac{\int_{\mathbb{C}^{N\times N}_{herm}} d\rho_N(H) f(H)}{\int_{\mathbb{C}^{N\times N}_{herm}} d\rho_N(H)}.$$

We are interested in the resolvent $(z - H)^{-1}$. Since $H^* = H$ we have $\sigma(H) \subset \mathbb{R}$ hence $z = E + i\varepsilon \in \rho(H)\ \forall E \in \mathbb{R}$ and $\varepsilon > 0$. Spectral properties of H can be inferred from the two averages

$$\langle (E + i\varepsilon - H)^{-1}_{xy}\rangle_N, \qquad \langle |(E + i\varepsilon - H)^{-1}_{xy}|^2\rangle_N.$$

These integrals are hard to control when $N \gg 1$ since the operation of inverting the matrix creates interactions between the (independently distributed) matrix elements. A dual representation exists but requires introducing anticommuting variables. The following duality does not require them.

$$\langle \frac{1}{\det(E+i\varepsilon-H)} \rangle_N = \frac{\sqrt{N}}{\sqrt{2\pi}} \int_{\mathbb{R}} da \; e^{-N\left(\frac{a^2}{2}+\ln(E+i\varepsilon-a)\right)}$$

On the left we have $O(N^2)$ variables, on the right we have only one real variable. The large number N is a parameter in the new measure and can be used to perform rigorous saddle analysis. This dual representation is obtained using complex Gaussian integrals.

1.2 Gaussian integrals

1.2.1 Scalar Gaussian integral

Theorem 1.1. *Fix $a \in \mathbb{C}$ with Re $a > 0$.*

(i) (Laplace-Fourier transform) For all $u, v \in \mathbb{C}$ we have

$$\begin{aligned} \int_{\mathbb{R}} \frac{d\varphi}{\sqrt{2\pi}} e^{-\frac{1}{2}a\varphi^2} e^{\varphi u} &= \frac{1}{\sqrt{a}} e^{\frac{1}{2}\frac{u^2}{a}} \\ \int_{\mathbb{C}} \frac{d\overline{\varphi} d\varphi}{2\pi} e^{-a\overline{\varphi}\varphi} e^{\overline{\varphi}u+\overline{v}\varphi} &= \frac{1}{a} e^{\frac{\overline{v}u}{a}} \end{aligned} \tag{1.3}$$

where we have taken the unique (complex) square root of a with positive real part. We will use the notation

$$\begin{aligned} d\mu_{\frac{1}{a}}(\varphi) &:= \sqrt{\frac{a}{2\pi}} d\varphi \; e^{-\frac{1}{2}a\varphi^2} \\ d\mu_{\frac{1}{a}}(\overline{\varphi},\varphi) &:= \frac{a}{2\pi} d\overline{\varphi} d\varphi \; e^{-a\overline{\varphi}\varphi} \end{aligned} \tag{1.4}$$

(ii) (integration by parts) We have

$$\begin{aligned} \int_{\mathbb{R}} d\mu_{\frac{1}{a}}(\varphi) \; \varphi \; f(\varphi) &= \frac{1}{a} \int_{\mathbb{R}} d\mu_{\frac{1}{a}}(\varphi) \; f'(\varphi) \\ \int_{\mathbb{C}} d\mu_{\frac{1}{a}}(\overline{\varphi},\varphi) \; \varphi \; f(\overline{\varphi},\varphi) &= \frac{1}{a} \int_{\mathbb{C}} d\mu_{\frac{1}{a}}(\overline{\varphi},\varphi) \; \partial_{\overline{\varphi}} f(\overline{\varphi},\varphi) \\ \int_{\mathbb{C}} d\mu_{\frac{1}{a}}(\overline{\varphi},\varphi) \; \overline{\varphi} \; f(\overline{\varphi},\varphi) &= \frac{1}{a} \int_{\mathbb{C}} d\mu_{\frac{1}{a}}(\overline{\varphi},\varphi) \; \partial_{\varphi} f(\overline{\varphi},\varphi), \end{aligned} \tag{1.5}$$

For all differentiable function f such that the above integrals exist in absolute value.

Remark 1 Setting $u = 0$ or $u = iy$, and $v = -iw$, in (1.3) we obtain the normalization and Fourier transform respectively

$$\begin{aligned} &\int_{\mathbb{R}} \frac{d\varphi}{\sqrt{2\pi}} e^{-\frac{1}{2}a\varphi^2} = \frac{1}{\sqrt{a}}, \qquad \int_{\mathbb{R}} \frac{d\varphi}{\sqrt{2\pi}} e^{-\frac{1}{2}a\varphi^2} e^{i\varphi y} = \frac{1}{\sqrt{a}} e^{-\frac{1}{2}\frac{y^2}{a}} \\ &\int_{\mathbb{C}} \frac{d\overline{\varphi} d\varphi}{2\pi} = \frac{1}{a}, \qquad \int_{\mathbb{C}} \frac{d\overline{\varphi} d\varphi}{2\pi} e^{i(\overline{\varphi}y+\overline{w}\varphi)} = \frac{1}{a} e^{-\frac{\overline{w}u}{a}}. \end{aligned} \tag{1.6}$$

Remark 2 The following identities follow directly from (1.5) is

$$\begin{aligned}
&\int_{\mathbb{R}} d\mu_{\frac{1}{a}}(\varphi)\varphi = 0, \qquad \int_{\mathbb{R}} d\mu_{\frac{1}{a}}(\varphi)\varphi^2 = \frac{1}{a}, \\
&\int_{\mathbb{C}} d\mu_{\frac{1}{a}}(\overline{\varphi},\varphi)\ \varphi = 0 = \int_{\mathbb{C}} d\mu_{\frac{1}{a}}(\overline{\varphi},\varphi)\ \overline{\varphi} \\
&\int_{\mathbb{C}} d\mu_{\frac{1}{a}}(\overline{\varphi},\varphi)\ \varphi^2 = 0 = \int_{\mathbb{C}} d\mu_{\frac{1}{a}}(\overline{\varphi},\varphi)\ \overline{\varphi}^2, \qquad \int_{\mathbb{C}} d\mu_{\frac{1}{a}}(\overline{\varphi},\varphi)\ \varphi\overline{\varphi} = \frac{1}{a}.
\end{aligned} \tag{1.7}$$

Definition 1.2. *Fix $a \in \mathbb{C}$ with Re $a > 0$.*

(i) We call $d\mu_{\frac{1}{a}}(\varphi)$ the normalized Gaussian measure on $\mathbb{R}$ with mean $\int_{\mathbb{R}} d\mu_{\frac{1}{a}}(\varphi)\varphi = 0$ and variance $\int_{\mathbb{R}} d\mu_{\frac{1}{a}}(\varphi)\varphi^2 = \frac{1}{a}$.

(ii) We call $d\mu_{\frac{1}{a}}(\overline{\varphi},\varphi)$ the normalized Gaussian measure on $\mathbb{C}$ with mean $\int_{\mathbb{C}} d\mu_{\frac{1}{a}}(\overline{\varphi},\varphi)\ \varphi = 0 = \int_{\mathbb{C}} d\mu_{\frac{1}{a}}(\overline{\varphi},\varphi)\ \overline{\varphi}$ and covariance $\int_{\mathbb{C}} d\mu_{\frac{1}{a}}(\overline{\varphi},\varphi)\ \varphi\overline{\varphi} = \frac{1}{a}$.

[1: 08.10.2024]

[2: 14.10.2024]

Proof. Proof of Theorem 1.1
Case 1: real variable $\varphi \in \mathbb{R}$.
• For $a > 0$ and $u \in \mathbb{R}$ we argue

$$\int_{\mathbb{R}} \frac{d\varphi}{\sqrt{2\pi}} e^{-\frac{1}{2}a\varphi^2} e^{\varphi u} = e^{\frac{1}{2}\frac{u^2}{a}} \int_{\mathbb{R}} \frac{d\varphi}{\sqrt{2\pi}} e^{-\frac{1}{2}a\left(\varphi - \frac{u}{a}\right)^2} = e^{\frac{1}{2}\frac{u^2}{a}} \int_{\mathbb{R}} \frac{d\varphi}{\sqrt{2\pi}} e^{-\frac{1}{2}a\varphi^2} = \frac{1}{\sqrt{a}} e^{\frac{1}{2}\frac{u^2}{a}},$$

where we used the coordinated change $\varphi \to \varphi - \frac{u}{a}$.

• Fix now $a, u \in \mathbb{C}$ with $\operatorname{Re} a > 0$. The integral is well defined since

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{d\varphi}{\sqrt{2\pi}} \left| e^{-\frac{1}{2}a\varphi^2} e^{\varphi u} \right| = \\
&\quad = \int_{\mathbb{R}} \frac{d\varphi}{\sqrt{2\pi}} e^{-\frac{1}{2}(\operatorname{Re} a)\varphi^2} e^{|\varphi u|} \le e^{\frac{u^2}{2\eta}} \int_{\mathbb{R}} \frac{d\varphi}{\sqrt{2\pi}} e^{-\frac{1}{2}(\operatorname{Re} a - \eta)\varphi^2} = e^{\frac{|u|^2}{2\eta}} \frac{1}{\sqrt{\operatorname{Re} a - \eta}} < \infty,
\end{aligned}$$

where we used $|\varphi u| \le \frac{1}{2}\left(\eta\varphi^2 + \frac{1}{\eta}|u|^2\right)$ with $0 < \eta < \operatorname{Re} a$.

We first prove (1.3) in the case $a > 0$ and $u \in \mathbb{C}$. Consider the two functions $F, G\colon \mathbb{C} \to \mathbb{C}$ defined by

$$F(z) := \int_{\mathbb{R}} \frac{d\varphi}{\sqrt{2\pi}} e^{-\frac{1}{2}a\varphi^2} e^{\varphi z}, \qquad G(z) := \frac{1}{\sqrt{a}} e^{\frac{1}{2}\frac{z^2}{a}}.$$

These two functions are holomorphic on $\mathbb{C}$ (exercise). Moreover $F(u) = G(u)\ \forall u \in \mathbb{R}$, hence by analytic continuation (see Thm. 1.5 below) $F(z) = G(z)\ \forall z \in \mathbb{C}$.

We now prove (1.3) in the general case $a, u \in \mathbb{C}$ with $\operatorname{Re} a > 0$. Fix $u \in \mathbb{C}$ and set

$$\mathbb{C}^+ := \{z \in \mathbb{C} |\ \operatorname{Re} z > 0\}. \tag{1.8}$$

Note that this set is open and connected. We consider the two functions $F, G\colon \mathbb{C}^+ \to \mathbb{C}$ defined by

$$F(z) := \int_{\mathbb{R}} \frac{d\varphi}{\sqrt{2\pi}} e^{-\frac{1}{2}z\varphi^2} e^{\varphi u}, \qquad G(z) := \frac{1}{\sqrt{z}} e^{\frac{1}{2}\frac{u^2}{z}}.$$

These two functions are holomorphic on $\mathbb{C}^+$ (exercise). Moreover $F(a) = G(a)$ $\forall a \in \mathbb{R}$, with $a > 0$ hence by analytic continuation (see Thm. 1.5 below) $F(z) = G(z)$ $\forall z \in \mathbb{C}^+$.

• To prove (1.5) use

$$\varphi e^{-\frac{1}{2}a\varphi^2} = -\frac{1}{a}\partial_\varphi e^{-\frac{1}{2}a\varphi^2}$$

and perform integration by parts.

Case 2: complex variable $\varphi \in \mathbb{C}$.
• The integral is well defined since

$$\begin{aligned}\int_{\mathbb{C}} \frac{d\overline{\varphi}d\varphi}{2\pi}\left|e^{-a\overline{\varphi}\varphi}e^{\overline{\varphi}u+\overline{v}\varphi}\right| &= \int_{\mathbb{R}^2}\frac{dxdy}{\pi}e^{-(\mathrm{Re}\,a(x^2+y^2)}e^{|(x+iy)|(|u|+|v|)}\\ &\leq e^{\frac{(|u|+|v|)^2}{2\eta}}\int_{\mathbb{R}^2}\frac{dxdy}{\pi}e^{-(\mathrm{Re}\,a-\eta/2)(x^2+y^2)} = e^{\frac{(|u|+|v|)^2}{2\eta}}\frac{1}{\mathrm{Re}\,a-\eta/2} < \infty,\end{aligned}$$

where we used again Young's inequality with $0 < \eta < 2\mathrm{Re}\,a$.

• Assume now $a > 0$. To prove (1.3) we argue, using Case 1,

$$\int_{\mathbb{C}}\frac{d\overline{\varphi}d\varphi}{2\pi}e^{-a\overline{\varphi}\varphi}e^{\overline{\varphi}u+\overline{v}\varphi} = \int_{\mathbb{R}^2}\frac{dxdy}{\pi}e^{-a(x^2+y^2)}e^{x(\overline{v}+u)+yi(\overline{v}-u)} = \frac{1}{a}e^{\frac{(\overline{v}+u)^2}{4a}-\frac{(\overline{v}-u)^2}{4a}} = \frac{1}{a}e^{\frac{\overline{v}u}{a}}.$$

• We prove now (1.3) in the general case $a, u, v \in \mathbb{C}$ with $\mathrm{Re}\,a > 0$. Remember the definition of $\mathbb{C}^+$ (1.8) and consider the two functions $F, G\colon \mathbb{C}^+ \to \mathbb{C}$ defined by

$$F(z) := \int_{\mathbb{C}}\frac{d\overline{\varphi}d\varphi}{2\pi}e^{-z\overline{\varphi}\varphi}e^{\overline{\varphi}u+\overline{v}\varphi}, \qquad G(z) := \frac{1}{z}e^{\frac{\overline{v}u}{z}}.$$

These two functions are holomorphic on $\mathbb{C}^+$ (exercise). Moreover $F(a) = G(a)$ $\forall a \in \mathbb{R}$, with $a > 0$ hence by analytic continuation (see Thm. 1.5 below) $F(z) = G(z)$ $\forall z \in \mathbb{C}^+$.

• To prove (1.5) use

$$\varphi e^{-a\overline{\varphi}\varphi} = -\frac{1}{a}\ \partial_{\overline{\varphi}}e^{-a\overline{\varphi}\varphi}, \qquad \overline{\varphi}e^{-a\overline{\varphi}\varphi} = -\frac{1}{a}\ \partial_{\varphi}e^{-a\overline{\varphi}\varphi}$$

and perform integration by parts (exercise).

□

1.2.2 Reminders of complex analysis

Definition 1.3. *Let $U \subset \mathbb{C}$ be an non-empty open set and $f\colon U \to \mathbb{C}$ a function.*

- *f is analytic on U if it has a power series representation at each point i.e. for all $w \in U$ there exists an open ball $B_r(w) \subset U$ and a power series $z \mapsto \sum_{n\geq 0} a_n(z-w)^n$ with convergence radius $\rho \geq r$ such that*

$$f(z) = \sum_{n\geq 0} a_n(z-w)^n \qquad \forall z \in B_r(w).$$

- *f is holomorphic on U if the complex derivative*

$$\partial_z f(z_0) := \lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}$$

exists in all points $z_0 \in U$ and f' defines a continuous function on U

Theorem 1.4 (important properties). *Let $U \subset \mathbb{C}$ be an non-empty open set and $f: U \to \mathbb{C}$ a function. The following statementas hold.*

(i) f is holomorphic on $U \Leftrightarrow f$ is analytic on U

(ii) Let $U_r\{(x,y) \in \mathbb{R}^2 | (x+iy) \in U\}$.
Then, f is holomorphic on $U \Leftrightarrow$ the function $F: U_r \to \mathbb{R}^2$ defined by $(x,y) \mapsto (Ref(x+iy), Imf(x+iy)$ is continuously differentiable and satisfies the Cauchy-Riemann equations

$$\partial_x Ref(x+iy) = \partial_y Imf(x+iy), \qquad \partial_y Ref(x+iy) = -\partial_x Imf(x+iy)$$

$\Leftrightarrow \partial_{\overline{z}} f := \frac{1}{2}(\partial_x + i\partial_y) f = 0.$

(iii) f complex differentiable $\Rightarrow \partial_z f = \frac{1}{2}(\partial_x - i\partial_y) f$.

(iv) f holomorphic on $U \Rightarrow f$ admits infinitely many complex derivatives.

Theorem 1.5 (analytic continuation). *Let $U \subset \mathbb{C}$ be an non-empty connected open set, $f, g: U \to \mathbb{C}$ two functions analytic on U. The following statements are equivalent:*

(i) $f(z) = g(z)\ \forall z \in U$;

(ii) there exists a set $V \subset U$ such that

(a) $f(z) = g(z)\ \forall z \in V$,

(b) V contains infinitely many points and an accumulation point in U;

(iii) there exists a point $z_0 \in U$ such that $f^{(n)}(z_0) = g^{(n)}(z_0)\ \forall n \geq 0$

Note that the complex derivatives are well defined since an analytic function is alwasy holomorphic.

1.2.3 Some properties of matrix spaces

To define Gaussian measures on vectors we replace $a\varphi^2$ (resp. $a\overline{\varphi}\varphi$) with a quadratic form $(\varphi, A\varphi)$ (resp. $(\overline{\varphi}, A\varphi)$). We will need some preliminary facts/definitions on complex-valued matrices. The analog of a real scalar is a self-adjoint matrix. In order to apply analytic continuation we need to parametrize a complex matrix A via variables that become real when A is self-adjoint. This is the content of the next result.

Lemma 1.6 (decomposition of complex matrices).

(i) Every matrix $A \in \mathbb{C}^{N \times N}$ can be decomposed as

$$A = A_1 + iA_2 \tag{1.9}$$

with A_1, A_2 self-adjoint matrices defined by

$$A_1 := \frac{1}{2}(A + A^*), \qquad A_2 := \frac{1}{2i}(A - A^*), \tag{1.10}$$

where $A^* = \overline{A}^t$. We define

$$\operatorname{Re} A := A_1 = \frac{1}{2}(A + A^*), \qquad \operatorname{Im} A := A_2 = \frac{1}{2}(A - A^*). \tag{1.11}$$

(ii) *Every matrix* $A \in \mathbb{C}^{N\times N}$ *can be uniquely identified by the set of* N^2 *complex variables*

$$\begin{aligned} z_{ii} &:= A_{ii} = (A_1)_{ii} + i(A_2)_{ii}, \qquad & i = 1, \dots N \\ z_{ij} &:= \mathrm{Re}\,(A_1)_{ij} + i\mathrm{Re}\,(A_2)_{ij}, \qquad & i < j \\ w_{ij} &:= \mathrm{Im}\,(A_1)_{ij} + i\mathrm{Im}\,(A_2)_{ij}, \qquad & i < j \end{aligned}$$

via the formula

$$A(z,w) = \begin{cases} A_{ii} = z_{ii} & \forall i = 1, \dots N \\ A_{ij} = z_{ij} + iw_{ij} & \forall i < j, \\ A_{ji} = z_{ij} - iw_{ij} & \forall i > j. \end{cases}$$

(iii) *The function* $z, w \mapsto A(z,w)$ *is analytic in each variable separateley, since* $\partial_{\overline{z}_{ij}} A(z,w) = 0 = \partial_{\overline{w}_{ij}} A(z,w)$.

(iv) $A^* = A$ *iff* $z_{ii}, z_{ij}, w_{ij} \in \mathbb{R}$.

Proof. exercise □

Remark 1 Since both A_1, A_2 are self-adjoint we have

$$|(\overline{\varphi}, A\varphi)| = |(\overline{\varphi}, A_1\varphi) + i(\overline{\varphi}, A_2\varphi)| = \sqrt{(\overline{\varphi}, A_1\varphi)^2 + (\overline{\varphi}, A_2\varphi)^2} \geq |(\overline{\varphi}, A_1\varphi)|. \tag{1.12}$$

Remark 2 A complex self-adjoint matrix $A = A^*$ is positive definite as a quadratic form $A > 0$ if

$$(\overline{\varphi}, A\varphi) > 0 \qquad \forall \varphi \in \mathbb{C}^N \setminus 0. \tag{1.13}$$

Since we are in finite dimension this is equivalent to find a number $\lambda > 0$ such that

$$(\overline{\varphi}, A\varphi) \geq \lambda(\overline{\varphi}, \varphi) = \lambda|\varphi|^2 \qquad \forall \varphi \in \mathbb{C}^N \setminus 0. \tag{1.14}$$

In the following we consider the space

$$\mathbb{C}_+^{N\times N} := \{A \in \mathbb{C}^{N\times N} |\ \mathrm{Re}\, A > 0\}. \tag{1.15}$$

Theorem 1.7. *The followig hold.*

(i) $\mathbb{C}_+^{N\times N}$ *is an open connected subset of* $\mathbb{C}^{N\times N}$.

(ii) $A \in \mathbb{C}_+^{N\times N}$ *iff* A *is invertible and* $A^{-1} \in \mathbb{C}_+^{N\times N}$.

Note that since we are in finite dimension all norms are equivalent so we do not need to specify under which norm the spaces are open.

Proof.
• We show that $\mathbb{C}_+^{N\times N}$ is an open subset of $\mathbb{C}^{N\times N}$ with respect to the operator norm.
Fix $A \in \mathbb{C}_+^{N\times N}$. Then $A = A_1 + iA_2$ and there is $\lambda > 0$ such that $(\overline{\varphi}, A_1\varphi) \geq \lambda|\varphi|^2 > 0$ $\forall \varphi \in \mathbb{C}^N \setminus 0$.
Our goal is to find $r > 0$ such that $B \in \mathbb{C}_+^{N\times N}$ for all $B \in \mathbb{C}^{N\times N}$ such that $\|A - B\| < r$.

Using the decomposition $B = B_1 + iB_2$ we have $B \in \mathbb{C}_+^{N\times N}$ iff $B_1 > 0$. We argue, using also (1.12),

$$(\overline{\varphi}, B_1\varphi) = (\overline{\varphi}, A_1\varphi) + (\overline{\varphi}, (B_1 - A_1)\varphi) \geq \lambda|\varphi|^2 - |(\overline{\varphi}, (B_1 - A_1)\varphi)|$$
$$\geq \lambda|\varphi|^2 - |(\overline{\varphi}, (B - A)\varphi)| \geq \lambda|\varphi|^2 - \|B - A\|\ |\varphi|^2 = (\lambda - \|B - A\|)|\varphi|^2.$$

Hence $B_\lambda(A) \subset \mathbb{C}_+^{N\times N}$.

• The set $\mathbb{C}_+^{N\times N}$ is convex and hence connected. Indeed, for all $A, B \in \mathbb{C}_+^{N\times N}$ and $t \in [0, 1]$ we have

$$(\overline{\varphi}, (tA + (1-t)B)_1\varphi) = t(\overline{\varphi}, A_1\varphi) + (1-t)(\overline{\varphi}, B_1\varphi) > 0$$

• Let $A \in \mathbb{C}_+^{N\times N}$. We claim that A is invertible and $A^{-1} \in \mathbb{C}_+^{N\times N}$.
Indeed, assume by contradiction A is not invertible. Then there exists $\varphi \in \mathbb{C}^N$ such that $A\varphi = 0$. But then $(\overline{\varphi}, A\varphi) = 0$ and hence $(\overline{\varphi}, A_1\varphi) = 0$ which contradicts $A_1 > 0$.
To prove $A^{-1} \in \mathbb{C}_+^{N\times N}$ we argue

$$2\ \mathrm{Re}\, A^{-1} = \left(A^{-1} + (A^{-1})^*\right) = \left(A^{-1} + (A^*)^{-1}\right) = \left(A^{-1} - (-A^*)^{-1}\right)$$
$$= (-A^*)^{-1}(-A^* - A)A^{-1} = (A^*)^{-1}(A + A^*)A^{-1},$$

where we used the resolvent identity

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1} = B^{-1}(B - A)A^{-1}.$$

Hence

$$\mathrm{Re}\, A^{-1} = (A^{-1})^*(\mathrm{Re}\, A)A^{-1}.$$

The result now follows from

$$(\overline{\varphi}, \mathrm{Re}\, A^{-1}\varphi) = (\overline{\varphi}, (A^{-1})^*(\mathrm{Re}\, A)A^{-1}\varphi) = (\overline{A^{-1}\varphi}, (\mathrm{Re}\, A)(A^{-1}\varphi)) > 0.$$

• Assume $A \in \mathbb{C}^{N\times N}$ is invertible and $A^{-1} \in \mathbb{C}_+^{N\times N}$. we claim that $A^{-1} \in \mathbb{C}_+^{N\times N}$. This follows from

$$\mathrm{Re}\, A = A^*(\mathrm{Re}\, A^{-1})A.$$

□

1.2.4 Vector Gaussian integral

Theorem 1.8. *Fix $N > 1$*

(i) (Laplace-Fourier transform)

(a) For all $A \in \mathbb{C}_+^{N\times N}$ with the additional condition $A^t = A$ it holds

$$\int_{\mathbb{R}^N} \prod_{j=1}^{N} \frac{d\varphi_j}{\sqrt{2\pi}}\ e^{-\frac{1}{2}(\varphi, A\varphi)} e^{(\varphi, v)} = \frac{1}{\sqrt{\det A}} e^{\frac{1}{2}(v, A^{-1}v)} \tag{1.16}$$

where $\sqrt{\det A}$ is defined via (see also Remark 3 below)

$$\sqrt{\det A} = \sqrt{\det A_1}\sqrt{\det(1 + iA_1^{-\frac{1}{2}}A_2A_1^{-\frac{1}{2}})}.$$

(b) For all $A \in \mathbb{C}_+^{N\times N}$, $v, w \in \mathbb{C}^N$ it holds

$$\int_{\mathbb{C}^N} \prod_{j=1}^{N} \frac{d\overline{\varphi}_j d\varphi_j}{2\pi} e^{-(\overline{\varphi},A\varphi)} e^{(\overline{\varphi}v)+(\overline{w},\varphi)} = \frac{1}{\det A} e^{(\overline{w},A^{-1}v)} \tag{1.17}$$

Setting $C := A^{-1}$, we will use the notation

$$d\mu_C(\varphi) := \sqrt{\det A} \, \prod_{j=1}^{N} \frac{d\varphi_j}{\sqrt{2\pi}} \, e^{-\frac{1}{2}(\varphi,A\varphi)} \tag{1.18}$$

$$d\mu_C(\overline{\varphi},\varphi) := \det A \, \prod_{j=1}^{N} \frac{\overline{\varphi}_j d\varphi_j}{2\pi} \, e^{-(\overline{\varphi},A\varphi)}$$

(ii) (integration by parts) We have

$$\int_{\mathbb{R}^N} d\mu_C(\varphi) \; \varphi_j \; f(\varphi) = \sum_{k=1}^{N} C_{jk} \int_{\mathbb{R}^N} d\mu_C(\varphi) \; \partial_{\varphi_k} f(\varphi) \tag{1.19}$$

$$\int_{\mathbb{R}^N} d\mu_C(\overline{\varphi},\varphi) \; \varphi_j \; f(\overline{\varphi},\varphi) = \sum_{k=1}^{N} C_{jk} \int_{\mathbb{R}^N} d\mu_C(\varphi) \; \partial_{\overline{\varphi}_k} f(\overline{\varphi},\varphi)$$

$$\int_{\mathbb{R}^N} d\mu_C(\overline{\varphi},\varphi) \; \overline{\varphi}_j \; f(\overline{\varphi},\varphi) = \sum_{k=1}^{N} C_{jk} \int_{\mathbb{R}^N} d\mu_C(\varphi) \; \partial_{\varphi_k} f(\overline{\varphi},\varphi)$$

For all differentiable function f such that the above integrals exist in absolute value.

Remark 1 Setting $v = 0$ we obtain the normalization

$$\int_{\mathbb{R}^N} \prod_{j=1}^{N} \frac{d\varphi_j}{\sqrt{2\pi}} \, e^{-\frac{1}{2}(\varphi,A\varphi)} = \frac{1}{\sqrt{\det A}}, \qquad \int_{\mathbb{C}^N} \prod_{j=1}^{N} \frac{d\overline{\varphi}_j d\varphi_j}{2\pi} e^{-(\overline{\varphi},A\varphi)} = \frac{1}{\det A}.$$

Remark 2 The following identities follow directly from (1.5) is

$$\int_{\mathbb{R}^N} d\mu_C(\varphi)\varphi_j = 0, \qquad \int_{\mathbb{R}} d\mu_C(\varphi)\varphi_j\varphi_k = C_{jk} \tag{1.20}$$

$$\int_{\mathbb{C}^N} d\mu_C(\overline{\varphi},\varphi)\varphi_j = 0 = \int_{\mathbb{C}^N} d\mu_C(\overline{\varphi},\varphi)\overline{\varphi}_j$$

$$\int_{\mathbb{R}} d\mu_C(\overline{\varphi},\varphi)\varphi_j\varphi_k = \int_{\mathbb{R}} d\mu_C(\overline{\varphi},\varphi)\overline{\varphi}_j\overline{\varphi}_k$$

$$\int_{\mathbb{R}} d\mu_C(\overline{\varphi},\varphi)\varphi_j\overline{\varphi}_k = C_{jk}$$

[2: 14.10.2024]

[3: 18.10.2024]

Remark 3 For all $A \in \mathbb{C}^{N\times N}_{+,sym}$ we have

$$A = A_1 + iA_2 \qquad A_1, A_2 \in \mathbb{R}^{N\times N}_{sym},\ A_1 > 0.$$

Since A_1 is real, symmetric and positive we have

$$A_1 = U_1^t \hat\lambda U_1, \quad U_1 \in O(N),\ \hat\lambda = \operatorname{diag}(\lambda_1, \dots \lambda_N), \quad \lambda_k > 0\ \forall k = 1, \dots, N.$$

On the other hand U_1 does not diagonalize the real symmetric matrix A_2 unless $[A_1, A_2] = 0$ (i.e. the matrices cannot be diagonalized simultaneously using the same orthogonal matrix). Since $\hat\lambda > 0$ we can define

$$\hat\lambda^{\frac{1}{2}} = \operatorname{diag}(\lambda_1^{\frac{1}{2}}, \dots \lambda_N^{\frac{1}{2}}).$$

The matrix $\hat\lambda^{\frac{1}{2}}$ is positive and invertible. We argue

$$\det A = \det U_1^t(\hat\lambda + iU_1A_2U_1^t)U_1^t = \det(\hat\lambda + iU_1A_2U_1^t) = \det\hat\lambda \det(1 + i\hat\lambda^{-\frac{1}{2}}U_1A_2U_1^t\hat\lambda^{-\frac{1}{2}}).$$

The matrix $\hat\lambda^{-\frac{1}{2}}U_1A_2U_1^t\hat\lambda^{-\frac{1}{2}}$ is real and symmetric, hence

$$\hat\lambda^{-\frac{1}{2}}U_1A_2U_1^t\hat\lambda^{-\frac{1}{2}} = U_2^t\hat\mu U_2$$

for some $U_2 \in O(N)$ and real diagonal matrix $\hat\mu$. Inserting this in the determinant we get

$$\det A = \prod_{j=1}^N \lambda_j \ \prod_{j=1}^N (1 + i\mu_j)$$

Since $\operatorname{Re}(1 + i\mu_j) = 1 > 0$ we can define

$$\sqrt{\det A} = \prod_{j=1}^N \sqrt{\lambda_j} \prod_{j=1}^N \sqrt{1 + i\mu_j}$$

where in each term we take the unique root with positive real part.

Proof. Proof of Theorem 1.8

• The assumptions on A imply the integrals are well defined (exercise)

Case 1: real variables $\varphi \in \mathbb{R}^N$.

• To prove (1.16) we argue, using Remark 3 above

$$A = A_1 + A_2 = U_1^t \left(\hat\lambda + iU_1A_2U_1^t\right) U_1$$

Inserting this in the quadratic form we obtain

$$\begin{aligned}(\varphi, A\varphi) &= \Big((U_1\varphi), \left(\hat\lambda + iU_1A_2U_1^t\right)(U_1\varphi)\Big) \\ &= \Big((\hat\lambda^{\frac{1}{2}}U_1\varphi), \left(1 + i\hat\lambda^{-\frac{1}{2}}U_1A_2U_1^t\hat\lambda^{-\frac{1}{2}}\right)(\hat\lambda^{\frac{1}{2}}U_1\varphi)\Big) \\ &= \Big((\hat\lambda^{\frac{1}{2}}U_1\varphi), \left(1 + iU_2^t\hat\mu U_2\right)(\hat\lambda^{\frac{1}{2}}U_1\varphi)\Big).\end{aligned}$$

We argue via three coordinate changes. In a first step we perform the rotation $\tilde{\varphi} := U_1\varphi$. Since this is an isometry there is no Jacobian. In a second step we perform the scaling $\tilde{\varphi}'_j := \lambda_j^{\frac{1}{2}}\tilde{\varphi}_j$, $\forall j \in \Lambda$. Finally we perform the rotation $\tilde{\varphi}'' := U_2\tilde{\varphi}'$. We obtain

$$\int_{\mathbb{R}^N} \prod_{j=1}^N \frac{d\varphi_j}{\sqrt{2\pi}}\ e^{-\frac{1}{2}(\varphi,A\varphi)} e^{(\varphi,v)} = \int_{\mathbb{R}^N} \prod_{j=1}^N \frac{d\varphi_j}{\sqrt{2\pi}}\ e^{-\frac{1}{2}(\varphi,\left(\hat{\lambda}+iU_1A_2U_1^t\right)\varphi)} e^{(\varphi,U_1v)}$$

$$= \frac{1}{\sqrt{\prod_j \lambda_j}} \int_{\mathbb{R}^N} \prod_{j=1}^N \frac{d\varphi_j}{\sqrt{2\pi}}\ e^{-\frac{1}{2}(\varphi,\left(1+i\hat{\lambda}^{-\frac{1}{2}}U_1A_2U_1^t\hat{\lambda}^{-\frac{1}{2}}\right)\varphi)} e^{(\varphi,\hat{\lambda}^{-\frac{1}{2}}U_1v)}$$

$$= \frac{1}{\sqrt{\prod_j \lambda_j}} \int_{\mathbb{R}^N} \prod_{j=1}^N \frac{d\varphi_j}{\sqrt{2\pi}}\ e^{-\frac{1}{2}(U_2\varphi,(1+i\hat{\mu})U_2\varphi)} e^{(U_2\varphi,U_2\hat{\lambda}^{-\frac{1}{2}}U_1v)}$$

$$= \frac{1}{\sqrt{\prod_j \lambda_j}} \prod_{j=1}^N \int_{\mathbb{R}} \frac{d\varphi_j}{\sqrt{2\pi}} e^{-\frac{1}{2}(1+i\mu_j)\varphi_j^2} e^{\varphi_j(U_2\hat{\lambda}^{-\frac{1}{2}}U_1v)_j}$$

$$= \frac{1}{\sqrt{\prod_{j=1}^N \lambda_j(1+i\mu_j)}} e^{\frac{1}{2}\sum_{j=1}^N \frac{((U_2\hat{\lambda}^{-\frac{1}{2}}U_1v)_j^2}{1+i\mu_j}} = \frac{1}{\sqrt{\det A}} e^{\frac{1}{2}(v,A^{-1}v)}.$$

• To prove (1.19) in the real case use

$$\varphi_j e^{-\frac{1}{2}(\varphi,A\varphi)} = -\sum_k (A^{-1})_{jk}\partial_{\varphi_k} e^{-\frac{1}{2}(\varphi,A\varphi)}$$

and perform integration by parts.

Case 1: complex variables $\varphi \in \mathbb{C}^N$.

• We prove (1.17) in the case $A = A^*$. Then there is a complex unitary matrix $U \in U(N)$ and a real diagonal matrix $\hat{\lambda} = \operatorname{diag}(\lambda_1, \dots, \lambda_N)$ such that $A = U^*\hat{\lambda}U$. Inserting this in the quadratic form we obtain

$$(\overline{\varphi}, A\varphi) = \sum_{j=1}^N \lambda_j |(U\varphi)_j|^2.$$

We perform the coordinated change (complex rotation) $\tilde{\varphi} := U\varphi$. Since this is an isometry there is no Jacobian. We obtain

$$\int_{\mathbb{C}^N} \prod_{j=1}^N \frac{d\overline{\varphi}_j d\varphi_j}{2\pi} e^{-(\overline{\varphi},A\varphi)} e^{(\overline{\varphi}v)+(\overline{w},\varphi)} = \prod_{j=1}^N \int_{\mathbb{C}} \frac{d\overline{\varphi}_j d\varphi_j}{2\pi} e^{-\frac{1}{2}\lambda_j|\varphi_j|^2} e^{\overline{\varphi}_j(Uv)_j + \overline{Uw}_j\varphi_j}$$

$$= \prod_{j=1}^N \frac{1}{\prod_{j=1}^N \lambda_j} e^{\frac{1}{2}\sum_{j=1}^N \overline{Uw}_j \frac{1}{\lambda_j}(Uv)_j} = \frac{1}{\det A} e^{\frac{1}{2}(\overline{w},A^{-1}v)}.$$

• We prove (1.17) in the general case $A \neq A^*$. We can repeat the strategy used in the case of a real matrix or argue via analytic deformation as follows.
We start by fixing all variables in A to be real except z_{11}. Let $A(z)$ be the matrix obtained where all z_{ii} (except z_{11}) and all $z_{ij,w_{ij}}$ are real and set

$$U := \{z \in \mathbb{C} | A_1(z) > 0\}.$$

Using Theorem 1.7 above we can show that U is open and connected. Consider the two functions

$$F(z) := \int_{\mathbb{C}^N} \prod_{j=1}^{N} \frac{d\overline{\varphi}_j d\varphi_j}{2\pi} e^{-(\overline{\varphi}, A(z)\varphi)} e^{(\overline{\varphi}v)+(\overline{w},\varphi)}, \qquad G(z) := \frac{1}{\det A(z)} e^{\frac{1}{2}(\overline{w}, A(z)^{-1}v)}.$$

These functions are analytic on U and coincide on $U \cap \mathbb{R}$, hence they coincide on U. Repeat making one variable at a time complex.

- To prove (1.19) in the complex case use

$$\varphi_j e^{-(\overline{\varphi}, A\varphi)} = -\sum_k (A^{-1})_{jk} \partial_{\overline{\varphi}_k} e^{-(\overline{\varphi}, A\varphi)}, \qquad \overline{\varphi}_j e^{-(\overline{\varphi}, A\varphi)} = -\sum_k (A^{-1})_{jk}\ \partial_{\varphi_k} e^{-(\overline{\varphi}, A\varphi)}$$

□

Theorem 1.9 (Sum of Gaussian variables)**.**

(i) Let $A_1, A_2 \in \mathbb{C}^{N\times N}_{+,sym}$ and set C_1, C_2 the corresponding inverse. We have

$$\int_{\mathbb{R}^{2N}} d\mu_{C_1}(\varphi) d\mu_{C_2}(\varphi')\ f(\varphi + \varphi') = \int_{\mathbb{R}^N} d\mu_{C_1 + C_2}(\varphi) f(\varphi)$$

for all functions f such that the integrals above are well-defined.

(ii) Let $A_1, A_2 \in \mathbb{C}^{N\times N}_{+}$ and set C_1, C_2 the corresponding inverse.

$$\int_{\mathbb{C}^{2N}} d\mu_{C_1}(\overline{\varphi}, \varphi) d\mu_{C_2}(\overline{\varphi}', \varphi') f(\varphi + \varphi') = \int_{\mathbb{C}^N} d\mu_{C_1 + C_2}(\overline{\varphi}, \varphi) f(\varphi)$$

for all functions f such that the integrals above are well-defined.

Proof. For a function $f \in L^2(\mathbb{R}^N)$ it is sufficient to consider $f = e^{i\varphi w}$ and then use the Fourier transform. For more general functions use the coordinate change

$$\varphi + \varphi' = u, \qquad \varphi - \varphi' = v$$

and perform the (Gaussian) integral with respect to v explicitely. The same argument works in the case of complex variables. □

We go back to the duality examples given in the introduction and use Gaussiam integrals to prove the formulas.

1.2.5 Example 1: spin $O(n)$ model

We consider the function $H_N : (\mathcal{S}^{n-1})^N \to \mathbb{R}$ defined via

$$H_N(S) = -\frac{1}{2N} \sum_{jk=1}^{N} S_j \cdot S_k - \frac{h}{\beta} \sum_{j=1}^{N} S_j \cdot \hat{e}$$

where $h, \beta > 0$ and $\hat{e} \in \mathcal{S}^{n-1}$ is a fixed direction. Set

$$dS := \prod_{j=1}^{N} dS_j$$

with dS_j the normalized Hausdorff measure $\mathcal{H}^{n-1}$ on $\mathcal{S}^{n-1}$. We are interested in the measure

$$dS\ e^{-\beta H_N(S)} = dS\ e^{\frac{\beta}{2N}\sum_{jk=1}^N S_j\cdot S_k} e^{h\sum_{j=1}^N S_j\cdot\hat{e}}$$

for N large.

Lemma 1.10. *We have*

$$\int_{(\mathcal{S}^{n-1})^N} dS e^{-\beta H(S)} = \left(\tfrac{N}{2\pi\beta}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} d\varphi\ e^{-N\left(\frac{|\varphi-h\hat{e}|^2}{2\beta}|-\ln I(\varphi)\right)} \tag{1.21}$$

where $d\varphi := \prod_{j=1}^n d\varphi_j$ *and*

$$I(\varphi) := \int_{\mathcal{S}^{n-1}} dS\ e^{S\cdot\varphi} > 0.$$

Proof.
We can reorganize $\beta H(S)$ as follows

$$-\beta H(S) = \frac{\beta}{2N}\sum_{jk} S_j\cdot S_k + h\sum_j S_j\cdot\hat{e} = \frac{\beta}{2N}|\sum_{j=1}^N S_j|^2 + h\left(\sum_{j=1}^N S_j\right)\cdot\hat{e}$$

We argue

$$e^{-\beta H(S)} = e^{\frac{\beta}{2N}|\sum_{j=1}^N S_j|^2} e^{h(\sum_{j=1}^N S_j)\cdot\hat{e}} = e^{h(\sum_{j=1}^N S_j)\cdot\hat{e}}\int_{\mathbb{R}^n} d\mu_{\frac{\beta}{N}\mathrm{Id}}(\varphi) e^{\sum_{j=1}^N S_j\cdot\varphi}$$

where $d\mu_{\frac{\beta}{N}\mathrm{Id}}(\varphi)$ is the vector Gaussian measure on $\mathbb{R}^n$ with mean zero and covariance $C = \frac{\beta}{N}\mathrm{Id}$
Inserting this in the integral we obtain

$$\begin{aligned}
\int_{(\mathcal{S}^{n-1})^N} dS e^{-\beta H(S)} &= \int_{(\mathcal{S}^{n-1})^N} dS\ \int_{\mathbb{R}^n} d\mu_{\frac{\beta}{N}\mathrm{Id}}(\varphi)\ e^{\sum_j S_j\cdot\varphi} e^{h\sum_j S_j\cdot\hat{e}} \\
&= \int_{\mathbb{R}^n} d\mu_{\frac{\beta}{N}\mathrm{Id}}(\varphi)\prod_j\int_{\mathcal{S}^{n-1}} dS_j\ e^{S_j\cdot(\varphi+h\hat{e})} = \int_{\mathbb{R}^n} d\mu_{\frac{\beta}{N}\mathrm{Id}}(\varphi)\ e^{N\ln I(\varphi+h\hat{e})} \\
&= \left(\tfrac{N}{\beta}\right)^{\frac{n}{2}}\int_{\mathbb{R}^n}\prod_{j=1}^n\frac{d\varphi_j}{\sqrt{2\pi}}\ e^{-N\left(\frac{|\varphi|^2}{2\beta}|-\ln I(\varphi+h\hat{e})\right)} = \left(\tfrac{N}{\beta}\right)^{\frac{n}{2}}\int_{\mathbb{R}^n}\prod_{j=1}^n\frac{d\varphi_j}{\sqrt{2\pi}}\ e^{-N\left(\frac{|\varphi-h\hat{e}|^2}{2\beta}|-\ln I(\varphi)\right)}
\end{aligned}$$

where we can exchange the integrals because we are integrating positive functions.

□

1.2.6 Example 2: average of the inverse determinant for GUE

We consider the measure on $\mathbb{C}_+^{N\times N}$

$$\prod_{j=1}^N dH_{jj} e^{-\frac{N}{2}H_{jj}^2}\prod_{i<j=1}^N d\overline{H}_{ij}dH_{ij}e^{-N\overline{H}_{ij}H_{ij}} = dH\ e^{-\frac{N}{2}\mathrm{Tr}\,H^2}$$

where $dH := \prod_{j=1}^N dH_{jj}\prod_{i<j=1}^N d\overline{H}_{ij}dH_{ij}$. We will use the notation

$$\langle f\rangle_N := \frac{1}{Z}\int_{\mathbb{C}_{herm}^{N\times N}} dH\ e^{-\frac{N}{2}\mathrm{Tr}\,H^2} f(H)$$

where $Z := \int_{\mathbb{C}^{N\times N}_{herm}} dH\ e^{-\frac{N}{2}\operatorname{Tr} H^2}$ is the constant normalizing the measure. We are interested in the resolvent $(z-H)^{-1}$. Since $H^* = H$ we have $\sigma(H) \subset \mathbb{R}$ hence $z = E + i\varepsilon \in \rho(H)\ \forall E \in \mathbb{R}$ and $\varepsilon > 0$. In particular

$$|\det(E + i\varepsilon - H)| \geq \varepsilon^N \qquad \forall H = H^*. \tag{1.22}$$

Lemma 1.11. *We have*

$$\langle \frac{1}{\det(E+i\varepsilon-H)} \rangle_N = \frac{\sqrt{N}}{\sqrt{2\pi}} \int_{\mathbb{R}} da\ e^{-N\left(\frac{a^2}{2} + \ln(E+i\varepsilon-a)\right)}$$

Note that by (1.22) the above integral is well defined.

[3: 18.10.2024]

[4: 22.10.2024]

Proof. We would like to use complex Gaussian integral to reformulate $(\det(E+i\varepsilon-H))^{-1}$ as a Gaussian integral. For this we need the real part of the matrix to be positive definite. Note that

$$\operatorname{Re}(E+i\varepsilon-H) = E - H.$$

This matrix has no sign! On the other hand

$$\operatorname{Re}[-i(E+i\varepsilon-H)] = \varepsilon > 0.$$

Apply formula (1.17) to $A = -i(E+i\varepsilon-H)$ we obtain

$$\frac{1}{\det(E+i\varepsilon-H)} = \frac{(-i)^N}{\det A} = \int_{\mathbb{C}^N} \left[\frac{d\overline{\varphi}d\varphi}{2\pi}\right]^N e^{-(\overline{\varphi},A\varphi)} = \int_{\mathbb{C}^N} \left[\frac{d\overline{\varphi}d\varphi}{2\pi}\right]^N e^{i(\overline{\varphi},(E+i\varepsilon-H)\varphi)}$$

where we defined

$$\left[\frac{d\overline{\varphi}d\varphi}{2\pi}\right]^N := \prod_{j=1}^N \frac{d\overline{\varphi}_j d\varphi_j}{2\pi}.$$

Inserting this in the average we obtain

$$\begin{aligned}
\langle \frac{1}{\det(E+i\varepsilon-H)} \rangle_N &= \frac{1}{Z} \int_{\mathbb{C}^{N\times N}_{herm}} dH\ e^{-\frac{N}{2}\operatorname{Tr} H^2} \frac{1}{\det(E+i\varepsilon-H)} \\
&= \frac{(-i)^N}{Z} \int_{\mathbb{C}^{N\times N}_{herm}} dH\ e^{-\frac{N}{2}\operatorname{Tr} H^2} \int_{\mathbb{C}^N} \left[\frac{d\overline{\varphi}d\varphi}{2\pi}\right]^N e^{i(\overline{\varphi},(E+i\varepsilon-H)\varphi)} \\
&= (-i)^N \int_{\mathbb{C}^N} \left[\frac{d\overline{\varphi}d\varphi}{2\pi}\right]^N e^{i(E+i\varepsilon)|\varphi|^2} \frac{1}{Z} \int_{\mathbb{C}^{N\times N}_{herm}} dH\ e^{-\frac{N}{2}\operatorname{Tr} H^2} e^{-i(\varphi,H\varphi)},
\end{aligned}$$

where we used

$$\int_{\mathbb{C}^N} \prod_{j=1}^N d\overline{\varphi}_j d\varphi_j \left| e^{-\frac{N}{2}\operatorname{Tr} H^2} e^{i(\overline{\varphi},(E+i\varepsilon-H)\varphi)} \right| = \int_{\mathbb{C}^N} \prod_{j=1}^N d\overline{\varphi}_j d\varphi_j \int_{\mathbb{C}^{N\times N}_{herm}} dH\ e^{-\varepsilon|\varphi|^2} e^{-\frac{N}{2}\operatorname{Tr} H^2} < \infty$$

to apply Fubini and exchange the integration order. We compute now

$$
\begin{aligned}
&\frac{1}{Z}\int_{\mathbb{C}^{N\times N}_{herm}} dH\; e^{-\frac{N}{2}\operatorname{Tr} H^2} e^{-i(\varphi,H\varphi)}\\
&=\frac{1}{Z}\prod_j \int_{\mathbb{R}} dH_{jj}\; e^{-\frac{N}{2}H_{jj}^2} e^{-iH_{jj}\overline{\varphi}_j\varphi_j} \prod_{j<k}\int_{\mathbb{C}} d\overline{H}_{jk}dH_{jk}\; e^{-N|H_{jk}|^2} e^{-i(H_{jk}\overline{\varphi}_j\varphi_k+\overline{H}_{jk}\overline{\varphi}_k\varphi_j)}\\
&=\prod_j e^{-\frac{1}{2N}(\overline{\varphi}_j\varphi_j)(\overline{\varphi}_j\varphi_j)} \prod_{j<k} e^{-\frac{1}{N}(\overline{\varphi}_j\varphi_k)(\overline{\varphi}_k\varphi_j)} = e^{-\frac{1}{2N}\sum_{jk}(\overline{\varphi}_j\varphi_k)(\overline{\varphi}_k\varphi_j)}.
\end{aligned}
$$

Therefore

$$
\langle\frac{1}{\det(E+i\varepsilon-H)}\rangle_N = (-i)^N \int_{\mathbb{C}^N}\left[\frac{d\overline{\varphi}d\varphi}{2\pi}\right]^N e^{i(E+i\varepsilon)|\varphi|^2} e^{-\frac{1}{2N}\sum_{jk}(\overline{\varphi}_j\varphi_k)(\overline{\varphi}_k\varphi_j)}.
$$

Using (1.3) we reorganize the quartic term as follows

$$
e^{-\frac{1}{2N}\sum_{jk}(\overline{\varphi}_j\varphi_k)(\overline{\varphi}_k\varphi_j)} = e^{-\frac{1}{2N}[\sum_j(\overline{\varphi}_j\varphi_j)]^2} = \frac{\sqrt{N}}{\sqrt{2\pi}}\int_{\mathbb{R}} da\; e^{-\frac{N}{2}a^2} e^{ia[\sum_j(\overline{\varphi}_j\varphi_j)]}.
$$

Inserting this above we get

$$
\langle\frac{1}{\det(E+i\varepsilon-H)}\rangle_N = (-i)^N \int_{\mathbb{C}^N}\left[\frac{d\overline{\varphi}d\varphi}{2\pi}\right]^N e^{i(E+i\varepsilon)|\varphi|^2}\frac{\sqrt{N}}{\sqrt{2\pi}}\int_{\mathbb{R}} da\; e^{-\frac{N}{2}a^2} e^{-ia[\sum_j(\overline{\varphi}_j\varphi_j)]}.
$$

Note that

$$
\int_{\mathbb{C}^N}\prod_{j=1}^N d\overline{\varphi}_j d\varphi_j \int_{\mathbb{R}} da\; \left| e^{i(E+i\varepsilon)|\varphi|^2} e^{-\frac{N}{2}a^2} e^{ia[\sum_j(\overline{\varphi}_j\varphi_j)]}\right| = \int_{\mathbb{C}^N}\prod_{j=1}^N d\overline{\varphi}_j d\varphi_j \int_{\mathbb{R}} da\; e^{-\varepsilon|\varphi|^2} e^{-\frac{N}{2}a^2} < \infty
$$

and hence by Fubini we can exchange the integration order. Finally we obtain

$$
\begin{aligned}
\langle\frac{1}{\det(E+i\varepsilon-H)}\rangle_N &= (-i)^N \frac{\sqrt{N}}{\sqrt{2\pi}}\int_{\mathbb{R}} da\; e^{-\frac{N}{2}a^2}\prod_{j=1}^N\int_{\mathbb{C}}\frac{d\overline{\varphi}_j d\varphi_j}{2\pi} e^{i(E+i\varepsilon-a)|\varphi_j|^2}\\
&= \sqrt{N}\int_R \frac{da}{\sqrt{2\pi}} e^{-\frac{N}{2}a^2}\frac{1}{(E+i\varepsilon-a)^N}.
\end{aligned}
$$

This completes the proof.

□

The second step in the proof above is sometimes called *Hubbard-Stratonovich* transformation.

1.2.7 Gaussian measures on infinite dimensional spaces

We start with an equivalent definition of Gaussian measure in finite dimension that can be generalized to infinite dimensional spaces.

Lemma 1.12. *Let $C \in \mathbb{R}^{N\times N}$ be a symmetrix positive definite matrix and $d\mu_C(\varphi)$ the Gaussian measure on $\mathbb{R}^N$ with mean zero and covariance C defined above.*

(i) $d\mu_C(\varphi)$ is the unique probability measure on $\mathbb{R}^N$ with Fourier transform

$$
\hat{\mu}_C(v) := \int_{\mathbb{R}^N} d\mu_C(\varphi) e^{i(\varphi,v)} = e^{\frac{1}{2}(v,Cv)}.
$$

(ii) The Fourier transform is the moment generating function, precisely

$$\int_{\mathbb{R}^N} d\mu_C(\varphi) \prod_{j=1}^N \varphi_j^{n_j} = \left[\prod_{j=1}^N \partial_{v_j}^{n_j} e^{\frac{1}{2}(v,Cv)} \right]_{v=0}$$

(iii) $d\mu_C(\varphi)$ is the unique probability measure on $\mathbb{R}^N$ such that $\int_{\mathbb{R}^N} d\mu_C(\varphi) \prod_{j=1}^N |\varphi_j|^{n_j} < \infty$ and

$$\int_{\mathbb{R}^N} d\mu_C(\varphi) \prod_{j=1}^N \varphi_j^{n_j} = \begin{cases} 1 & \text{if } \sum_j n_j = 0 \\ 0 & \text{if } \sum_j n_j \text{ odd} \\ \sum_{G(2m)} \prod_{l \in G} C_{x_l,y_l} & \text{if } \sum_j n_j = 2m \text{ even} \end{cases} \tag{1.23}$$

where $G(2m)$ denotes the set of all partitions of the $2m \sum_j n_j$ terms into subsets of two elements (pairs).

Proof. (not done in class)
(i) Holds since a probability measure is uniquely defined by its Fourier transform.

(ii) Direct computation.

(iii) We show first that the moments of $d\mu_C(\varphi)$ are absolutely integrable and given by (1.23). For all $\eta > 0$ we have the bound

$$|\varphi_j|^{n_j} = \left(\varphi_j^{2n_j}\right)^{\frac{1}{2}} = \left(\frac{1}{\eta^{n_j}} (\varphi_j^2 \eta)^{n_j}\right)^{\frac{1}{2}} \leq \left(\frac{n_j!}{\eta^{n_j}} e^{\eta \varphi_j^2}\right)^{\frac{1}{2}}.$$

Hence

$$\int_{\mathbb{R}^N} d\mu_C(\varphi) \prod_{j=1}^N |\varphi_j|^{n_j} \leq \prod_{j=1}^N \left(\frac{n_j!}{\eta^{n_j}}\right)^{\frac{1}{2}} \int_{\mathbb{R}^N} d\mu_C(\varphi) e^{\eta |\varphi|^2}.$$

The last integral is finite for all $\eta > 0$ such that $C^{-1} - 2\eta > 0$. Such an η exists since $C > 0$.

Formula (1.23) follows applying several times (1.19).

Assume now μ is a probability measure with integrable moments given by (1.19). The Fourier transform of the measure $\hat{\mu}_C(v)$ is infinitely often differentiable in all variables since

$$\int_{\mathbb{R}^N} d\mu_C(\varphi) \prod_{j=1}^N |\varphi_j|^{n_j} \left| e^{i(v,\varphi)} \right| = \int_{\mathbb{R}^N} d\mu_C(\varphi) \prod_{j=1}^N |\varphi_j|^{n_j} < \infty.$$

We show now that the Fourier series $S(v)$ of $\hat{\mu}_C(v)$ around the point $v = 0$ is absolutely convergent. Indeed

$$|S(v)| = \sum_{n_1,\dots,n_N} \left| \prod_{j=1}^N \frac{v_j^{n_j}}{n_j!} \hat{\mu}_C^{(n_1,\dots,n_N)}(0) \right| = 1 + \sum_{m=1}^\infty \sum_{\sum_j n_j = 2m} \prod_{j=1}^N \frac{|v_j|^{n_j}}{n_j!} \sum_{G(2m)} \prod_{l \in G} C_{x_l,y_l}$$

We have $\prod_{j=1}^N |v_j|^{n_j} \leq |v|_\infty^{2m}$ and

$$\sum_{G(2m)} \prod_{l \in G} C_{x_l,y_l} \leq |C|_\infty^m \ (2m-1)!!$$

where $|v|_\infty := \max_j |v_j|$ and $|C|_\infty := \max_{jk} |C_{jk}|$. Hence

$$|S(v)| \leq 1 + \sum_{m=1}^{\infty} \tfrac{(2m-1)!!}{(2m)!} |v|_\infty^{2m} |C|_\infty^m \sum_{\sum_j n_j = 2m} \tfrac{(2m)!}{\prod_j n_j!}$$
$$= 1 + \sum_{m=1}^{\infty} \left(\tfrac{|v|_\infty^2 |C|_\infty N^2}{2}\right)^m \frac{1}{m!} = e^{\frac{|v|_\infty^2 |C|_\infty N^2}{2}} < \infty,$$

where we used
$$\frac{(2m-1)!!}{(2m)!} = \frac{1}{2^m m!}, \qquad \sum_{\sum_{j=1}^N n_j = 2m} \frac{(2m)!}{\prod_j n_j!} = N^{2m}.$$

It follows that the function is real analytic on $\mathbb{R}^N$ and hence $\hat{\mu}_C(v)$ is determined by the Taylor series in $v = 0$.

□

This characterization can be extended to infinite dimensional spaces as follows.
The covariance $C = A^{-1}$ with $A \in \mathbb{R}^{N \times N}$ is replaced by $C = A^{-1}$ where $A\colon D(A) \to \mathcal{H}$ is a positive self-adjoint (unbounded) operator defined on a subset $D(A)$ of a Hilbert space $\mathcal{H}$, with $0 \in \rho(A)$. To make sense of the formal expression

$$\int_X d\mu_C(\varphi) \prod_{j=1}^{n} \varphi(x_j)$$

we replace $\mathcal{H}$ with $\mathcal{H}_\infty = \cap_{n=-\infty}^{\infty} D(A+I)^n$, the test point x_j with a test function $\xi_j \in \mathcal{H}_\infty$ and the function φ with an element in the dual space $\varphi \in \mathcal{H}_\infty^*$. To define all this properly we need the notion of *nuclear space*. The measure is then defined on $\mathcal{H}_\infty^*$ and the covariance is now a bilinear form $C : \mathcal{H}_\infty \times \mathcal{H}_\infty \to \mathbb{R}$ defined via

$$C(\varphi, \psi) := (\varphi, C\psi)_{\mathcal{H}}.$$

For the proper construction and definitions see the book by Glimm and Jaffe *Quantum Physics, a functional integral point of view.*

Definition 1.13. *A measure on $\mathcal{H}_\infty^*$ is Gaussian with mean zero and covariance C is $\forall \xi_1, \dots, \xi_n \in \mathcal{H}_\infty$ test functions we have*

$$\int d\mu_C(\varphi) \prod_{j=1}^{n} \varphi(\xi_j) = \begin{cases} 1 & \text{if} n = 0 \\ 0 & \text{if} n \text{ odd} \\ \sum_{G(2m)} \prod_{l \in G} (\xi_{x_l}, C\xi_{y_l}) & \text{if} n = 2m \text{ even} \end{cases}$$

Theorem 1.14. *There is a unique Gaussian measure on $\mathcal{H}_\infty^*$ with mean 0 and covariance C*

Proof. See Glimm-Jaffe. □

This construction is not easily generalizable to complex covariances.

[4: 22.10.2024]

[5: 25.10.2024]

2 Grassmann variables

2.1 Definition

Definition 2.1. *A Grassmann algebra is a real (or complex) unital algebra whose generators anticommute. More precisely let $\mathcal{V}$ be a finite dimensional $\mathbb{K}-$vector space with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We introduce the antisymmetric tensor product*

$$\begin{aligned} \Lambda : \quad \mathcal{V} \times \mathcal{V} \quad &\to \mathcal{V} \otimes_{as} \mathcal{V} \\ (v, w) \quad &\mapsto v \wedge w = v\ w. \end{aligned} \tag{2.1}$$

This (Grassmann) product is bilinear associative and anticommuting i.e.

$$v\ w = -\ w\ v \qquad \text{and} \quad v\ v = v^2 = 0 \qquad \forall v, w \in V.$$

The Grassmann algebra *(also called* exterior *or* $\mathbb{Z}^2-$graded algebra*) on $\mathbb{K}$ generated by $\mathcal{V}$ is the associative algebra with unit defined by*

$$\mathcal{G}_{\mathbb{K}}[\mathcal{V}] := \bigoplus_{n=0}^{\dim \mathcal{V}} \Lambda^n \mathcal{V}, \qquad \text{with} \tag{2.2}$$

$$\Lambda^0 \mathcal{V} := \mathbb{K}, \qquad \Lambda^1 \mathcal{V} := \mathcal{V}, \qquad \Lambda^n \mathcal{V} := \underbrace{\mathcal{V} \otimes_{as} \mathcal{V} \otimes_{as} \ldots \otimes_{as} \mathcal{V}}_{n \text{ times}}, \quad n \geq 2. \tag{2.3}$$

We define

$$\mathcal{G}_{\mathbb{K}}^{\text{even}}[\mathcal{V}] := \bigoplus_{\substack{0 \leq n \leq \dim \mathcal{V} \\ n \text{ even}}} \Lambda^n \mathcal{V}, \quad \mathcal{G}_{\mathbb{K}}^{\text{odd}}[\mathcal{V}] := \bigoplus_{\substack{0 \leq n \leq \dim \mathcal{V} \\ n \text{ odd}}} \Lambda^n \mathcal{V} \tag{2.4}$$

its even and its odd subspace, respectively.
In particular, $\mathbb{K} = \Lambda^0 \mathcal{V} \subset \mathcal{G}_{\mathbb{K}}^{\text{even}}[\mathcal{V}]$ and $\mathcal{V} = \Lambda^1 \mathcal{V} \subset \mathcal{G}_{\mathbb{K}}^{\text{odd}}[\mathcal{V}]$.

Elements in $\mathcal{G}_{\mathbb{K}}^{\text{even}}[\mathcal{V}]$ are called even *or* Bosonic *variables.*

Elements in $\mathcal{G}_{\mathbb{K}}^{\text{odd}}[\mathcal{V}]$ are called odd *or* Fermionic *or* Grassmann *variables.*

If $\{\psi_1, \ldots, \psi_N\}$ is a basis for the $\mathbb{K}-$vector space $\mathcal{V}$, we write $\mathcal{G}_{\mathbb{K}}[\mathcal{V}] = \mathcal{G}_{\mathbb{K}}[\psi_1, \ldots, \psi_N]$.

Examples

1. Set $N = 1$ and $\mathcal{V} = \text{span}\{\psi_1\}$. Then $\mathcal{G} = \Lambda^0 \mathcal{V} \bigoplus \Lambda^1 \mathcal{V}$, $\mathcal{G}^{\text{even}} = \Lambda^0 \mathcal{V} = \mathbb{K}$, $\mathcal{G}^{\text{odd}} = \Lambda^1 \mathcal{V} = \mathcal{V}$.

Note that $\Lambda^n \mathcal{V} = \{0\}$ $\forall n \geq 2$ since $\psi_1^2 = 0$.

2. Set $N = 2$ and $\mathcal{V} = \text{span}\{\psi_1, \psi_2\}$. Then $\mathcal{G} = \Lambda^0 \mathcal{V} \bigoplus \Lambda^1 \mathcal{V} \bigoplus \Lambda^2 \mathcal{V}$, $\mathcal{G}^{\text{even}} = \Lambda^0 \mathcal{V} \bigoplus \Lambda^2 \mathcal{V}$, $\mathcal{G}^{\text{odd}} = \Lambda^1 \mathcal{V} = \mathcal{V}$.

Note that $\Lambda^n \mathcal{V} = \{0\}$ $\forall n \geq 3$ since $\psi_1 \psi_2 \psi_1 = 0 = \psi_1 \psi_2 \psi_2$. Moreover

$$\begin{aligned} v \in \mathcal{G}^{\text{even}} &\Leftrightarrow v = x + a\psi_1 \psi_2, \qquad x, a \in \mathbb{K} \\ v \in \mathcal{G}^{\text{odd}} &\Leftrightarrow v = a_1 \psi_1 + a_2 \psi_2, \qquad a_1, a_2 \in \mathbb{K}. \end{aligned}$$

The next result extends this to general N.

Proposition 2.2. *Let $\{\psi_1, \ldots, \psi_N\}$ be a basis for the $\mathbb{K}-$vector space $\mathcal{V}$. Set $\mathcal{G} = \mathcal{G}_{\mathbb{K}}[\mathcal{V}]$, and $\mathcal{I}_N = \{1, 2, \ldots, N\}$. Later $\mathcal{I}_N$ will be replaced by a finite subset of $\mathbb{Z}^d$.*
For each $I \subset \mathcal{I}_N$ we choose some ordering $<_I$ and define

$$\psi_I = \prod_{i \in I} \psi_i$$

where the product is performed according to the ordering. It holds

(i) *For all permutation $\sigma \in \mathcal{P}_{|I|}$ we have*

$$\psi_I = \epsilon^\sigma \prod_{i \in \sigma(I)} \psi_{\sigma(i)},$$

where ϵ^σ is the sign of the permutation and the last product is performed in the permuted order.

(ii) *$v \in \mathcal{G}$ admits a unique decomposition*

$$v = \sum_{I \subset \mathcal{I}_n} v_I \psi_I$$

where $v_I \in \mathbb{K}$ is an antisymmetric tensor $v_I = (v_{i_1, \ldots, v_{i|I|}})$.

Every element $v \in \mathcal{G}^{\text{even}}$ (resp $\mathcal{G}^{\text{odd}}$) admits a unique decomposition

$$v = \sum_{I \subset \mathcal{I}_n,\ |I| \text{even}} v_I \psi_I, \qquad \text{resp} \qquad v = \sum_{I \subset \mathcal{I}_n,\ |I| \text{odd}} v_I \psi_I.$$

(iii) *$v, w \in \mathcal{G}^{\text{odd}} \Rightarrow vw \in \mathcal{G}^{\text{even}}$ and $vw = -wv$. In particular $v^2 = 0\ \forall v \in \mathcal{G}^{\text{odd}}$.*
Note that $\mathcal{G}^{\text{odd}}$ is not a subalgebra.

(iv) *$v, w \in \mathcal{G}^{\text{even}} \Rightarrow vw \in \mathcal{G}^{\text{even}}$ and $vw = wv$.*
In particular $\mathcal{G}^{\text{even}}$ is a subalgebra.

(v) *$v \in \mathcal{G}^{\text{even}}$ $w \in \mathcal{G}^{\text{odd}} \Rightarrow vw \in \mathcal{G}^{\text{odd}}$ and $vw = wv$.*

(vi) *$v \in \mathcal{G}^{\text{even}}$ admits the unique decomposition*

$$v = x + n, \qquad \text{where } x \in \Lambda^0 \mathcal{V} = \mathbb{K}, \quad n \in \bigoplus_{n \geq 1} \Lambda^{2n} \mathcal{V}. \tag{2.5}$$

In particular n is nilponent i.e. $\exists k \leq \frac{N}{2}$ st $n^k \neq 0$ and $n^{k+1} = 0$.

Proof. Use the definition of $\mathcal{G}$ and the following fact (excercise): for all $I, I' \subset \mathcal{I}_n$ with $|I| = m$, $|I'| = m'$ we have

$$\psi^I \psi^{I'} = (-1)^{mm'} \psi^{I'} \psi^I.$$

□

Definition 2.3. *Fix $v \in \mathcal{G}^{\text{even}}$ and let $v = x + n$ be the unique decomposition introduced in* (2.5)*. We call x the* body *and n the* soul *of v:*

$$\text{body}(v) := x \qquad \text{soul}(v) := n.$$

We say that $v \in U \subset \mathbb{K}$ if body$(v) \in U$*.*

Note that an even element is almost a standard real or complex number (it has a domain if definition and commutes with everything) except for the additional nilpotent part.

2.2 Functions

Definition 2.4 (Functions I)**.** *Every alement $v \in \mathcal{G}$ can be seen as a function of the basis elements $\{\psi_1, \dots, \psi_N\}$. Every such function is a polynomial of degree at most 1 in each variable:*

$$f(\psi_1, \dots, \psi_N) = \sum_{I \subset \mathcal{I}_n} v_I \psi^I. \tag{2.6}$$

Definition 2.5 (Functions II)**.** *Every function $f \in C^\infty(\mathbb{K}, \mathbb{K})$ can be upgraded to a function mapping $\mathcal{G}^{\text{even}} \to \mathcal{G}^{\text{even}}$ as follows*

$$\begin{aligned} f\colon \quad & \mathcal{G}^{\text{even}}_{\mathbb{K}}[\mathcal{V}] && \to \mathcal{G}^{\text{even}}_{\mathbb{K}}[\mathcal{V}] \\ & v = x + n && \mapsto f(x+n) := \textstyle\sum_{k \geq 0} \frac{f^{(k)}(x)}{k!} n^k \end{aligned} \tag{2.7}$$

Remarks

• Since n is nilpotent, the sum above is finite. Precisely, setting $N := \dim \mathcal{V}$, we have $2k \leq N$. Therefore we only need $f \in C^{[N/2]}(\mathbb{K}, \mathbb{K})$.

• The same construction works for $f \in C^{[N/2]}(U, \mathbb{K})$ for some open set $U \subset \mathbb{K}$. In this case we have to replace $\mathcal{G}^{\text{even}}_{\mathbb{K}}[\mathcal{V}]$ in the domain of the function with

$$\mathcal{G}^{\text{even}}_{\mathbb{K}}[\mathcal{V}] \cap U := \{v \in \mathcal{G}^{\text{even}}_{\mathbb{K}} | \text{ body}(v) \in U\}.$$

• We have $\text{body} f(v) = f(\text{body}(v))$

Example 1: the exponential function. Using the definition above we get

$$e^v = e^{x+n} := \sum_{k \geq 0} e^x \frac{n^k}{k!}.$$

With this definition we have

$$e^{v_1} e^{v_2} = e^{v_1 + v_2} \qquad \forall v_1, v_2 \in \mathcal{G}^{\text{even}}. \tag{2.8}$$

Note that we could use the Taylor expansion around zero to define e^v for $v \in \mathcal{G}^{\text{odd}}$ too. We would get

$$e^v = 1 + v$$

since $v^2 = 0 \ \forall v \in \mathcal{G}^{\text{odd}}$. This definition does not satisfy $e^{v_1} e^{v_2} = e^{v_1+v_2}$ since $e^{v_1} e^{v_2} = 1 + v_1 + v_2 + v_1 v_2$ while $e^{v_1+v_2} = 1 + v_1 + v_2$.

Example 2: the scalar and matrix inverse function. We define $\forall v \in \mathcal{G}^{\text{even}}$ with $x = \text{body} v \neq 0$

$$v^{-1} = (x+n)^{-1} := \frac{1}{x} + \sum_{k \geq 1} (-1)^k \frac{1}{x^{k+1}} n^k$$

With this definition we have (exercise) $v^{-1} v = v v^{-1} = 1$. Moreover, if $n^2 = 0$ we have

$$e^{-n} = 1 - n = (1+n)^{-1}.$$

Note that an odd element admits no inverse! By contradiction assume $v \in \mathcal{G}^{\text{odd}}$ and let $w \in \mathcal{G}$ such that $vw = 1$. That means $vw \in \mathcal{G}^{\text{even}}$ with $\text{body}(vw) = 1$ and $\text{soul}(vw) = 0$. To get $vw \in \mathcal{G}^{\text{even}}$ we need $w \in \mathcal{G}^{\text{odd}}$ too, hence $v, w \in \bigoplus_{n\geq 1} \Lambda^n \mathcal{V}$. As a result

$$vw \in \bigoplus_{n\geq 2} \Lambda^n \mathcal{V}$$

and therefore $\text{body}(vw) = 0$ which gives a contradiction.
To define the analog in the case of a matrix consider $A = A_0 + A_1 \in (\mathcal{G}^{\text{even}})^{m\times m}$ with $A_0 = \text{body} A \in \mathbb{C}^{m\times m}$ invertible. We define

$$A^{-1} = (1 + A_0^{-1}A_1)^{-1}A_0^{-1} := \sum_{k\geq 0} (-1)^k (A_0^{-1}A_1)^k A_0^{-1}.$$

With this definition we have (exercise) $AA^{-1} = A^{-1}A = 1$.

Example 3: the scalar logarithm. We define $\forall v = x + n \in \mathcal{G}^{\text{even}}$

$$\ln v = \ln(x + n) := \ln x - \sum_{k\geq 1} \frac{(-1)^k}{kx^k} n^k.$$

With this definition we have (exercise)

$$e^{(\ln v)} = v, \qquad \ln(v_1 v_2) = \ln v_1 + \ln v_2.$$

In particular, if $x = 1$,

$$\ln(1 + n) = -\sum_{k\geq 1} \frac{(-1)^k}{k} n^k,$$

and $\text{body}(\ln(1 + n)) = 0$.

Example 4: the matrix exponential and logarithm. Let $A \in \mathbb{C}^{m\times m}$. Remember that, there are two ways of defining exponential and logarithm for this matrix.
If $A^* = A$, then $A = U^* \hat{\lambda} U$, with $U^*U = 1$ and $\hat{\lambda} = \text{diag}\,\{\lambda_1, \ldots, \lambda_m\}$, $\lambda_j \in \mathbb{R}\ \forall j = 1, \ldots, m$. In this case we define

$$\begin{aligned} e^A &:= U^* \widehat{e^{\lambda}}\, U, \qquad \widehat{e^{\lambda}} := \text{diag}\,\{e^{\lambda_1}, \ldots, e^{\lambda_m}\} \\ \ln A &:= U^* \widehat{\ln \lambda}\, U, \qquad \widehat{\ln \lambda} := \text{diag}\,\{\ln \lambda_1, \ldots, \ln \lambda_m\}. \end{aligned}$$

For general A (not necessarily hermitian), we can use the Taylor expansion

$$\begin{aligned} e^A &:= \sum_{j\geq 0} \frac{1}{j!} A^j \\ \ln A &:= -\sum_{j\geq 1} \frac{(-1)^j}{j} (A - 1)^j, \qquad \text{with } \|A - 1\| < 1. \end{aligned}$$

For $A = A^*$ the two definitions above are equivalent (for the log we also need to require $\|A-1\| < 1$). Whenever $\ln A$ is well defined we also have

$$\ln \det A = \text{tr}\ \ln A. \tag{2.9}$$

This relation implies in particular $\operatorname{tr}\, \ln(A_1A_2) = \operatorname{tr}\,(\ln A_1 + \ln A_2)$ whenever the logarithms are well defined.

Consider now $A = A_0 + A_1 \in (\mathcal{G}^{\text{even}})^{m\times m}$ with $A_0 = \text{body}A$. We define

$$e^A := \sum_{j\geq 0} \frac{1}{j!} A^j = \sum_{j\geq 0} \frac{1}{j!}(A_0 + A_1)^j$$

$$\ln A = \ln(A_0 + A_1) := -\sum_{j\geq 1} \frac{(-1)^j}{j}(A_0 + A_1 - 1)^j \qquad \text{with } \|A_0 - 1\| < 1$$

Since A_1 is nilpotent there is some k such that $\prod_{l=1}^{n} A_0^{n_l} A_1^{n'_l} = 0$ whenever $n'_1 + \cdots + n'_n > k$ for any $n \geq 1$. Using this fact one can show that the above sums are still convergent (exercise). Moreover, $\text{body}(e^A) = e^{A_0},\ \text{body}(\ln(A)) = \ln A_0$ and

$$e^{\ln(A)} = A.$$

2.3 Derivative

Lemma 2.6. *Fix ψ_j. An element $v \in \mathcal{G}$ admits a unique deconposition*

$$v = v_1 + \psi_j v_2^l = v_1 + v_2^r \psi_j \tag{2.10}$$

where $v_1, v_2^l, v_2^r \in \mathcal{G}$ are independent of ψ_j.

Proof. exercise ☐

Definition 2.7. *Fix ψ_j.*
The left derivative *of $v \in \mathcal{G}$ with respect to ψ_j is $\overrightarrow{\partial}_{\psi_j} v := v_2^l$*

The right derivative *of $v \in \mathcal{G}$ with respect to ψ_j is $v\overleftarrow{\partial}_{\psi_j} := v_2^r$*

We will mostly use the left derivative and note is by ∂ instead of $\overrightarrow{\partial}$. We will need the right derivative when defining the Jacobian of a coordinate change.

Lemma 2.8.

(i) (anticommutation property) *We have*

$$\overrightarrow{\partial}_{\psi_j} \overrightarrow{\partial}_{\psi_k} = -\overrightarrow{\partial}_{\psi_k} \overrightarrow{\partial}_{\psi_j} \qquad \forall j, k.$$

In particular $\overrightarrow{\partial}^2_{\psi_j} = 0$. The same holds for the right derivative.

(ii) (product rule)

(a) Assume v is homogeneous of degree $\pi(v)$. Then we have

$$\overrightarrow{\partial}_{\psi_j}(vw) = (\overrightarrow{\partial}_{\psi_j} v)w \ +\ \pi(v)\ v\ (\overrightarrow{\partial}_{\psi_j} w)$$

(b) Assume w is homogeneous of degree $\pi(w)$. Then we have

$$(vw)\overleftarrow{\partial}_{\psi_j} = v\ (w\overleftarrow{\partial}_{\psi_j}) +\ \pi(w)\ (v\overleftarrow{\partial}_{\psi_j})\ w$$

Proof.
(*i*) Let $j < k$. Each v admits the unique decompositiiton $v = v_1 + \psi_j v_2 + \psi_k v_3 + \psi_j\psi_k v_4$, where v_j is independent of both ψ_j and ψ_k. We compute

$$\partial_{\psi_k}\partial_{\psi_j} v = \partial_{\psi_k}\partial_{\psi_j}\psi_j\psi_k v_4 = v_4$$
$$\partial_{\psi_j}\partial_{\psi_k} v = \partial_{\psi_j}\partial_{\psi_k}\psi_j\psi_k v_4 = -\partial_{\psi_j}\partial_{\psi_k}\psi_k\psi_j v_4 = -v_4.$$

(*ii*) We prove the identity for the left derivative. We have $v = v_1 + \psi_j v_2$ and $w = w_1 + \psi_j w_2$. Moreover, since v is homogeneous of degree $\pi(v)$, v_1 is homogeneous of degree $\pi(v)$, and v_2 is homogeneous of degree $\pi(v) - 1$.
We compute

$$vw = v_1w_1 + \psi_j v_2 w_1 + v_1\psi_j w_2 = v_1w_1 + \psi_j(v_2w_1 + (-1)^{\pi(v)}v_1w_2)$$

Hence

$$\begin{aligned}\partial_{\psi_j}(vw) &= v_2w_1 + (-1)^{\pi(v)}v_1w_2 = (\partial_{\psi_j}v)w_1 + (-1)^{\pi(v)}v_1(\partial_{\psi_j}w)\\ &= (\partial_{\psi_j}v)w + (-1)^{\pi(v)}v(\partial_{\psi_j}w) - \Big[v_2\psi_j w_2 + (-1)^{\pi(v)}\psi_j v_2 w_2\Big]\\ &= (\partial_{\psi_j}v)w + (-1)^{\pi(v)}v(\partial_{\psi_j}w)\end{aligned}$$

where we used

$$v_2\psi_j w_2 = -(-1)^{\pi(v)}\psi_j v_2 w_2.$$

□

2.4 Integration

To motivate a notion of integral for Grassmann variables consider the following properties of the standard integral on $\mathbb{R}$[1]

- The integral is a linear map $\int_{\mathbb{R}} dx(f + \lambda g) = \int_{\mathbb{R}} dxf \;+\; \lambda\int_{\mathbb{R}} dxg$.
- The integral maps a function into a number.
- The integral of a derivative is zero: $\int_{\mathbb{R}} dxf' = 0$.

Each function of ψ_j can be written as $f(\psi_j) = f_1 + \psi_j f_2$. By linearity we get

$$\int d\psi_j f = (\int d\psi_j 1)f_1 + (\int d\psi_j\psi_j)f_2.$$

Therefore we only need to define $\int d\psi_j 1$ and $\int d\psi_j\psi_j$. The integral of a derivative must be zero hence we define

$$\int d\psi_j 1 := 0.$$

It remains to fix $\int d\psi_j\psi_j$. Two frequently used conventions are $\int d\psi_j\psi_j := 1$ and $\int d\psi_j\psi_j := 1/\sqrt{2\pi}$. We will use the first one.

Definition 2.9. *For $I \subset \mathcal{I}$ and $f(\psi) \in \mathcal{G}$ we define*

$$\int d\psi^I f(\psi) := \partial^I_\psi f(\psi) = \prod_{j\in I}\partial_{\psi_j} f(\psi).$$

[1]This informal motivation for the definition of Grassmann integral is taken from a post on the web page of Terry Tao.

Example Consider $f := e^{-a\psi_1\psi_2}$ with $a \in \mathbb{C}$. Since $\psi_1\psi_2$ is an even element the exponential is well defined. We compute

$$\int d\psi_1 d\psi_2 e^{-a\psi_1\psi_2} = \partial_{\psi_1}\partial_{\psi_2} e^{-a\psi_1\psi_2} = \partial_{\psi_1}\partial_{\psi_2}(1 - a\psi_1\psi_2) = a.$$

This result is the analog of the scalar Gaussian integral

$$\int_{\mathbb{C}} \frac{d\overline{\varphi}d\varphi}{2\pi} e^{-a|\varphi|^2} = \frac{1}{a}$$

which is only true if $\mathrm{Re}a > 0$.

[5: 25.10.2024]

[6: 29.10.2024]

2.5 Grassmann Gaussian integral

Remember the formulas (1.16)(1.17)

$$\int_{\mathbb{C}^N} \prod_{j=1}^{N} \frac{d\overline{\varphi}_j d\varphi_j}{2\pi} e^{-(\overline{\varphi}, A\varphi)} = \frac{1}{\det A} \qquad \forall A \in \mathbb{C}_+^{N\times N}, \tag{2.11}$$

$$\int_{\mathbb{R}^N} \prod_{j=1}^{N} \frac{d\varphi_j}{\sqrt{2\pi}}\ e^{-\frac{1}{2}(\varphi, A\varphi)} = \frac{1}{\sqrt{\det A}} \qquad \forall A \in \mathbb{C}_{+,sym}^{N\times N} \tag{2.12}$$

The next theorem states the analog results in the case of Grassmann variables.

Theorem 2.10.

(i) Let $\mathcal{V} := \mathrm{span}\{\overline{\psi}_1, \psi_1, \dots, \overline{\psi}_N, \psi_N\}$ *with* $\dim \mathcal{V} = 2N$*. We have*

$$\int \prod_{j=1}^{N} d\overline{\psi}_j d\psi_j\ e^{-(\overline{\psi}, A\psi)} = \det A \qquad \forall A \in \mathbb{C}^{N\times N}, \tag{2.13}$$

where

$$(\overline{\psi}, A\psi) := \sum_{jk} \overline{\psi}_j A_{jk} \psi_k.$$

(ii) Let $\mathcal{V} := \mathrm{span}\{\psi_1, \dots, \psi_N\}$ *with* $\dim \mathcal{V} = N$*. We have*

$$\int \prod_{j=1}^{N} d\psi_j\ e^{-(\psi, A\psi)} = \mathrm{Pf}A \qquad \forall A \in \mathbb{C}_{skew}^{N\times N} \text{ i.e. } A^t = -A, \tag{2.14}$$

where

$$(\psi, A\psi) := \sum_{jk} \psi_j A_{jk} \psi_k$$

and the Pfaffian *of the matrix* $A \in \mathbb{C}^{N\times N}$ *is defined via*

$$\mathrm{Pf}A := \begin{cases} 0 & \text{if } N \text{ odd} \\ 2^{-N/2} \frac{1}{(N/2)!} \sum_{\sigma \in \mathcal{P}(N)} \varepsilon^{\sigma} A_{\sigma(1)\sigma(2)} \cdots A_{\sigma(N-1)\sigma(N)} & \text{if } N \text{ even,} \end{cases} \tag{2.15}$$

where $\mathcal{P}(N)$ *is the set of permutations of* $\{1, \dots, N\}$*.*

Remarks

• The variable $\overline{\psi}$ is not a complex conjugate!

• $(\overline{\psi}, A\psi) := \sum_{jk} \overline{\psi}_j A_{jk} \psi_k \in \mathcal{G}_{\mathbb{C}}^{\text{even}}[\mathcal{V}]$ hence $e^{-(\overline{\psi}, A\psi)}$ is well defined and again an element in $\mathcal{G}_{\mathbb{C}}^{\text{even}}[\mathcal{V}]$. The same holds for $(\psi, A\psi)$.

• Using the anticommutativity of the ψ variables and $A^t = -A$ we argue

$$(\psi, A\psi) := \sum_{jk} \psi_j A_{jk} \psi_k = 2 \sum_{j<k} \psi_j A_{jk} \psi_k.$$

• Since $d\overline{\psi}_j d\psi_j$ is homogeneous of even degree the order in the product $\prod_{j=1}^{N} d\overline{\psi}_j d\psi_j$ is irrelevant

$$d\overline{\psi}_1 d\psi_1 \; d\overline{\psi}_2 d\psi_2 = d\overline{\psi}_2 d\psi_2 \; d\overline{\psi}_1 d\psi_1.$$

On the other hand the order in the product $\prod_{j=1}^{N} d\psi_j$ is relevant

$$d\psi_1 d\psi_2 = -d\psi_2 d\psi_1.$$

• While in (2.11) we need $\mathrm{Re}A > 0$ (in particular A is invertible) no condition on A is required for (2.13).

• While in (2.12) we need $\mathrm{Re}A > 0$ (in particular A is invertible) and $A^t = A$, for (2.14) we require $A^t = -A$ but no invertibility.

• Formula (2.13) remains true when $\dim \mathcal{V} = 2N + 2N'$ with $N' \geq 1$, $\{\overline{\psi}_1, \psi_1, \ldots, \overline{\psi}_N, \psi_N, \overline{\xi}_1, \xi_1, \ldots, \overline{\xi}_{N'}, \xi_{N'}\}$ is a corresponding basis and the matrix element $A_{jk} \in \mathbb{C}$ is replaced with $A_{jk} \in \mathcal{G}_{\mathbb{C}}^{\text{even}}[\overline{\xi}_1, \xi_1, \ldots, \overline{\xi}_{N'}, \xi_{N'}]$. Formula (2.14) remains true when $\dim \mathcal{V} = N + N'$ with $N' \geq 1$, $\{\psi_1, \ldots, , \psi_N, \xi_1, \ldots, \xi_{N'}\}$ is a corresponding basis and the matrix element $A_{jk} \in \mathbb{C}$ is replaced with $A_{jk} \in \mathcal{G}_{\mathbb{C}}^{\text{even}}[\xi_1, \ldots, \xi_{N'}]$.

• We will see later that $\mathrm{Pf}(A)^2 = \det A$ for all skew-symmetric matrix A.

Proof of Theorem 2.10.

(i) Using the definition of exponential we have

$$e^{-(\overline{\psi}, A\psi)} = \sum_{n \geq 0} \frac{(-1)^n}{n!} (\overline{\psi}, A\psi)^n.$$

It holds

$$(\overline{\psi}, A\psi)^n = 0 \qquad \forall n > N,$$

since $(\overline{\psi}, A\psi)$ contains at least one ψ variable, we have at most N different ψ and $\psi_j^2 = 0 \; \forall j$. Therefore

$$\int \prod_{j=1}^{N} d\overline{\psi}_j d\psi_j \; e^{-(\overline{\psi}, A\psi)} = \sum_{n=0}^{N} \frac{(-1)^n}{n!} I_n,$$

where

$$I_n := \int \prod_{j=1}^{N} d\overline{\psi}_j d\psi_j \; (\overline{\psi}, A\psi)^n = \prod_{j=1}^{N} \partial_{\overline{\psi}_j} \partial_{\psi_j} \; (\overline{\psi}, A\psi)^n.$$

We have $I_n = 0 \; \forall 0 \leq n < N$ since in this case we have N derivatives but only $n < N$ ψ variables. Therefore

$$\int \prod_{j=1}^{N} d\overline{\psi}_j d\psi_j \; e^{-(\overline{\psi}, A\psi)} = \frac{(-1)^N}{N!} I_N.$$

To compute I_N note that

$$\begin{aligned}
(\overline{\psi}, A\psi)^N &= \left(\sum_{jk} \overline{\psi}_j A_{jk} \psi_k \right)^N = \sum_{j_1,\dots,j_N} \sum_{k_1,\dots,k_N} \overline{\psi}_{j_1} \psi_{k_1} \cdots \overline{\psi}_{j_N} \psi_{k_N} \; A_{j_1 k_1} \cdots A_{j_N k_N} \\
&= \sum_{\sigma,\tau \in \mathcal{P}(N)} \prod_{i=1}^{N} \overline{\psi}_{\sigma(i)} \psi_{\tau(i)} \prod_{i=1}^{N} A_{\sigma(i)\tau(i)}
\end{aligned}$$

where we used the fact that, since $\psi_j^2 = \overline{\psi}_j^2 = 0$ we cannot have $j_i = j_{i'}$ or $k_i = k_{i'}$ for $i \neq i'$.

Claim (exercise) We have

$$\prod_{i=1}^{N} \overline{\psi}_{\sigma(i)} \psi_{\tau(i)} = \epsilon^{\sigma} \epsilon^{\tau} \prod_{i=1}^{N} \overline{\psi}_i \psi_i \qquad \forall \sigma, \tau \in \mathcal{P}(N). \tag{2.16}$$

Inserting all this in I_N we get

$$\begin{aligned}
I_N &= \prod_{j=1}^{N} \partial_{\overline{\psi}_j} \partial_{\psi_j} \; (\overline{\psi}, A\psi)^N = \prod_{j=1}^{N} \partial_{\overline{\psi}_j} \partial_{\psi_j} \sum_{\sigma,\tau \in \mathcal{P}(N)} \prod_{i=1}^{N} \overline{\psi}_{\sigma(i)} \psi_{\tau(i)} \prod_{i=1}^{N} A_{\sigma(i)\tau(i)} \\
&= \prod_{j=1}^{N} \partial_{\overline{\psi}_j} \partial_{\psi_j} \; \prod_{i=1}^{N} \overline{\psi}_i \psi_i \sum_{\sigma,\tau \in \mathcal{P}(N)} \epsilon^{\sigma} \epsilon^{\tau} \prod_{i=1}^{N} A_{\sigma(i)\tau(i)}.
\end{aligned}$$

We compute

$$\prod_{j=1}^{N} \partial_{\overline{\psi}_j} \partial_{\psi_j} \; \prod_{i=1}^{N} \overline{\psi}_i \psi_i = \prod_{j=1}^{N} \left[\partial_{\overline{\psi}_j} \partial_{\psi_j} \overline{\psi}_j \psi_j \right] = \prod_{j=1}^{N} \left[-\partial_{\overline{\psi}_j} \partial_{\psi_j} \psi_j \; \overline{\psi}_j \right] = (-1)^N$$

and

$$\sum_{\sigma,\tau \in \mathcal{P}(N)} \epsilon^{\sigma} \epsilon^{\tau} \prod_{i=1}^{N} A_{\sigma(i)\tau(i)} = \sum_{\sigma,\tau \in \mathcal{P}(N)} \epsilon^{\tau \circ \sigma^{-1}} \prod_{i=1}^{N} A_{i\tau(\sigma^{-1}(i))} = N! \sum_{\tau \in \mathcal{P}(N)} \epsilon^{\tau} \prod_{i=1}^{N} A_{i\tau(i)} = N! \; \det A.$$

Finally

$$\int \prod_{j=1}^{N} d\overline{\psi}_j d\psi_j \; e^{-(\overline{\psi}, A\psi)} = \frac{(-1)^N}{N!} I_N = \det A,$$

which concludes the proof of (i).

(ii) exercise.

□

In the following we will mostly use (2.13).

Lemma 2.11 (moments)**.** *Let $A \in \mathbb{C}^{N\times N}$. Remember that for $I, J \subset \{1,\dots,N\}$ we define $\psi^I := \prod_{i\in I}\psi_i$, $\overline{\psi}^J := \prod_{j\in J}\overline{\psi}_j$ where the product is performed according to the ordering in I, J.*

(i) For all $I, J \subset \{1,\dots,N\}$ non empty sets we have

$$\int \prod_{j=1}^N d\overline{\psi}_j d\psi_j \; e^{-(\overline{\psi},A\psi)}\psi^I = 0$$
$$\int \prod_{j=1}^N d\overline{\psi}_j d\psi_j \; e^{-(\overline{\psi},A\psi)}\overline{\psi}^J = 0.$$

Moreover, if $|I| \neq |J|$ we also have

$$\int \prod_{j=1}^N d\overline{\psi}_j d\psi_j \; e^{-(\overline{\psi},A\psi)}\psi^I\overline{\psi}^J = 0.$$

(ii) Assume $|I| = |J| = p \geq 1$. Let

$$I = \{i_1,\dots,i_p\}, \qquad J = \{j_1,\dots,j_p\},$$

with the ordering $i_1 < i_2 < \cdots < i_p$, $j_1 < j_2 < \cdots < j_p$. We define

$$\psi_I\overline{\psi}_J := \psi_{i_1}\overline{\psi}_{j_1}\cdots\psi_{i_p}\overline{\psi}_{j_p} = \prod_{l=1}^p \psi_{i_l}\overline{\psi}_{j_l}.$$

Moreover we define $A_{J^cI^c} \in \mathbb{C}^{(N-p)\times(N-p)}$ the matrix obtained by removing from A the rows corresponding to the indices J and the columns corresponding to the indices I. With this notation we have

$$\int \prod_{j=1}^N d\overline{\psi}_j d\psi_j \; e^{-(\overline{\psi},A\psi)}\psi_I\overline{\psi}_J = (-1)^{\sum I + \sum J}\det A_{J^cI^c}, \tag{2.17}$$

where

$$\sum I := \sum_{l=1}^p i_l, \qquad \sum J := \sum_{l=1}^p j_l.$$

In particular we have, for all $i, j \in \{1,\dots,N\}$

$$\int \prod_{j=1}^N d\overline{\psi}_j d\psi_j \; e^{-(\overline{\psi},A\psi)}\psi_i\overline{\psi}_j = (-1)^{i+j}\det A_{\{j\}^c\{i\}^c} = Cof(A)_{ij}.$$

Proof.
(i) The first identity follows from

$$\prod_{j=1}^N \partial_{\overline{\psi}_j}\partial_{\psi_j}\; (\overline{\psi}, A\psi)^k\psi^I = 0 \qquad \forall k \geq 0,$$

which holds since we have k powers of $\overline{\psi}$ and $k+|I|>k$ powers of ψ. Similar arguments work for the other two identities.

(ii) Note that, for any function $f(\psi)$ we have

$$\psi_j f(\psi)=\psi_j[f_{|\psi_j=0}+\psi_j\partial_{\psi_j}f]=\psi_j f_{|\psi_j=0}. \tag{2.18}$$

Hence

$$\psi_I\overline{\psi}_J e^{-(\overline{\psi},A\psi)}=\psi_I\overline{\psi}_J e^{-(\overline{\psi},A\psi)_{|\psi_i=0\forall i\in I,\overline{\psi}_j=0\forall j\in J}},$$

where

$$(\overline{\psi},A\psi)_{|\psi_i=0\forall i\in I,\overline{\psi}_j=0\forall j\in J}=(\overline{\psi}_{|J^c},A_{J^cI^c}\psi_{|I^c}).$$

We can reorganize the product of differentials as follows (exercise)

$$\prod_{j=1}^N d\overline{\psi}_j d\psi_j\ =(-1)^{\sum I+\sum J}d\overline{\psi}_J d\psi_I\ d\overline{\psi}_{J^c}d\psi_{I^c}$$

where

$$d\overline{\psi}_J d\psi_I=\prod_{l=1}^p d\overline{\psi}_{j_l}d\psi_{i_l},\qquad d\overline{\psi}_{J^c}d\psi_{I^c}=\prod_{l=1}^{N-p}d\overline{\psi}_{j_l^c}d\psi_{i_l^c},$$

where in the second product we organized also the elements in I^c,J^c in growing order. Putting all this together we get

$$\int\prod_{j=1}^N d\overline{\psi}_j d\psi_j\ e^{-(\overline{\psi},A\psi)}\psi_I\overline{\psi}_J=(-1)^{\sum I+\sum J}\int d\overline{\psi}_J d\psi_I\ (\psi_I\overline{\psi}_J)\int d\overline{\psi}_{J^c}d\psi_{I^c}e^{-(\overline{\psi}_{|J^c},A_{J^cI^c}\psi_{|I^c})}$$
$$=(-1)^{\sum I+\sum J}\det A_{J^cI^c}\int d\overline{\psi}_J d\psi_I\ (\psi_I\overline{\psi}_J).$$

Finally we compute

$$\int d\overline{\psi}_J d\psi_I\ (\psi_I\overline{\psi}_J)=\prod_{l=1}^p\partial_{\overline{\psi}_{j_l}}\partial_{\psi_{i_l}}\prod_{l=1}^p(\psi_{i_l}\overline{\psi}_{j_l})=\prod_{l=1}^p\left[\partial_{\overline{\psi}_{j_l}}\partial_{\psi_{i_l}}\psi_{i_l}\overline{\psi}_{j_l}\right]=1.$$

This concludes the proof of the lemma.

□

Definition 2.12. *Assume $A\in\mathbb{C}^{N\times N}$ and invertible. Set $C:=A^{-1}$. The normalized Grassmann Gaussian measure with mean zero and covariance $C=A^{-1}$ is*

$$d\mu_C(\overline{\psi},\psi):=\frac{1}{\det A}\prod_{j=1}^N d\overline{\psi}_j d\psi_j\ e^{-(\overline{\psi},A\psi)}. \tag{2.19}$$

Note that this is not a true measure!

Theorem 2.13. *Assume $A\in\mathbb{C}^{N\times N}$ and invertible. Set $C:=A^{-1}$.*

(i) We have

$$\int d\mu_C(\overline{\psi},\psi)1=1,\qquad\int d\mu_C(\overline{\psi},d\psi)\psi_i=\int d\mu_C(\overline{\psi},\psi)\overline{\psi}_j=0,\qquad\int d\mu_C(\overline{\psi},\psi)\psi_i\overline{\psi}_j=C_{ij}.$$

(ii) (Laplace-Fourier transform)

Let $dim(\mathcal{V}) = 2N + 2N'$ *with* $N' \geq N$. *Let* $\{\overline{\psi}_1, \psi_1, \ldots, \overline{\psi}_N, \psi_N, \overline{\xi}_1, \xi_1, \ldots \overline{\xi}_{N'}, \xi_{N'}\}$ *be a basis for* $\mathcal{V}$. *For any*

$$\eta = (\eta_1, \ldots, \eta_N) \in \mathcal{G}_{\mathbb{K}}^{\text{odd}}[\overline{\xi}_1, \xi_1, \ldots \overline{\xi}_{N'}, \xi_{N'}]^N,$$
$$\tilde{\eta} = (\tilde{\eta}_1, \ldots, \tilde{\eta}_N) \in \mathcal{G}_{\mathbb{K}}^{\text{odd}}[\overline{\xi}_1, \xi_1, \ldots \overline{\xi}_{N'}, \xi_{N'}]^N$$

we define

$$(\overline{\psi}, \eta) := \sum_{j=1}^{N} \overline{\psi}_j \eta_j, \qquad (\tilde{\eta}, \psi) := \sum_{j=1}^{N} \tilde{\eta}_j \psi_j.$$

With this notation

$$\int d\mu_C(\overline{\psi}, \psi) e^{v(\overline{\psi}, \eta)} e^{w(\tilde{\eta}, \psi)} = e^{vw(\tilde{\eta}, C\eta)} \qquad \forall v, w \in \mathbb{C}. \tag{2.20}$$

[6: 29.10.2024]

[7: 05.11.2024]

Proof.
(i) Follows from Lemma 2.11 together with

$$(-1)^{i+j} \frac{\det A_{\{j\}^c \{i\}^c}}{\det A} = (A^{-1})_{ij} = C_{ij}.$$

(ii) One may expand both sides in $\xi, \overline{\xi}$ (exercise). We will use instead a coordinate change (later).

□

2.6 Coordinate changes

Definition 2.14 (generators)**.** *Let* $\dim \mathcal{V} = N$. *A set* $\chi_1, \ldots, \chi_N \in \mathcal{G}_{\mathbb{K}}^{\text{odd}}[\mathcal{V}]$ *is a set of* generators *for* $\mathcal{G} = \mathcal{G}_{\mathbb{K}}[\mathcal{V}]$ *if every element* $v \in \mathcal{G}$ *admits a unique decomposition*

$$v = \sum_{I \subset \{1, \ldots, N\}} v_I \ \chi^I.$$

In this case we write $\mathcal{G}_{\mathbb{K}}[\mathcal{V}] = \mathcal{G}_{\mathbb{K}}[\chi_1, \ldots, \chi_N]$.

A basis for $\mathcal{V}$ is a natural set of generators.

2.6.1 Grassmann translation

To define the translation by an odd element we need to enlarge the algebra. Let $\{\psi_1, \ldots, \psi_N, \xi_1, \ldots, \xi_{N'}\}$ be a basis for $\mathcal{V}$. We can generalize the definintion of function as follows.

Definition 2.15 (Functions III)**.** *Let* $\{\psi_1, \ldots, \psi_N, \xi_1, \ldots, \xi_{N'}\}$ *be basis for* $\mathcal{V}$. *A function of the variables* $\{\psi_1, \ldots, \psi_N\}$ *taking values in* $\mathcal{G}_{\mathbb{K}}[\mathcal{V}]$ *is any element of the algebra. Every such function is a polynomial of degree at most 1 in each variable:*

$$f(\psi_1, \ldots, \psi_N) = \sum_{\substack{I \subset \{1, \ldots, N\} \\ I' \subset \{1, \ldots, N'\}}} v_{I,I'} \ \psi^I \ \xi^{I'}. \tag{2.21}$$

where $v_{I,I'} \in \mathbb{K}$.

Let now $\alpha_1, \ldots, \alpha_N \in \mathcal{G}_{\mathbb{K}}^{\mathrm{odd}}[\xi_1, \ldots, \xi_{N'}]$, i.e. $\alpha_j = \alpha_j(\xi)$ is an odd function of the ξ variables taking values in $\mathcal{G}_{\mathbb{K}}[\xi_1, \ldots, \xi_{N'}]$.
We define $\chi_j := \psi_j + \alpha_j$ $j = 1, \ldots, N$. Then $\{\chi_1, \ldots, \chi_N, \xi_1, \ldots, \xi_{N'}\}$ is a set of generators for $\mathcal{G}_{\mathbb{K}}[\mathcal{V}]$ (exercise) but $\chi_j \notin \mathcal{V}$ in general.

Under the coordinate change $\chi = \psi + \alpha(\xi)$ the function $f(\psi) \in \mathcal{G}_{\mathbb{K}}[\psi_1, \ldots, \psi_N, \xi_1, \ldots, \xi_{N'}]$ transforms as

$$f(\psi) = \sum_{\substack{I \subset \{1,\ldots,N\} \\ I' \subset \{1,\ldots,N'\}}} v_{I,I'}\ \psi^I\ \xi^{I'} \mapsto \tilde{f}(\chi) := f(\chi - \alpha) = \sum_{\substack{I \subset \{1,\ldots,N\} \\ I' \subset \{1,\ldots,N'\}}} v_{I,I'}\ (\chi - \alpha)^I\ \xi^{I'}.$$

Proposition 2.16. *Let $\mathcal{V} = \mathrm{span}\{\psi_1, \ldots, \psi_N, \xi_1, \ldots, \xi_{N'}\}$, $f(\psi)$ a function taking values in $\mathcal{G}_{\mathbb{K}}[\mathcal{V}]$, $\alpha_1, \ldots, \alpha_N \in \mathcal{G}_{\mathbb{K}}^{\mathrm{odd}}[\xi_1, \ldots, \xi_{N'}]$ and $\chi_j := \psi_j + \alpha_j$ $j = 1, \ldots, N$ the corresponding translation. We have*

$$\int \prod_{j=1}^N d\psi_j\ f(\psi) = \int \prod_{j=1}^N d\chi_j\ f(\chi - \alpha).$$

Proof. Remember that

$$\int \prod_{j=1}^N d\psi_j\ f(\psi) = \prod_{j=1}^N \partial_{\psi_j}\ f(\psi)$$

We perform the derivarives one at a time.

$$\partial_{\psi_N}\ f(\psi) = \partial_{\psi_N}\ [f_0 + \psi_N f_1] = f_1$$

where f_0, f_1 are independent of ψ_N. We argue

$$\partial_{\chi_N}\ f(\chi - \alpha) = \partial_{\chi_N}(f_0 + (\chi_N - \alpha_N) f_1) = f_1.$$

The result follows repeating for all j. □

As a first application we use this result to prove the second statement in Theorem 2.13.

Proof of Theorem 2.13 (ii). We have

$$\int d\mu_C(\overline{\psi}, \psi) e^{v(\overline{\psi},\eta)} e^{w(\tilde{\eta},\psi)} = \frac{1}{\det A} \int \prod_{j=1}^N d\overline{\psi}_j d\psi_j\ e^{-(\overline{\psi}, A\psi)} e^{v(\overline{\psi},\eta)} e^{w(\tilde{\eta},\psi)}.$$

We compute

$$(\overline{\psi}, A\psi) - v(\overline{\psi}, \eta) - w(\tilde{\eta}, \psi) = \Big((\overline{\psi} - \tilde{\alpha}), A(\psi - \alpha)\Big)\ -\ vw\ (\tilde{\eta}, A^{-1}\eta)$$

where we defined

$$\alpha_j := v(A^{-1}\xi)_j, \qquad \tilde{\alpha}_j := w((A^{-1})^t \tilde{\eta})_j.$$

The result now follows inserting this in the integral and using Lemma 2.16. □

Note that, in the proof above $\tilde{\alpha}_j$ is NOT the complex conjugate of α_j. To define more general coordinate changes we need to introduce the composition of functions.

2.6.2 Chain rule with Grassmann variables

Proposition 2.17. *Let $\{\psi_1, \dots, \psi_N\}$ be a basis for $\mathcal{V}$.*

(i) Let $\chi_1(\psi), \dots, \chi_N(\psi) \in \mathcal{G}_{\mathbb{K}}^{\text{odd}}[\mathcal{V}]$, N odd functions of $\psi_1, \dots, \psi_N$.
Then, for any $f(\psi) = f(\psi_1, \dots, \psi_N) \in \mathcal{G}_{\mathbb{K}}[\mathcal{V}]$, we have

$$\partial_{\psi_j} f(\chi_1(\psi), \dots, \chi_N(\psi)) = \sum_{k=1}^{N} \partial_{\psi_j} \chi_k(\psi)\ \partial_{\tilde{\psi}_k} f(\tilde{\psi})_{|\tilde{\psi}=\chi(\psi)} \tag{2.22}$$

(ii) Let $g_1(\psi), \dots, g_p(\psi) \in \mathcal{G}_{\mathbb{K}}^{\text{even}}[\mathcal{V}]$, p even functions of $\psi_1, \dots, \psi_N$. In particular $g_j(\psi)$ admits the unique decomposition $g_j(\psi) = x_j + n_j(\psi)$ where $x_j \in \mathbb{K}$ and $n_j(\psi)$ is an even nilpotent function of ψ.
Then, for any $f \in C^N(U; \mathbb{C})$ with $U \subset \mathbb{K}^p$ open and $2N \geq \dim \mathcal{V}$

$$\partial_{\psi_j} f(g(\psi)) = \sum_{k=1}^{p} \partial_{\psi_j} g_k(\psi)\ \partial_{z_k} f(z)_{|z=g(\psi)} \tag{2.23}$$

Proof.
(i) Since every function $f(\psi)$ can be written as

$$f(\psi) = \sum_{I \subset \{1,\dots,N\}} v_I \psi^I$$

it is sufficient to consider the case $f = \psi^I = \prod_{l=1}^{p} \psi_{i_l}$ with $i_1 < i_2 < \dots < i_p$. We argue by induction on p.
For $p = 1$ we have $f(\chi(\psi)) = \chi_{i_1}(\psi)$ and

$$\begin{aligned} \partial_{\psi_j} f(\chi(\psi)) &= \partial_{\psi_j} \chi_{i_1}(\psi) = \partial_{\psi_j} \chi_{i_1}(\psi) \partial_{\tilde{\psi}_{i_1}} \tilde{\psi}_{i_1} \\ &= \sum_{k=1}^{N} \partial_{\psi_j} \chi_k(\psi)\ \partial_{\tilde{\psi}_k} \tilde{\psi}_{i_1} = \sum_{k=1}^{N} \partial_{\psi_j} \chi_k(\psi)\ \partial_{\tilde{\psi}_k} f(\tilde{\psi})_{|\tilde{\psi}=\chi(\psi)}, \end{aligned}$$

where we used $\partial_{\tilde{\psi}_k} \tilde{\psi}_{i_1} = \delta_{k,i_1}$. Assume the statement is true for $p \geq 1$ and consider $f(\psi) = \prod_{l=1}^{p+1} \psi_{i_l}$. We can write f as a product of two functions

$$f(\psi) = f_1(\psi) f_2(\psi), \qquad f_1(\psi) := \psi_{i_1},\ f_2(\psi) := \prod_{l=2}^{p+1} \psi_{i_l}.$$

We argue now, using Lemma 2.8 and the fact that χ_{i_1} is odd,

$$\partial_{\psi_j} f(\chi(\psi)) = \partial_{\psi_j} (f_1 f_2) = \partial_{\psi_j} \Big(\chi_{i_1}(\psi) f_2(\chi(\psi)) \Big) = \Big(\partial_{\psi_j} \chi_{i_1}(\psi) \Big) f_2(\chi(\psi)) - \chi_{i_1}(\psi) \partial_{\psi_j} f_2(\chi(\psi)).$$

Using $\partial_{\tilde{\psi}_k} f_1(\tilde{\psi}) = \partial_{\tilde{\psi}_k} \tilde{\psi}_{i_1} = \delta_{k,i_1}$, we write

$$\partial_{\psi_j} \chi_{i_1}(\psi) = \partial_{\psi_j} \chi_{i_1}(\psi) = \sum_{k=1}^{N} \partial_{\psi_j} \chi_k(\psi)\ \partial_{\tilde{\psi}_k} f_1(\tilde{\psi})_{|\tilde{\psi}=\chi(\psi)}.$$

The induction hypothesis gives

$$\partial_{\psi_j} f_2(\chi(\psi)) = \sum_{k=1}^{N} \partial_{\psi_j}\chi_k(\psi)\ \partial_{\tilde\psi_k} f_2(\tilde\psi)_{|\tilde\psi=\chi(\psi)}.$$

Inserting this above and using that $\partial_{\psi_j}\chi_k(\psi)$ is even we obtain

$$\begin{aligned}\partial_{\psi_j} f(\chi(\psi)) = \partial_{\psi_j}(f_1 f_2) &= \sum_{k=1}^{N} \partial_{\psi_j}\chi_k(\psi)\ \Big[(\partial_{\tilde\psi_k} f_1(\tilde\psi))\ f_2(\tilde\psi) - f_1(\tilde\psi)\ (\partial_{\tilde\psi_k} f_2(\tilde\psi))\Big]_{|\tilde\psi=\chi(\psi)}\\ &= \sum_{k=1}^{N} \partial_{\psi_j}\chi_k(\psi)\ \partial_{\tilde\psi_k}(f_1 f_2)_{|\tilde\psi=\chi(\psi)},\end{aligned}$$

which concludes the proof of (i).

(ii) Using the definition 2.5 we write

$$f(g(\psi)) = f(x_1+n_1,\dots,x_p+n_p) = \sum_{q_1,\dots q_N\geq 0} \frac{f^{(q_1,\dots q_N)}(x)}{q_1!\cdots q_N!} n_1^{q_1}\cdots n_N^{q_N}.$$

In the case $p=1$ we compute

$$\partial_{\psi_j} f(g(\psi)) = \partial_{\psi_j} \sum_{q\geq 0} \frac{f^{(q)}(x)}{q!} n(\psi)^{(q)} = \sum_{q\geq 0} \frac{f^{(q)}(x)}{q!} \partial_{\psi_j} n(\psi)^q.$$

We argue, using Lemma 2.8 and the fact that $n(\psi)$ is even,

$$\partial_{\psi_j} n^q = q\ (\partial_{\psi_j} n)\ n^{q-1} = q\ (\partial_{\psi_j} g(\psi))\ n^{q-1},$$

where in the last step we used $\partial_{\psi_j} g(\psi) = \partial_{\psi_j}(x+n(\psi)) = \partial_{\psi_j} n$. Hence

$$\partial_{\psi_j} f(g(\psi)) = (\partial_{\psi_j} g(\psi))\ \sum_{q\geq 1} \frac{f^{(q)}(x)}{(q-1)!}\ n^{q-1} = (\partial_{\psi_j} g(\psi))\ \sum_{q\geq 0} \frac{(f')^{(q)}(x)}{q!} n^q = (\partial_{\psi_j} g(\psi))\ f'(x+n).$$

The case of general p works in the same way (exercise).

□

2.6.3 Linear transformation of Grassmann variables

Let $\{\psi_1,\dots,\psi_N,\xi_1,\dots,\xi_{N'}\}$ be a basis for $\mathcal{V}$. We consider the Grassmann algebra $\mathcal{G}_{\mathbb{K}}[\mathcal{V}]$.
Let $Q\in\ \mathcal{G}^{\text{even}}_{\mathbb{K}}[\xi_1,\dots,\xi_{N'}]^{N\times N}$ such that $\text{body}(Q) = \{\text{body}(Q_{ij})\}_{i,j=1}^N \in \mathbb{K}^{N\times N}$ is invertible. Then $\chi := Q\psi$ is an invertible linear transformation of ψ. In particular $\chi_j\in\mathcal{G}^{\text{odd}}_{\mathbb{K}}[\mathcal{V}]\ \forall j$ and $\{\chi_1,\dots,\chi_N,\xi_1,\dots,\xi_{N'}\}$ is a set of generators for $\mathcal{G}_{\mathbb{K}}[\mathcal{V}]$ (exercise).

Proposition 2.18. *Let $\{\psi_1,\dots,\psi_N,\xi_1,\dots,\xi_{N'}\}$ be a basis for $\mathcal{V}$. We consider the Grassmann algebra $\mathcal{G}_{\mathbb{K}}[\mathcal{V}]$. Let $Q\in\ \mathcal{G}^{\text{even}}_{\mathbb{K}}[\xi_1,\dots,\xi_{N'}]^{N\times N}$ such that* body(Q) *is invertible and set $\chi(\psi) := Q\psi$.*
For any function $f = f(\chi) = f(\chi_1,\dots,\chi_N)\in\mathcal{G}_{\mathbb{K}}[\mathcal{V}]$ we have

$$\int \prod_{j=1}^{N} d\chi_j\ f(\chi) = \frac{1}{\det Q}\int \prod_{j=1}^{N} d\psi_j\ f(Q\psi).$$

Remark Note that in the case of N real variables $x = x_1, \dots, x_N$ we have

$$\int_{\mathbb{R}^N} \prod_{j=1}^N dx_j \; f(x) = \det Q \int_{\mathbb{R}^N} \prod_{j=1}^N dy_j \; f(Qy).$$

In the Grassmann case we have instead of $\frac{1}{\det Q}$ $\det Q$. This is due to the fact integrating in a Grassmann variables is the same as deriving in the variable.

Proof. Remember

$$\int \prod_{j=1}^N d\psi_j \; f(Q\psi) = \prod_{j=1}^N \partial_{\psi_j} \; f(Q\psi) = \prod_{j=1}^N \partial_{\psi_j} \; f(\chi(\psi))$$

where we defined $\chi_j(\psi) := (Q\psi)_j$. Using Prop. 2.17 we argue

$$\partial_{\psi_N} f(\chi(\psi)) = \sum_k (\partial_{\psi_N} \chi_k(\psi)) \partial_{\chi_k} f(\chi)_{|\chi(\psi)} = \sum_k Q_{kN} \partial_{\chi_k} f(\chi)_{|\chi(\psi)},$$

where we used $\partial_{\psi_N} \chi_k(\psi) = \partial_{\psi_N}(Q\psi)_k = Q_{kN}$. Repeting for each derivative we get

$$\begin{aligned}
\prod_{j=1}^N \partial_{\psi_j} \; f(Q\psi) &= \sum_{k_1,\dots,k_N} Q_{k_1 1} \cdots Q_{k_N N} \partial_{\chi_{k_1}} \cdots \partial_{\chi_{k_N}} f(\chi)_{|\chi = Q\psi} \\
&= \sum_{\sigma \in \mathcal{P}(N)} \prod_{j=1}^N Q_{\sigma(j),j} \prod_{j=1}^N \partial_{\chi_{\sigma(j)}} f(\chi) \\
&= \sum_{\sigma \in \mathcal{P}(N)} \varepsilon^\sigma \prod_{j=1}^N Q_{\sigma(j),j} \prod_{j=1}^N \partial_{\chi_j} f(\chi) = \det Q \int \prod_{j=1}^N d\chi_j \; f(\chi)
\end{aligned}$$

where in the second line we removed the information $\chi = Q\psi$ since the result of N derivatives in χ is independent of χ. This concludes the proof. □

[7: 05.11.2024]

[8: 07.11.2024]

2.6.4 Translations by even elements

Definition 2.19. *Let $U \subset \mathbb{R}^k$ be an open subset and $n = (n_1, \dots n_k)$ with $n_j \in \mathcal{G}^{\text{even}}_{\mathbb{R}}[\mathcal{V}] \setminus \mathbb{R} \; \forall j$ a nilpotent even element for all j. We define*

$$U + n := \{v = (v_1, \dots v_k) \in (\mathcal{G}^{\text{even}})^k | \; v_j = x_j + n_j, x = (x_1, \dots, x_k) \in U\}.$$

For any function $f \in C^N(U;\mathbb{K})$ with $2N \geq \dim \mathcal{V}$, and $f^{(q_1,\dots,q_k)} \in L^1(U) \; \forall \sum_j q_j \leq N$, we define

$$\int_{U+n} dv f(v) := \int_U dx \; f(x+n) = \sum_{q_1,\dots q_N} \frac{\prod_j n_j^{q_j}}{\prod_j q_j!} \int_U dx \; f^{(q_1,\dots,q_k)}(x)$$

where the sum is automatically restricted to $\sum_j q_j \leq N$ since $\prod_j n_j^{q_j} = 0$ otherwise.

Proposition 2.20. *Let $U \subset \mathbb{R}^k$ be an open ans bounded subset with smooth boundary and $n = (n_1, \dots n_k)$ with $n_j \in \mathcal{G}_{\mathbb{R}}^{\mathrm{even}}[\mathcal{V}] \setminus \mathbb{R}\ \forall j$ a nilpotent even element for all j. Let $k_j \geq 0$ the unique integer such that $n_j^{k_j} \neq 0$ and $n_j^{k_j+1} = 0$. Let $f \in C^N(\overline{U})$ with $N \geq \overline{k} := \sum_j k_j$.*
If in addition $f_{\partial U}^{(q_1,\dots,q_k)} = 0$ for all $0 \leq q_j \leq k_j$ then

$$\int_{U+n} dv f(v) = \int_U dx f(x).$$

The same result holds for unbounded domain as long as the function and all the relevant derivatives are integrable and vanish at infinity.

Proof. Apply Gauss theorem and the fact that the function and enough derivatives vanish on the boundary of U.

□

Application: Fourier/Laplace transform of a real/complex Gaussian measure

Theorem 2.21.

(i) *Assume $A \in \mathbb{C}^{N\times N}_{+,sym}$ and set $C := A^{-1}$. Then, for all $v \in \mathcal{G}_{\mathbb{C}}^{\mathrm{even}}[\mathcal{V}]^N$ we have*

$$\int_{\mathbb{R}^N} d\mu_C(\varphi) e^{(\varphi,v)} = e^{\frac{1}{2}(v,C,v)}.$$

(ii) *Assume $A \in \mathbb{C}^{N\times N}_{+}$ and set $C := A^{-1}$. Then, for all $v, w \in \mathcal{G}_{\mathbb{C}}^{\mathrm{even}}[\mathcal{V}]^N$ we have*

$$\int_{\mathbb{C}^N} d\mu_C(\varphi,\overline{\varphi}) e^{(\overline{\varphi},v)+(w,\varphi)} = e^{(w,C,v)}.$$

Proof.
(i) Since $v_j \in \mathcal{G}_{\mathbb{C}}^{\mathrm{even}}[\mathcal{V}]\ \forall j$ we have $(\varphi, v) = \sum_{j=1}^N \varphi_j v_j \in \mathcal{G}_{\mathbb{C}}^{\mathrm{even}}[\mathcal{V}]$ and hence $e^{(\varphi,v)} \in \mathcal{G}_{\mathbb{C}}^{\mathrm{even}}[\mathcal{V}]$. Each v_j admits the unique decomposition $v_j = x_j + n_j$ with $x_j \in \mathbb{C}$ and n_j nilpotent. Hence

$$(\varphi, v) = (\varphi, x) + (\varphi, n).$$

Inserting this in the integral we get

$$\int_{\mathbb{R}^N} d\mu_C(\varphi) e^{(\varphi,v)} = \sqrt{\det A} \int_{\mathbb{R}^N} \prod_{j=1}^N d\varphi_j\ F(\varphi)\ e^{(\varphi,n)},$$

with

$$F(\varphi) := e^{-\frac{1}{2}(\varphi,A\varphi)}\ e^{(\varphi,x)}.$$

We argue, by completing the square,

$$F(\varphi)\, e^{(\varphi,x)} = e^{-\frac{1}{2}((\varphi-A^{-1}n),A(\varphi-A^{-1}n))} e^{\frac{1}{2}(n,Cn)} e^{((\varphi-A^{-1}n),x)} e^{(n,Cx)} = F(\varphi - A^{-1}n) e^{\frac{1}{2}(n,Cn)+(n,Cx)}.$$

Inserting this in the integral above we obtain

$$\int_{\mathbb{R}^N} d\mu_C(\varphi) e^{(\varphi,v)} = e^{\frac{1}{2}(n,Cn)+(n,Cx)} \sqrt{\det A} \int_{\mathbb{R}^N} \prod_{j=1}^N d\varphi_j \; F(\varphi - A^{-1}n)$$

$$= e^{\frac{1}{2}(n,Cn)+(n,Cx)} \sqrt{\det A} \int_{\mathbb{R}^N} \prod_{j=1}^N d\varphi_j \; F(\varphi)$$

$$= e^{\frac{1}{2}(n,Cn)+(n,Cx)} \int_{\mathbb{R}^N} d\mu_C(\varphi) e^{(\varphi,x)} = e^{\frac{1}{2}(n,Cn)+(n,Cx)+\frac{1}{2}(x,Cx)} = e^{\frac{1}{2}(v,Cv)}$$

where in the second line we used Prop. 2.20. This is applicable since $-A^{-1}n$ is even and F together with all its derivatives is integrable and vanishes at infinity.

(ii) reformulate the integral in terms of real and imaginary part of φ and argue as in (i).

□

2.6.5 Fubini

Proposition 2.22. *Let $\{\psi_1, .., \psi_N\}$ be a family of generators for the Grassmann algebra $\mathcal{G}_{\mathbb{K}}[\mathcal{V}]$ and $f(\psi) = \sum_{I \subset \{1,\ldots,N\}} v_I \psi^I$ a function. Let $U \subset \mathbb{R}^k$ and assume $v_I \in L^1(U;\mathbb{K})$ for all $I \subset \{1, \ldots, N\}$. We write $f = f(\psi, x)$. Then, for all $J \subset \{1, \ldots, N\}$ we have*

$$\int \prod_{j \in J} d\psi_j \int_U dx \; f(\psi, x) = \int_U dx \int \prod_{j \in J} d\psi_j \; f(\psi, x).$$

Proof. The result follows from

$$\prod_{j \in J} \partial_{\psi_j} \int_U dx \; (v_I(x) \psi^I) = \left(\int_U dx \; v_I(x) \right) \prod_{j \in J} \partial_{\psi_j} \psi^I.$$

□

2.7 Average of the determinant for GUE

We consider the measure on $\mathbb{C}_+^{N \times N}$

$$\prod_{j=1}^N dH_{jj} e^{-\frac{N}{2} H_{jj}^2} \prod_{i<j=1}^N d\overline{H}_{ij} dH_{ij} e^{-N \overline{H}_{ij} H_{ij}} = dH \; e^{-\frac{N}{2} \operatorname{Tr} H^2}$$

where $dH := \prod_{j=1}^N dH_{jj} \prod_{i<j=1}^N d\overline{H}_{ij} dH_{ij}$. We will use the notation

$$\langle f \rangle_N := \frac{1}{Z} \int_{\mathbb{C}_{herm}^{N \times N}} dH \; e^{-\frac{N}{2} \operatorname{Tr} H^2} f(H)$$

where $Z := \int_{\mathbb{C}_{herm}^{N \times N}} dH \; e^{-\frac{N}{2} \operatorname{Tr} H^2}$ is the constant normalizing the measure. We have seen in Lemma 1.11 that, for all $z \in \mathbb{C}_+$ we have

$$\langle \frac{1}{\det(z - H)} \rangle_N = \frac{\sqrt{N}}{\sqrt{2\pi}} \int_{\mathbb{R}} da \; e^{-N \frac{a^2}{2}} \frac{1}{(z-a)^N}.$$

We will use now the Grassmann calculus to prove a dual formula for the average of the determinant.

Lemma 2.23. *For all $z \in \mathbb{C}$ it holds*

$$\langle \det(z-H) \rangle_N = \frac{\sqrt{N}}{\sqrt{2\pi}} \int_{\mathbb{R}} db\ e^{-N\frac{b^2}{2}} (z-ib)^N.$$

Proof. Using Theorem 2.10 we write

$$\det(z-H) = \int \prod_{j=1}^N d\overline{\psi}_j d\psi_j\ e^{-(\overline{\psi},(z-H)\psi)},$$

hence

$$\begin{aligned}\langle \det(z-H) \rangle_N &= \frac{1}{Z} \int_{\mathbb{C}^{N\times N}_{herm}} dH \int \prod_{j=1}^N d\overline{\psi}_j d\psi_j\ e^{-\frac{N}{2}\operatorname{Tr} H^2} e^{-(\overline{\psi},(z-H)\psi)} \\ &= \frac{1}{Z} \int_{\mathbb{C}^{N\times N}_{herm}} dH \int \prod_{j=1}^N d\overline{\psi}_j d\psi_j\ f(\psi, H),\end{aligned}$$

where

$$f(\psi,H) = e^{-\frac{N}{2}\operatorname{Tr} H^2} e^{-(\overline{\psi},(z-H)\psi)} = \sum_{I,\overline{I} \subset \{1,\dots,N\}} v_{I,\overline{I}}(H) \psi^I \overline{\psi}^{\overline{I}}.$$

Each function $v_{I,\overline{I}}(H)$ has the form $Pol(H,z)e^{-\frac{N}{2}\operatorname{Tr} H^2}$, where $Pol(H,z)$ is a polynome in the matrix entries and z. Hence $v_{I,\overline{I}} \in L^1(\mathbb{C}^{N\times N}_{herm})$ and, by Prop 2.22, we can exchange the integration order. We otain

$$\langle \det(z-H) \rangle_N = \int \prod_{j=1}^N d\overline{\psi}_j d\psi_j e^{-z(\overline{\psi},\psi)} \langle e^{(\overline{\psi},H\psi)} \rangle_N.$$

We compute, using Theorem 2.21,

$$\begin{aligned}\langle e^{(\overline{\psi},H\psi)} \rangle_N &= \frac{1}{Z} \prod_j \int_{\mathbb{R}} dH_{jj}\ e^{-\frac{N}{2}H_{jj}^2} e^{H_{jj}\overline{\psi}_j\psi_j} \prod_{j<k} \int_{\mathbb{C}} d\overline{H}_{jk} dH_{jk}\ e^{-N|H_{jk}|^2} e^{(H_{jk}\overline{\psi}_j\psi_k + \overline{H}_{jk}\overline{\psi}_k\psi_j)} \\ &= \prod_j e^{\frac{1}{2N}(\overline{\psi}_j\psi_j)^2} \prod_{j<k} e^{\frac{1}{N}(\overline{\psi}_j\psi_k)(\overline{\psi}_k\psi_j)} = e^{\frac{1}{2N}\sum_{jk}(\overline{\psi}_j\psi_k)(\overline{\psi}_k\psi_j)}.\end{aligned}$$

Using $(\overline{\psi}_j\psi_k)(\overline{\psi}_k\psi_j) = -(\overline{\psi}_j\psi_j)(\overline{\psi}_k\psi_k)$ and Theorem 2.21 we reorganize the quartic term as follows

$$e^{\frac{1}{2N}\sum_{jk}(\overline{\psi}_j\psi_k)(\overline{\psi}_k\psi_j)} = e^{-\frac{1}{2N}[\sum_j(\overline{\psi}_j\psi_j)]^2} = \frac{\sqrt{N}}{\sqrt{2\pi}} \int_{\mathbb{R}} db\ e^{-\frac{N}{2}b^2} e^{ib[\sum_j(\overline{\psi}_j\psi_j)]}.$$

Inserting this above we get

$$\begin{aligned}\langle \det(z-H) \rangle_N &= \int \prod_{j=1}^N d\overline{\psi}_j d\psi_j e^{-z(\overline{\psi},\psi)} \frac{\sqrt{N}}{\sqrt{2\pi}} \int_{\mathbb{R}} db\ e^{-\frac{N}{2}b^2} e^{ib[\sum_j(\overline{\psi}_j\psi_j)]} \\ &= \frac{\sqrt{N}}{\sqrt{2\pi}} \int \prod_{j=1}^N d\overline{\psi}_j d\psi_j \int_{\mathbb{R}} db\ f(\psi,b),\end{aligned}$$

where

$$f(\psi,b) = e^{-z(\overline{\psi},\psi)} e^{-\frac{N}{2}b^2} e^{ib[\sum_j(\overline{\psi}_j\psi_j)]} = \sum_{I,\overline{I} \subset \{1,\dots,N\}} v_{I,\overline{I}}(b) \psi^I \overline{\psi}^{\overline{I}},$$

with

$$v_{I,\bar{I}}(b) = e^{-\frac{N}{2}b^2} Pol(b,z) \in L^1(\mathbb{R}) \qquad \forall I \subset \{1,\dots,N\}.$$

Hence, by Prop 2.22, we can exchange the integration order. We obtain

$$\langle \det(z-H)\rangle_N = \frac{\sqrt{N}}{\sqrt{2\pi}} \int_{\mathbb{R}} db\; e^{-\frac{N}{2}b^2} \int \prod_{j=1}^N d\overline{\psi}_j d\psi_j e^{-(z-ib)(\overline{\psi},\psi)} = \frac{\sqrt{N}}{\sqrt{2\pi}} \int_{\mathbb{R}} db\; e^{-\frac{N}{2}b^2} (z-ib)^N.$$

This completes the proof.

□

Proposition 2.24. *For all $z = E + i\varepsilon$, with $E \in \mathbb{R}$ and $\varepsilon > 0$ it holds*

$$\langle (E+i\varepsilon - H)^{-1}_{xy}\rangle_N =$$
$$(-i)\int_{\mathbb{C}^N} [d\overline{\varphi}d\varphi]^N \int [d\overline{\psi}d\psi]^N\; \varphi_x\overline{\varphi}_y e^{iz\sum_{j=1}^N(\overline{\varphi}_j\varphi_j+\overline{\psi}_j\psi_j)} e^{-\frac{1}{2N}\sum_{jk=1}^N(\overline{\varphi}_j\varphi_k+\overline{\psi}_j\psi_k)(\overline{\varphi}_k\varphi_j+\overline{\psi}_k\psi_j)},$$

where we defined

$$[d\overline{\varphi}d\varphi]^N := \prod_{j=1}^N \frac{d\overline{\varphi}_j d\varphi_j}{2\pi}, \qquad [d\overline{\psi}d\psi]^N := \prod_{j=1}^N \overline{\psi}_j d\psi_j.$$

Proof. We argue, since $\mathrm{Re}A := -i(z-H) = \varepsilon > 0$

$$\begin{aligned}
(E+i\varepsilon-H)^{-1}_{xy} &= (-i)A^{-1}_{xy} = (-i)\det A \int_{\mathbb{C}^N} [d\overline{\varphi}d\varphi]^N\; \varphi_x\overline{\varphi}_y\; e^{-(\overline{\varphi},A\varphi)} \\
&= (-i)\int_{\mathbb{C}^N} [d\overline{\varphi}d\varphi]^N \int [d\overline{\psi}d\psi]^N\; \varphi_x\overline{\varphi}_y e^{-(\overline{\varphi},A\varphi)} e^{-(\overline{\psi},A\psi)} \\
&= \; (-i)\int_{\mathbb{C}^N} [d\overline{\varphi}d\varphi]^N \int [d\overline{\psi}d\psi]^N\; \varphi_x\overline{\varphi}_y e^{iz\sum_{j=1}^N(\overline{\varphi}_j\varphi_j+\overline{\psi}_j\psi_j)} \\
&\quad \cdot \prod_j e^{-\frac{1}{2N}H_{jj}(\overline{\varphi}_j\varphi_j+\overline{\psi_j}\psi_j)}, \prod_{j<k} e^{-\frac{1}{N}\left(H_{jk}(\overline{\varphi}_j\varphi_k+\overline{\psi}_j\psi_k)+\overline{H_{jk}}(\overline{\varphi}_k\varphi_j+\overline{\psi}_k\psi_j)\right)}
\end{aligned}$$

Inserting this in the average and exchanging the integrals (argue as in previous lemma) we obtain the result. □

[8: 07.11.2024]

[9: 12.11.2024]

2.8 Average of the resolvent for random Schrödinger with Cauchy distribution

Fubini does not always apply. As an example consider the matrix $H \in \mathbb{R}^{N\times N}_{sym}$ of the form

$$H = T + \lambda V$$

where $T \in \mathbb{R}^{N\times N}_{sym}$ is a deterministic matrix, $V = \mathrm{diag}\,(\lambda_1,\dots,\lambda_N)$ is a diagonal matrix with random diagonal entries and $\lambda > 0$ is a parameter. When $\{1,\dots,N\}$ is replaced by $\Lambda \subset\subset \mathbb{Z}^d$ and T by $-\Delta$ with $\Delta =$ the lattice Laplacian this is called discrete random Schrödinger operator, or Anderson model.

In the following we assume the random variables $V_1, \dots, V_N$ are independent identically distributed with probability measure absolutely contibuous with respect to Lebesgue i.e.

$$d\rho(V) = \prod_{j=1}^{N} dV_j \rho(V_j),$$

where $\rho \in L^1(\mathbb{R}; [0,\infty))$ and $\int_{\mathbb{R}} dx \rho(x) = 1$.
We denote by $\langle f(V) \rangle_V = \int_{\mathbb{R}^N} d\rho(V) f(V)$ the corresponding average.

Theorem 2.25. *Assume $\rho(x) = \frac{1}{\pi}\frac{1}{1+x^2}$ (Cauchy distribution).*
For all $z \in \mathbb{C}_+$ the following identities hold.

(i) $\langle \frac{1}{\det(z-H)} \rangle_V = \frac{1}{\det(z+i\lambda-T)}$,

(ii) $\langle (z-H)^{-1}_{xy} \rangle_V = (z+i\lambda-T)^{-1}_{xy}$.

In particular, setting $z = E + i\varepsilon$ the limits $\varepsilon \to 0+$ are well defined

$$\lim_{\varepsilon \to 0+} \langle \frac{1}{\det(z-H)} \rangle_V = \frac{1}{\det(E+i\lambda-T)}, \qquad \lim_{\varepsilon \to 0+} \langle (z-H)^{-1}_{xy} \rangle_V = (E+i\lambda-T)^{-1}_{xy}.$$

Remark 1 The integrals above are well defined for all $\varepsilon > 0$. Indeed $z \in \mathbb{C}_+ \Rightarrow z = E + i\varepsilon$ with $\varepsilon > 0$. Since the eigenvalues of H are all real we have

$$\frac{1}{|\det(E+i\varepsilon-H)|} \leq \varepsilon^{-N}.$$

Moreover, using Cauchy-Schwartz and $\|e_x\| = \|e_y\| = 1$,

$$|(z-H)^{-1}_{xy}| = |(\delta_x, (z-H)^{-1}\delta_y)| \leq ((z-H)^{-1}\delta_y, (z-H)^{-1}\delta_y)^{\frac{1}{2}} = (\delta_y, ((E-H)^2+\varepsilon^2)^{-1}\delta_y)^{\frac{1}{2}}.$$

Since $(E-H)^2 = (E-H)^*(E-H) \geq 0$ as a quadratic form we conlcude $(E-H)^2 + \varepsilon^2 \geq \varepsilon^2$ and hence

$$|(z-H)^{-1}_{xy}| \leq (\delta_y, \frac{1}{\varepsilon^2}\delta_y)^{\frac{1}{2}} = \frac{1}{\varepsilon}.$$

Remark 2 Instead of reformulating the averages above as a new integral we obtain exact formulas, hence the oservables $\det(z-H)^{-1}$ and $(z-H)^{-1}_{xy}$ are called integrable. Note that there is no exact formula for $\langle |(z-H)^{-1}_{xy}|^2 \rangle_V$. This average gives information on the spectral type of H in the limit $N \to \infty$.

Facts on the Cauchy distribution

- Normalization: $\frac{1}{\pi}\int_{\mathbb{R}} dx \frac{1}{1+x^2} = \frac{1}{\pi} \arctan x|_{-\infty}^{\infty} = 1$.
- Moments: the integral $\frac{1}{\pi}\int_{\mathbb{R}} dx \frac{|x|^\alpha}{1+x^2}$ is finite for all $0 < \alpha < 1$ and diverges for all $\alpha \geq 1$. In particular the first moment is not well defined.
- The Laplace transform is not well defined.

- Fourier transform: $\hat{\rho}(t) := \frac{1}{\pi}\int_{\mathbb{R}} dx \frac{1}{1+x^2} e^{itx} = e^{-|t|}$ $\forall t \in \mathbb{R}$. This function satisfies $\hat{\rho} \in C^\infty(\mathbb{R}\setminus\{0\})$.

  Proof The function $(z^2+1)^{-1}e^{itz}$ is holomorphic on $\mathbb{C}\setminus\{i,-i\}$, therefore we use contour deformation. For all $t\in\mathbb{R}$ we have

$$\hat{\rho}(t) = \frac{1}{\pi}\int_{\mathbb{R}} dx \frac{1}{1+x^2} e^{itx} = \lim_{R\to\infty}\frac{1}{\pi}\int_{-R}^{R} dx \frac{1}{1+x^2} e^{itx}.$$

  Assume now $t>0$ and consider the contour $\gamma_R = [-R,R]\cup C_R$ where $C_R = \gamma[0,\pi]$ with $\theta \mapsto \gamma(\theta) := Re^{i\theta}$. For $R>1$ we have, using the Cauchy formula,

$$\oint_{\gamma_R} \frac{e^{itz}}{1+z^2} = \oint_{\gamma_R} \frac{f(z)}{z-i} = e^{-t},$$

  where $f(z) := e^{itz}(z+i)^{-1}$. Finally

$$\left|\int_{C_R} \frac{e^{itz}}{1+z^2}\right| \leq \frac{R}{R^2-1}\int_0^\pi d\theta\ e^{-tR\sin\theta} \leq \frac{\pi R}{R^2-1} \to_{R\to\infty} 0.$$

  For $t<0$ we repeat the argument with $\gamma_R = [-R,R]\cup C_R$ where $C_R = \gamma[0,\pi]$ with $\theta\mapsto\gamma(\theta) := Re^{-i\theta}$.

- Extension to the even Grassmann subalgebra: since $\hat{\rho}\in C^\infty(\mathbb{R}\setminus\{0\})$ we can define $\hat{\rho}(v)$ $\forall v \in \mathcal{G}^{\text{even}}_{\mathbb{R}}[\mathcal{V}]$ such that $\text{body}(v)\neq 0$. In particular, for all $v = t+n$ with $t\in\mathbb{R}\setminus\{0\}$ and $n^2=0$ we have

$$\hat{\rho}(t+n) = \hat{\rho}(t) + \hat{\rho}'(t)n.$$

Remark 3 Since $\det(z-H)$ is a polynome in $V_1,\dots V_N$ the corresponding average is not well defined.

Proof of Theorem 2.25.
(*i*) $z\in\mathbb{C}_+ \Rightarrow z = E+i\varepsilon$ with $\varepsilon>0$. Set $A := -i(z-H)$. We compute

$$\frac{1}{\det(E+i\varepsilon-H)} = \frac{(-i)^N}{\det A} = (-i)^N\int_{\mathbb{C}^N}[d\overline{\varphi}d\varphi]^N\ e^{-(\overline{\varphi},A\varphi)} = (-i)^N\int_{\mathbb{C}^N}[d\overline{\varphi}d\varphi]^N\ e^{i(\overline{\varphi},(z-T)\varphi)}\prod_{j=1}^N e^{-iV_j\lambda|\varphi_j|^2}.$$

Inserting this in the average we obtain

$$\langle\tfrac{1}{\det(z-H)}\rangle_V = (-i)^N\frac{1}{\pi^N}\int_{\mathbb{R}^N}\prod_j dV_j\int_{\mathbb{C}^N}[d\overline{\varphi}d\varphi]^N\ F(\varphi,V),$$

with

$$F(\varphi,V) = e^{i(\overline{\varphi},(z-T)\varphi)}\prod_j\left(\tfrac{1}{1+V_j^2}e^{-iV_j\lambda|\varphi_j|^2}\right).$$

The corresponding absolute value is

$$|F(\varphi,V)| = \prod_j\left(e^{-\varepsilon|\varphi_j|^2}\tfrac{1}{1+V_j^2}\right)$$

and is integrable both in V and φ. Therefore we can exchange the integrals and obtain

$$\begin{aligned}
&\langle \tfrac{1}{\det(z-H)} \rangle_V = (-i)^N \int_{\mathbb{C}^N} [d\overline{\varphi} d\varphi]^N \;{}^{i(\overline{\varphi},(z-T)\varphi)} \prod_j \hat{\rho}(\lambda|\varphi_j|^2) \\
&\quad = (-i)^N \int_{\mathbb{C}^N} [d\overline{\varphi} d\varphi]^N \;{}^{i(\overline{\varphi},(z-T)\varphi)} \prod_j e^{-\lambda|\varphi_j|^2} = (-i)^N \int_{\mathbb{C}^N} [d\overline{\varphi} d\varphi]^N \;{}^{i(\overline{\varphi},(z+i\lambda-T)\varphi)} \\
&\quad\quad = \tfrac{(-i)^N}{\det -i(z+i\lambda-T)} = \tfrac{1}{\det(z+i\lambda-T)}
\end{aligned}$$

where in the second line we used $|\lambda|\varphi_j|^2| = \lambda|\varphi_j|^2$ since $\lambda > 0$. This concludes the proof of (i).
(ii) We argue as above

$$\begin{aligned}
&(E + i\varepsilon - H)^{-1}_{xy} = (-i) A^{-1}_{xy} = (-i) \det A \int_{\mathbb{C}^N} [d\overline{\varphi} d\varphi]^N \;\; \varphi_x \overline{\varphi}_y e^{-(\overline{\varphi}, A\varphi)} \\
&= (-i) \int_{\mathbb{C}^N} [d\overline{\varphi} d\varphi]^N \int [d\overline{\psi} d\psi]^N \;\; \varphi_x \overline{\varphi}_y e^{-(\overline{\varphi}, A\varphi) - (\overline{\psi}, A\psi)} \\
&= (-i) \int_{\mathbb{C}^N} [d\overline{\varphi} d\varphi]^N \int [d\overline{\psi} d\psi]^N \;\; \varphi_x \overline{\varphi}_y e^{i(\overline{\varphi},(z-t)\varphi) + i(\overline{\psi},(z-t)\psi)} \prod_j e^{iV_j \lambda (\overline{\varphi}_j \varphi_j + \overline{\psi}_j \psi_j)}.
\end{aligned}$$

Inserting this in the average we obtain

$$\langle (E + i\varepsilon - H)^{-1}_{xy} \rangle_V = (-i) \frac{1}{\pi^N} \int_{\mathbb{R}^N} \prod_j dV_j \int_{\mathbb{C}^N} [d\overline{\varphi} d\varphi]^N \;\; \int [d\overline{\psi} d\psi]^N \sum_{I, \overline{I} \subset \{1,\ldots,N\}} v_{I,\overline{I}}(\varphi, V) \psi^I \overline{\psi}^{\overline{I}}$$

with

$$v_{I,\overline{I}}(\varphi, V) = e^{i(\overline{\varphi},(z-T)\varphi)} \prod_j \left(\tfrac{1}{1+V_j^2} e^{-iV_j\lambda|\varphi_j|^2} \right) Pol_{I,\overline{I}}(V),$$

where $Pol_{I,\overline{I}}(V)$ is a polynome of degree one in each variable $V_1, \ldots, V_N$. The absolute value is

$$|v_{I,\overline{I}}(\varphi, V)| = \prod_j \left(\tfrac{1}{1+V_j^2} e^{-\varepsilon|\varphi_j|^2} \right) \left| Pol_{I,\overline{I}}(V) \right|.$$

This function is not integrable in V, hence we cannot exchange the integrals. To solve the problem we introduce a regularization of the V distribution. Note that

$$e^{-iV_j \lambda \overline{\psi}_j \psi_j} = 1 - iV_j \lambda \overline{\psi}_j \psi_j,$$

therefore we only need to regularize enough to make the fist moment in each variable finite.

[9: 12.11.2024]

[10: 15.11.2024]

We replace ρ with ρ_η, $\eta > 0$ defined via

$$\rho_\eta(x) := \rho(x) \frac{1}{(1 + \eta x^2)} = \frac{1}{\pi} \frac{1}{(1 + x^2)(1 + \eta x^2)}.$$

We have $0 < \rho_\eta(x) \le \rho(x) \in L^1$ hence, by dominated convergence

$$\lim_{\eta \to 0} \int_{\mathbb{R}} dx \;\rho_\eta(x) f(x) = \int_{\mathbb{R}} dx \;\rho(x) f(x),$$

for any function such that $\int_{\mathbb{R}} dx \;\rho(x)|f(x)| < \infty$.

Facts on ρ_η

- Normalization: $\int_{\mathbb{R}} dx\, \rho_\eta \leq \int_{\mathbb{R}} dx\, \rho = 1$.
- Moments: the integral $\int_{\mathbb{R}} dx\, |x|^\alpha\, \rho_\eta$ is finite for all $0 < \alpha < 3$ and diverges for all $\alpha \geq 3$. In particular the first moment is well defined.
- The Laplace transform is not well defined.
- Fourier transform: for all $0 < \eta < 1, t \in \mathbb{R}$ we have

  $$\hat{\rho}_\eta(t) := \int_{\mathbb{R}} dx\, \rho_\eta(x)\, e^{itx} = \frac{1}{1-\eta}\left(e^{-|t|} - \eta e^{-\frac{1}{\sqrt{\eta}}|t|}\right).$$

  This function satisfies $\hat{\rho}_\eta \in C^\infty(\mathbb{R} \setminus \{0\})$.

  Proof exercise. Use again contour deformation but note that this time there are four poles.

- Extension to the even Grassmann subalgebra: since $\hat{\rho}_\eta \in C^\infty(\mathbb{R}\setminus\{0\})$ we can define $\hat{\rho}_\eta(v)$ $\forall v \in \mathcal{G}^{\text{even}}_{\mathbb{R}}[\mathcal{V}]$ such that $\text{body}(v) \neq 0$. In particular, for all $v = t + n$ with $t \in \mathbb{R}\setminus\{0\}$ and $n^2 = 0$ we have

  $$\hat{\rho}_\eta(t+n) = \hat{\rho}_\eta(t) + \hat{\rho}'_\eta(t)n.$$

- Bounds on $\hat{\rho}_\eta$ and $\hat{\rho}'_\eta(t)$: we have

  $$\lim_{\eta\to 0+} \hat{\rho}_\eta(t) = \hat{\rho}(t), \qquad \lim_{\eta\to 0+} \hat{\rho}'_\eta(t) = \hat{\rho}'(t) = \sigma(t)\, e^{-|t|}$$

  pointwise a.e., where $\sigma(t)$ is the sign of t. Moreover, for all $t \neq 0$ and $0 < \eta \ll 1$

  $$|\hat{\rho}'(t)| = \hat{\rho}(t) \leq 1, \qquad |\hat{\rho}_\eta(t)| \leq 2\hat{\rho}_\eta(t) \leq 2, \qquad |\hat{\rho}'_\eta(t)| \leq 2\hat{\rho}_\eta(t) \leq 2.$$

We use now the above facts to complete the proof of (ii). With the notation

$$\langle f(V_1,\ldots,V_N)\rangle_\eta := \int_{\mathbb{R}^N} \prod_j d\rho_\eta(V_j)\; f(V_1,\ldots,V_N)$$

we have

$$\begin{aligned}
&\langle (E+i\varepsilon - H)^{-1}_{xy}\rangle_V = \lim_{\eta\to 0+} \langle (E+i\varepsilon - H)^{-1}_{xy}\rangle_\eta \\
&= (-i) \lim_{\eta\to 0+} \int_{\mathbb{C}^N} [d\overline{\varphi} d\varphi]^N \int [d\overline{\psi} d\psi]^N\; \varphi_x \overline{\varphi}_y e^{i(\overline{\varphi},(z-t)\varphi) + i(\overline{\psi},(z-t)\psi)} \prod_j \hat{\rho}_\eta(\lambda(\overline{\varphi}_j\varphi_j + \overline{\psi}_j\psi_j)) \\
&= (-i) \lim_{\eta\to 0+} \int_{\mathbb{C}^N} [d\overline{\varphi} d\varphi]^N \int [d\overline{\psi} d\psi]^N\; F_\eta(\varphi,\psi)
\end{aligned}$$

where we used the fact that for all $\eta > 0$ we can exhange the integrals. The function $F_\eta(\varphi,\psi)$ can be written as

$$F_\eta(\varphi,\psi) = \sum_{I,\overline{I}\subset\{1,\ldots,N\}} v_{\eta,I,\overline{I}}(\varphi)\psi^I \overline{\psi}^{\overline{I}},$$

with $v_{\eta,I,\bar{I}}(\varphi)$ given by

$$v_{\eta,I,\bar{I}}(\varphi) = e^{i(\overline{\varphi},(z-T)\varphi)} Pol(z-T) \sum_{n_1,\dots,n_N=0,1} \prod_j \lambda^{n_j} \hat{\rho}_\eta^{(n_j)}(\lambda|\varphi_j|^2).$$

Using the bounds on $\hat{\rho}_\eta$ and $\hat{\rho}'_\eta$ we obtain

$$|v_{\eta,I,\bar{I}}(\varphi, V)| \leq \; Const(z,T,N,\lambda) \prod_j e^{-\varepsilon|\varphi_j^2|}.$$

Therefore we can bring the limit inside the integral. As a result we get

$$\begin{aligned}\langle (E+i\varepsilon - H)^{-1}_{xy}\rangle_V &= (-i)\int_{\mathbb{C}^N} [d\overline{\varphi} d\varphi]^N \int [d\overline{\psi} d\psi]^N \; \varphi_x \overline{\varphi}_y e^{i(\overline{\varphi},(z-t)\varphi)+i(\overline{\psi},(z-t)\psi)} \prod_j \hat{\rho}(\lambda(\overline{\varphi}_j\varphi_j + \overline{\psi}_j\psi_j)) \\ &= (-i)\int_{\mathbb{C}^N} [d\overline{\varphi} d\varphi]^N \int [d\overline{\psi} d\psi]^N \; \varphi_x \overline{\varphi}_y e^{i(\overline{\varphi},(z-t)\varphi)+i(\overline{\psi},(z-t)\psi)} e^{-\lambda(\overline{\varphi},\varphi)-\lambda(\overline{\psi},\psi)} = (z+i\lambda - T)^{-1}_{xy}.\end{aligned}$$

This concludes the proof of the theorem. □

3 Asymptotic analysis of purely bosonic integrals

3.1 Scalar Laplace principle

Proposition 3.1 (Laplace's principle (I))**.** *Let $f, g \in C^\infty(\mathbb{R})$ be two given functions. Assume*

(a) f admits a unique global minimum in x_0 and $f''(x_0) > 0$,

(b) $\inf_{\substack{x \text{ local min} \\ x \neq x_0}} f(x) - f(x_0) > 0$

(c) $\exists N_0 > 0$ such that $\int_{\mathbb{R}} dx \; e^{-N_0 f(x)} < \infty$ and $\int_{\mathbb{R}} dx \; e^{-N_0 f(x)} |g(x)| < \infty$.

Then for $N \to \infty$ we have

(i) $\int_{\mathbb{R}} dx \; e^{-Nf(x)} = e^{-Nf(x_0)} \frac{\sqrt{2\pi}}{\sqrt{Nf''(x_0)}} \left(1 + O\left(\frac{1}{N}\right)\right).$

(ii) $\langle g \rangle := \frac{\int_{\mathbb{R}} dx \; e^{-Nf(x)} g(x)}{\int_{\mathbb{R}} dx \; e^{-Nf(x)}} = g(x_0) + \frac{1}{2N}\left[\frac{g''(x_0)}{f''(x_0)} - \frac{g'(x_0)f^{(3)}(x_0)}{f''(x_0)^2} - \frac{g(x_0)f^{(4)}(x_0)}{4f''(x_0)^2}\right] + o\left(\frac{1}{N}\right)$

If we have k global minima $x_1, \dots, x_k$, under the same assumptions for each minimum, we obtain

(i)′ $\int_{\mathbb{R}} dx \; e^{-Nf(x)} = \sum_{j=1}^k e^{-Nf(x_j)} \frac{\sqrt{2\pi}}{\sqrt{Nf''(x_j)}} \left(1 + O\left(\frac{1}{N}\right)\right),$

(ii)′ $\langle g \rangle = \frac{1}{\sum_{j=1}^k \frac{1}{\sqrt{f''(x_j)}}} \sum_{j=1}^k \frac{1}{\sqrt{f''(x_j)}} \left(g(x_j) + \frac{1}{2N}\left[\frac{g''(x_j)}{f''(x_j)} - \frac{g'(x_j)f^{(3)}(x_j)}{f''(x_j)^2} - \frac{g(x_j)f^{(4)}(x_j)}{4f''(x_j)^2}\right] + o\left(\frac{1}{N}\right)\right)$

Informal proof of (i) For $N \gg 1$ the measure concentrates on a small region near the minimum point x_0

$$\int_{\mathbb{R}} dx \; e^{-Nf(x)} \simeq \int_{|x-x_0|<\varepsilon} dx \; e^{-Nf(x)}.$$

For small ε the function is well approximated by its Taylor expansion

$$f(x) \simeq f(x_0) + f''(x_0)\frac{(x-x_0)^2}{2}.$$

Hence

$$\int_{\mathbb{R}} dx\ e^{-Nf(x)} \simeq \int_{|x-x_0|<\varepsilon} dx\ e^{-Nf(x)} \simeq e^{-Nf(x_0)} \int_{|x-x_0|<\varepsilon} dx\ e^{-Nf''(x_0)\frac{(x-x_0)^2}{2}}.$$

We claim we can replace above the integral on $\mathbb{R}$

$$\int_{\mathbb{R}} dx\, e^{-Nf(x)} \simeq e^{-Nf(x_0)} \int_{|x-x_0|<\varepsilon} dx\, e^{-Nf''(x_0)\frac{(x-x_0)^2}{2}} \simeq e^{-Nf(x_0)} \int_{\mathbb{R}} dx\, e^{-Nf''(x_0)\frac{x^2}{2}} = e^{-Nf(x_0)} \frac{\sqrt{2\pi}}{\sqrt{Nf''(x_0)}}.$$

Proof.
The integral in (i) is well defined $\forall N \geq N_0$ since

$$\begin{aligned}\int_{\mathbb{R}} dx\ e^{-Nf(x)} &= e^{-Nf(x_0)} \int_{\mathbb{R}} dx\ e^{-N(f(x)-f(x_0))} \leq e^{-Nf(x_0)} \int_{\mathbb{R}} dx\ e^{-N_0(f(x)-f(x_0))} \\ &= e^{-(N-N_0)f(x_0)} \int_{\mathbb{R}} dx\ e^{-N_0 f(x)} < \infty,\end{aligned}$$

where we used $f(x) - f(x_0) \geq 0$. The same argument shows that the integrals in (ii) are well defined $\forall N \geq N_0$. In the following we can assume $f(x_0) = 0$. If $f(x_0) \neq 0$ we consider $\tilde{f} := f - f(x_0)$.

(i) We will show the weaker estimate $\int_{\mathbb{R}} dx\ e^{-Nf(x)} = e^{-Nf(x_0)} \frac{\sqrt{2\pi}}{\sqrt{Nf''(x_0)}} \left(1 + o\left(\frac{1}{N^{\frac{1}{2}}}\right)\right)$.
We define

$$I_N := \int_{\mathbb{R}} dx\ e^{-Nf(x)}, \quad c_2 := f''(x_0), \quad I_\infty := \frac{\sqrt{2\pi}}{\sqrt{c_2}} = \int_{\mathbb{R}} dy\ e^{-\frac{c_2}{2}y^2}.$$

With this notation we need to prove

$$\sqrt{N} I_N = I_\infty \left(1 + o(N^{-\frac{1}{2}})\right). \tag{3.1}$$

We look for ε_N such that

- $\lim_{N\to\infty} \varepsilon_N = 0$,
- $\sqrt{N} \int_{|x-x_0|>\varepsilon_N} dx\ e^{-Nf(x)} = o(N^{-\frac{1}{2}})$
- $\sqrt{N} \int_{|x-x_0|<\varepsilon_N} dx\ e^{-Nf(x)} = I_\infty \left(1 + o(N^{-\frac{1}{2}})\right)$.

[10: 15.11.2024]

[11: 19.11.2024]

Region far from the minimum
By (a) and (b), there exists $\varepsilon_0 > 0$ such that $\forall 0 < \varepsilon \leq \epsilon_0$ we have

$$f(x) = f(x) - f(x_0) \geq \min\{f(x_0 + \varepsilon), f(x_0 - \varepsilon)\} \qquad \forall |x - x_0| \geq \varepsilon. \tag{3.2}$$

For $\varepsilon \ll 1$ we have

$$f(x_0 \pm \varepsilon) = \frac{c_2}{2}(x - x_0)^2 + O(|x - x_0|^3) = \frac{c_2}{2}\varepsilon^2 + O(\varepsilon^3) \geq \frac{c_2}{4}\varepsilon^2.$$

Inserting this bound in the integral and choosing $N \geq 2N_0$ we get

$$\sqrt{N}\int_{|x-x_0|>\varepsilon_N} dx\ e^{-Nf(x)} \leq \sqrt{N}\sup_{|x-x_0|\geq\varepsilon} e^{-\frac{N}{2}f(x)}\int_{|x-x_0|>\varepsilon_N} dx\ e^{-N_0 f(x)}$$
$$\leq \sqrt{N}\sup_{|x-x_0|>\varepsilon_N} e^{-\frac{N}{2}f(x)}\int_{\mathbb{R}} dx\ e^{-N_0 f(x)} = C_f\sqrt{N}\sup_{|x-x_0|>\varepsilon_N} e^{-\frac{N}{2}f(x)} \leq C_f\sqrt{N}e^{-\frac{c_2}{8}N\varepsilon_N^2}$$

We take

$$\varepsilon_N := \frac{N^\delta}{N^{\frac{1}{2}}}, \qquad 0<\delta<\frac{1}{2}. \tag{3.3}$$

Then $\lim_{N\to\infty}\varepsilon_N = 0$, and

$$\sqrt{N}\int_{|x-x_0|>\varepsilon_N} dx\ e^{-Nf(x)} \leq C_f\sqrt{N}e^{-\frac{c_2}{8}N^\delta} = o(N^{-\alpha}) \qquad \forall\alpha>0.$$

Region near the minimum: first try
We write

$$Nf(x) = \frac{c_1}{2}N(x-x_0)^2 + R_N(x), \qquad \text{with } R_N(x) := Nf(x) - \frac{c_2}{2}N(x-x_0)^2.$$

We have, for all $|x-x_0|\leq\varepsilon$,

$$|R_N(x)| \leq CN\varepsilon^3 = CN^{-\frac{1}{2}+3\delta},$$

where $C>0$ is a constant independent of ε and N. To ensure R_N is a small correction we require

$$\lim_{N\to\infty} N\varepsilon_N^3 = 0.$$

This holds if we assume $\delta < \frac{1}{6}$. With this assumption we argue

$$\sqrt{N}\int_{|x-x_0|<\varepsilon_N} dx\ e^{-Nf(x)} = \sqrt{N}\int_{|x-x_0|<\varepsilon_N} dx\ e^{-\frac{c_2}{2}N(x-x_0)^2}e^{-R_N(x)}.$$

Performing the coordinate change $y=\sqrt{N}(x-x_0)$ we obtain

$$\sqrt{N}\int_{|x-x_0|<\varepsilon_N} dx\ e^{-Nf(x)} = \int_{|y|<\sqrt{N}\varepsilon_N} dy\ e^{-\frac{c_2}{2}y^2}e^{-\tilde{R}_N(y)}.$$

Note that $\lim_{N\to\infty}\sqrt{N}\varepsilon_N = \infty$ hence the integral is well approximated by the integral on $\mathbb{R}$. To make this precise we argue

$$\int_{|y|<\sqrt{N}\varepsilon_N} dy\ e^{-\frac{c_2}{2}y^2}e^{-\tilde{R}_N(y)} = I_a + I_b$$

with

$$I_a := \int_{|y|<\sqrt{N}\varepsilon_N} dy\ e^{-\frac{c_2}{2}y^2} = I_\infty - \int_{|y|>\sqrt{N}\varepsilon_N} dy\ e^{-\frac{c_2}{2}y^2}$$
$$I_b := \int_{|y|<\sqrt{N}\varepsilon_N} dy\ e^{-\frac{c_2}{2}y^2}\left(e^{-\tilde{R}_N(y)}-1\right).$$

The first integral equals $I_\infty(1+Err)$, where the error term Err is bounded by

$$\frac{1}{I_\infty}\int_{|y|>\sqrt{N}\varepsilon_N} dy\ e^{-\frac{c_2}{2}y^2} \le e^{-\frac{c_2}{4}N\varepsilon_N^2} Const = e^{-\frac{c_2}{4}N^\delta} Const = o(N^{-\alpha}) \qquad \forall \alpha>0.$$

The second integral is bounded by

$$|I_b| \le Const\ N\varepsilon_N^3 = O(N^{-\frac{1}{2}+3\delta}).$$

Putting all this together we obtain

$$\sqrt{N} I_N = I_\infty \left(1 + O(N^{-\frac{1}{2}+3\delta}) + O(e^{-N^\delta})\right).$$

There is no choice of δ ensuring $R_N = o(N^{-\frac{1}{2}})$ hence we cannot get anything better than $O(N^{-\frac{1}{2}+3\delta})$ in the correction term above. To obtain a smaller correction we must expand more.

Region near the minimum: correct argument
Setting $c_3 := \frac{f'''(x_0)}{3!}$ we can write

$$Nf(x) = \frac{c_2}{2}N(x-x_0)^2 + c_3 N(x-x_0)^3 + R_N(x), \qquad \text{with } R_N(x) := Nf(x) - \frac{c_2}{2}N(x-x_0)^2 - c_3N(x-x_0)^3.$$

We have, for all $|x-x_0|\le\varepsilon$,

$$|R_N(x)| \le CN\varepsilon^4 = CN^{-1+4\delta},$$

where $C>0$ is a constant independent of ε and N. To ensure R_N is $o(N^{-\frac{1}{2}})$ we need $-1+4\delta < -\frac{1}{2}$. This holds for $\delta < \frac{1}{8}$.
We argue

$$\begin{aligned}\sqrt{N}\int_{|x-x_0|<\varepsilon_N} dx\ e^{-Nf(x)} &= \sqrt{N}\int_{|x-x_0|<\varepsilon_N} dx\ e^{-\frac{c_2}{2}N(x-x_0)^2} e^{-c_3N(x-x_0)^3} e^{-R_N(x)} \\ &= \int_{|y|<\sqrt{N}\varepsilon_N} dy\ e^{-\frac{c_2}{2}y^2} e^{-c_3\frac{1}{\sqrt{N}}y^3} e^{-\tilde{R}_N(y)} = I_a + I_b\end{aligned}$$

where

$$\begin{aligned} I_a &= \int_{|y|<\sqrt{N}\varepsilon_N} dy\ e^{-\frac{c_2}{2}y^2} e^{-c_3\frac{1}{\sqrt{N}}y^3} \\ I_b &= \int_{|y|<\sqrt{N}\varepsilon_N} dy\ e^{-\frac{c_2}{2}y^2} e^{-c_3\frac{1}{\sqrt{N}}y^3}\left(e^{-\tilde{R}_N(y)} - 1\right).\end{aligned}$$

The second integral is bounded by

$$|I_b| \le K\ N\varepsilon_N^4 e^{CN\varepsilon_N^3}\int_{\mathbb{R}} dy\ e^{-\frac{c_2}{2}y^2} = K\ N\varepsilon_N^4 e^{CN\varepsilon_N^3} I_\infty = o(N^{-\frac{1}{2}}),$$

for some constants $K, C>0$. To study the first integral we argue as follows.

$$
\begin{aligned}
I_a &= \int_{|y|<\sqrt{N}\varepsilon_N} dy\; e^{-\frac{c_2}{2}y^2} + \int_{|y|<\sqrt{N}\varepsilon_N} dy\; e^{-\frac{c_2}{2}y^2}\left(e^{-c_3\frac{1}{\sqrt{N}}y^3} \;-1\right)\\
&= \int_{|y|<\sqrt{N}\varepsilon_N} dy\; e^{-\frac{c_2}{2}y^2} - c_3\frac{1}{\sqrt{N}}\int_{|y|<\sqrt{N}\varepsilon_N} dy\; e^{-\frac{c_2}{2}y^2}y^3 + O\left((N\varepsilon_N^3)^2\right)\\
&= \int_{|y|<\sqrt{N}\varepsilon_N} dy\; e^{-\frac{c_2}{2}y^2} + O\left((N\varepsilon_N^3)^2\right)\\
&= I_\infty\left(1+O(e^{-N^\delta})+o(N^{-\frac{1}{2}})+O\left((N\varepsilon_N^3)^2\right)\right)
\end{aligned}
$$

where in the second line the middle integral vanishes by symmetry and we used in the last line

$$
(N\varepsilon_N^3)^2 = N^{-1+6\delta} < N^{-\frac{1}{2}} \qquad \forall 0<\delta<\frac{1}{12}.
$$

This shows that the error term is $o(N^{-\frac{1}{2}})$. To obtain $O(N^{-1})$ write the explicit terms of the Taylor expansion up to order 4

$$
f(x) = \frac{c_2}{2}(x-x_0)^2 + c_3(x-x_0)^3 + c_4(x-x_0)^5 + O(|x-x_0|^5),
$$

then expand in the region near the saddle up to order $1/N$ (exercise).

(ii) Since $g(x) = g(x_0) + g(x) - g(x_0)$, proving (ii) is equivalent to prove

$$
N\langle (g(x)-g(x_0))\rangle = \frac{1}{2}\left[\frac{g''(x_0)}{f''(x_0)} - \frac{g'(x_0)f'''(x_0)}{f''(x_0)^2}\right] + o\,(1)\,.
$$

We will show that

$$
\sqrt{\frac{Nc_2}{2\pi}}N\int_{\mathbb{R}} dx\,[g(x)-g(x_0)]\; e^{-Nf(x)} = \frac{1}{2}\left[\frac{g''(x_0)}{f''(x_0)} - \frac{g'(x_0)f'''(x_0)}{f''(x_0)^2}\right] + o\,(1)\,. \tag{3.4}
$$

Together with

$$
\sqrt{\frac{Nc_2}{2\pi}}\int_{\mathbb{R}} dx\, e^{-Nf(x)} = 1 + O\left(\frac{1}{N}\right),
$$

which holds by (i), this yelds the result. To prove (3.4) we set $\varepsilon = N^{\delta-\frac{1}{2}}$ with $0<\delta<1$ small enough and distinguish the region far and near the minimum.

Region far from the minimum
As in (i) we estimate

$$
\sqrt{\frac{Nc_2}{2\pi}}N\int_{|x-x_0|>\varepsilon_N} dx\,|g(x)-g(x_0)|\; e^{-Nf(x)} \le KN^{\frac{3}{2}}e^{-CN^\delta} = o(N^{-\alpha}) \qquad \forall\alpha>0.
$$

Region near the minimum

$$
\begin{aligned}
&\sqrt{\frac{Nc_2}{2\pi}}N\int_{|x-x_0|<\varepsilon_N} dx\,[g(x)-g(x_0)]\;e^{-Nf(x)} = \sqrt{\frac{Nc_2}{2\pi}}\int_{|x-x_0|<\varepsilon_N} dx \\
&\qquad\cdot\left[Ng'(x_0)(x-x_0)+\frac{g''(x_0)}{2}N(x-x_0)^2+O(N\varepsilon_N^3)\right]\;e^{-\frac{c_2}{2}N(x-x_0)^2}e^{-c_3N(x-x_0)^3}e^{O(N\varepsilon_N^4)} \\
&= \sqrt{\frac{c_2}{2\pi}}\int_{|y|<\sqrt{N}\varepsilon_N} dy\; e^{-\frac{c_2}{2}y^2}e^{-c_3\frac{1}{\sqrt{N}}y^3}e^{O(N\varepsilon_N^4)}\left[\sqrt{N}g'(x_0)y+\frac{g''(x_0)}{2}y^2+O(N\varepsilon_N^3)\right] \\
&= \sqrt{\frac{c_2}{2\pi}}\int_{|y|<\sqrt{N}\varepsilon_N} dy\; e^{-\frac{c_2}{2}y^2}e^{O(N\varepsilon_N^4)} \\
&\qquad\cdot\left[1-c_3\frac{1}{\sqrt{N}}y^3+O\left((N\varepsilon_N^3)^2\right)\right]\;\left[\sqrt{N}g'(x_0)y+\frac{g''(x_0)}{2}y^2+O(N\varepsilon_N^3)\right] \\
&= \sqrt{\frac{c_2}{2\pi}}\int_{|y|<\sqrt{N}\varepsilon_N} dy\; e^{-\frac{c_2}{2}y^2}\left[1+O(N\varepsilon_N^4)\right] \\
&\qquad\cdot\left[\sqrt{N}g'(x_0)y+\frac{1}{2}\left(g''(x_0y^2-2c_3g'(x_0)y^4\right)+O\left((N\varepsilon_N^3)(N\varepsilon_N^2)\right)+O\left((N\varepsilon_N^3)^2N\varepsilon_N\right)\right] \\
&= \sqrt{\frac{c_2}{2\pi}}\int_{\mathbb{R}} dy\; e^{-\frac{c_2}{2}y^2}\left[\frac{1}{2}\left(g''(x_0y^2-2c_3g'(x_0)y^4\right)+o(1)\right],
\end{aligned}
$$

where we used $\int_{|y|<\sqrt{N}\varepsilon_N} dy\; e^{-\frac{c_2}{2}y^2}y=0$ by symmetry. Inserting now the values of c_2, c_3 and performing the Gaussian integral yelds the result. □

[11: 19.11.2024]

[12: 22.11.2024]

3.2 Application 1: mean field Ising model

Consider the model introduced in Section 1.2.5, with $n=1$. To each spin configuration $\sigma=(\sigma_1,\dots,\sigma_N)\in\{-1,1\}^N$ we associate the weight

$$\mu_{N,h}(\sigma)=e^{\frac{\beta}{2N}\sum_{jk=1}^N\sigma_j\sigma_k}e^{h\sum_{j=1}^N\sigma_j},$$

where $\beta=\frac{1}{T}$ is the inverse temperature and $h\in\mathbb{R}$ is the magnetic field. The average of a function $f(\sigma)$ is defined by

$$\mathbb{E}_N^{h,\beta}[f]:=\tfrac{1}{Z_N(h)}\tfrac{1}{2^N}\sum_{\sigma\in\{-1,1\}^N}\mu_{N,h}(\sigma)f(\sigma),$$

where

$$Z_N(h):=\tfrac{1}{2^N}\sum_{\sigma\in\{-1,1\}^N}\mu_{N,h}(\sigma). \tag{3.5}$$

In particular, the magnetization is defined by

$$M_N(h)=M_N(h,\beta):=\tfrac{1}{N}\mathbb{E}_N^{h,\beta}\left[\sum_{j=1}^N\sigma_j\right].$$

Note that when $h = 0$ $\mu_{N,0}(-\sigma) = \mu_{N,0}(\sigma)$ and hence $M_N(0) = 0$. This means the spins σ_j have no preferred value. Moreover $\mathbb{E}_N^{-h,\beta}[f(\sigma)] = \mathbb{E}_N^{h,\beta}[f(-\sigma)]$ hence $M_N(-h) = -M_N(h)$. We show now that $M(h) > 0$ for all $h > 0$.

$$\begin{aligned} M_N(h) &= \tfrac{1}{NZ_N(h)}\tfrac{1}{2^N} \sum_{\sigma\in\{-1,1\}^N} \mu_{N,h}(\sigma)(\textstyle\sum_j \sigma_j) = \tfrac{1}{NZ_N(h)}\tfrac{1}{2}\tfrac{1}{2^N} \displaystyle\sum_{\sigma\in\{-1,1\}^N} (\textstyle\sum_j \sigma_j)(\mu_{N,h}(\sigma) - \mu_{N,h}(-\sigma)) \\ &= \tfrac{1}{NZ_N(h)}\tfrac{1}{2^N} \sum_{\sigma\in\{-1,1\}^N} e^{\frac{\beta}{2N}\sum_{jk=1}^N \sigma_j\sigma_k}(\textstyle\sum_j \sigma_j)\sinh\left(h\sum_j \sigma_j\right). \end{aligned}$$

For $h > 0$ we have $x\sinh(xh) > 0$ $\forall x \in \mathbb{R}\setminus\{0\}$. Hence $M_N(h) > 0$. Finally the map $h \mapsto M_N(h)$ is continuous and hence $\lim_{h\to 0} M_N(h) = M_N(0) = 0$.

Question 1: does the limit $M(h) := \lim_{N\to\infty} M_N(h)$ exist?

Question 2: in case the limit exists do we have $\lim_{h\to 0} M(h) = 0$?

The answer to the first question is yes. The answer to the second question depends on the value of β. This is the content of the next theorem.

Theorem 3.2.

(i) For all $\beta > 0$ and $h \in \mathbb{R}$ the limit $M(h) := \lim_{N\to\infty} M_N(h)$ exists.
For $h = 0$ the limit is $M(0) = 0$.
For $h > 0$ the limit is

$$M(h) = \tanh(x(h,\beta)),$$

where $x(h,\beta)$ is the largest positive solution of

$$\frac{x-h}{\beta} = \tanh x.$$

For $h < 0$ the limit is $M(h) = -M(-h) = -\tanh(x(-h,\beta))$.
The function $h \mapsto M(h)$ is continuous on $\mathbb{R}\setminus\{0\}$ and satisfies

$$\lim_{\beta\to\infty} M(h) = \pm 1 \qquad \textit{for } h \gtrless 0.$$

(ii) For all $0 < \beta \le 1$ the map $h \mapsto M(h)$ is continuous in 0 i.e. $\lim_{h\to 0} M(h) = M(0) = 0$.
For all $1 < \beta$ the map $h \mapsto M(h)$ is discontinuous in 0

$$\lim_{h\downarrow 0} M(h) = M_+(\beta) = \tanh x_\beta,$$

where $x_\beta > 0$ is the unique stricly positive solution of

$$\frac{x}{\beta} = \tanh x.$$

Moreover

$$\lim_{\beta\to\infty} x_\beta = \infty, \qquad \lim_{\beta\to\infty} M_+(\beta) = 1$$
$$\lim_{\beta\downarrow 1} x_\beta = 0, \qquad \lim_{\beta\downarrow 1} M_+(\beta) = 0.$$

Remark. For $\beta > 1$ the measure μ_N exhibits spontaneous symmetry breaking as $N \to \infty$ since there are at least two possible limit measures, one favoring positive spins, the other favoring negative spins.

Proof. We reformulate $M_N(h)$ as a dual integral.

$$M_N(h) = \tfrac{1}{N}\mathbb{E}_N^{h,\beta}\left[\sum_{j=1}^N \sigma_j\right] = \frac{1}{N}\partial_h \ln Z_N(h),$$

where $Z_N(h)$ is the partition function defined in (3.5). We proved in Lemma 1.10 the identity

$$\tfrac{1}{2^N}\sum_{\sigma\in\{-1,1\}^N} \mu_{N,h}(\sigma) = \left(\tfrac{N}{2\pi\beta}\right)^{\frac{1}{2}} \int_{\mathbb{R}} d\varphi\ e^{-N\left(\frac{(\varphi-h)^2}{2\beta} - \ln I(\varphi)\right)},$$

where

$$I(\varphi) := \frac{1}{2}\sum_{\sigma=\pm 1} e^{\sigma\varphi} = \cosh\varphi.$$

Therefore

$$M_N(h) = \langle g\rangle_N := \frac{\int_{\mathbb{R}} d\varphi\ e^{-Nf(\varphi)} g(\varphi)}{\int_{\mathbb{R}} d\varphi\ e^{-Nf(\varphi)}},$$

with

$$f(\varphi) := \frac{(\varphi-h)^2}{2\beta} - \ln\cosh\varphi, \qquad g(\varphi) := \frac{\varphi-h}{\beta}.$$

Both functions are smooth and

$$\int_{\mathbb{R}} d\varphi\ e^{-Nf(\varphi)} < \infty, \qquad \int_{\mathbb{R}} d\varphi\ e^{-Nf(\varphi)}|g(\varphi)| < \infty$$

hold for all $N > 0$. To check if we can apply Proposition 3.1 we need to study the minimum points of f.
We have

$$f'(\varphi) = \frac{\varphi-h}{\beta} - \tanh\varphi, \qquad f''(\varphi) = \frac{1}{\beta} - \frac{1}{(\cosh\varphi)^2}.$$

Note that f'' is independent of h.
Case 1: $h = 0$. In this case we have $f(-\varphi) = f(\varphi)$ while $g(-\varphi) = -g(\varphi)$, and hence $\langle g\rangle_N = 0$. We try to recover this result via Proposition 3.1. We have

$$0 = f'(\varphi) = \frac{\varphi}{\beta} - \tanh\varphi \quad \Leftrightarrow \quad \frac{\varphi}{\beta} = \tanh\varphi.$$

Comparing the two curves $\frac{\varphi}{\beta}$ and $\tanh\varphi$ we see that, if $\frac{1}{\beta} \geq 1$ there is only one solution $\varphi = 0$, while for $\frac{1}{\beta} < 1$ there are three solutions: $\varphi = 0$ and $\varphi = \pm x_\beta$ with $x_\beta > 0$. Using $\cosh\varphi \geq 1$ we argue

$$f''(\varphi) \geq \frac{1}{\beta} - 1.$$

• For $\beta \leq 1$ we have $f''(\varphi) > 0\ \forall\varphi \neq 0$ and hence f has a unique (global) mininum in $\varphi = 0$. Moreover, for $\frac{1}{\beta} > 1$ we have $f''(0) > 0$, hence by Proposition 3.1

$$\langle g\rangle_N = g(0) + O(N^{-1}) = O(N^{-1}) \qquad \text{for } \beta < 1.$$

• In the case $\beta > 1$ f has two (global) minimum points in $\varphi = \pm x_\beta$ and one local maximum in $\varphi = 0$. Therefore we must have

$$f''(\pm x_\beta) = \frac{1}{\beta} - \frac{1}{(\cosh x_\beta)^2} \geq 0.$$

We show that $f''(x_\beta) > 0$. By contradiction assume $f''(x_\beta) = 0$. Then

$$f(x) - f(x_\beta) = f'''(x_\beta)\frac{(x - x_\beta)^3}{3!} + o(|x - x_\beta|^3).$$

By direct computation $f'''(x_\beta) > 0$, hence we would obtain $f(x) < f(x_\beta)$ for $x < x_\beta$, which contradicts the fact that x_β is a minimum point. Therefore $f''(x_\beta) = f''(-x_\beta) > 0$ and Proposition 3.1 yields

$$\langle g\rangle_N = g(x_\beta) + g(-x_\beta) + O(N^{-1}) = O(N^{-1}),$$

since $g(x_\beta) + g(-x_\beta) = 0$.

Case 2: $h > 0$. We have

$$0 = f'(\varphi) = \frac{\varphi - h}{\beta} - \tanh\varphi \quad \Leftrightarrow \quad \frac{\varphi - h}{\beta} = \tanh\varphi.$$

• For $\beta \leq 1$ the function f has a unique (global) minimum in $\varphi_m = x(h, \beta) > 0$. Since $f''(\varphi) > 0$ except eventually in $\varphi = 0$ we also have $f''(\varphi_m) > 0$ and hence Proposition 3.1 yields

$$\langle g\rangle_N = g(x(h)) + O(N^{-1}) = \frac{x(h,\beta) - h}{\beta} + O(N^{-1}) = \tanh x(h, \beta) + O(N^{-1}).$$

This implies that the limit $N \to \infty$ is well defined and

$$M(h) = \lim_{N\to\infty} M_N(h) = \tanh x(h, \beta).$$

The function $h \mapsto x(h, \beta)$ is continuous on $\mathbb{R}$ and

$$\lim_{h\downarrow 0} x(h, \beta) = 0.$$

hence

$$\lim_{h\downarrow 0} M(h) = 0.$$

• For $\beta > 1$ there exists a $h_\beta > 0$ such that:

for $0 < h < h_\beta$ the function f has 3 critical points $x_-(h) < x_0(h) < 0 < x_+(h)$,

for $h > h_\beta$ the unique critical point is $x_+(h)$,

for $h = h_\beta$ there are two critical point in $x_-(h_\beta) < 0 < x_+(h_\beta)$.

Note that $\lim_{h\downarrow 0} x_0(h) = 0$, $\lim_{h\downarrow 0} x_\pm(h) = \pm x_\beta$, and $\lim_{h\uparrow h_\beta} x_0(h) = \lim_{h\uparrow h_\beta} x_-(h)$.
For all $h > 0$ the function has a unique global minimum in $x_+(h)$ plus eventually a local minimum in $x_-(h)$, therefore $f''(x_+(h)) \geq 0$. Arguing as in the case $h = 0$ we obtain $f''(x_+(h)) > 0$, hence Proposition 3.1 yields

$$\langle g\rangle_N = g(x_+(h)) + O(N^{-1}) = \frac{x_+(h) - h}{\beta} + O(N^{-1}) = \tanh x_+(h) + O(N^{-1}).$$

This implies that the limit $N \to \infty$ is well defined and

$$M(h) = \lim_{N\to\infty} M_N(h) = \tanh x_+(h).$$

The function $h \mapsto x_+(h)$ is continuous on $\mathbb{R} \setminus \{0\}$ and

$$\lim_{h\downarrow 0} x_+(h) = x_\beta > 0, \qquad \lim_{\beta\to\infty} x_+(h) = \infty,$$

hence

$$\lim_{h\downarrow 0} M(h) = \tanh x_\beta > 0, \qquad \lim_{\beta\to\infty} M(h) = 1.$$

□

[12: 22.11.2024]

[13: 25.11.2024]

3.3 Application 2: mean field $O(n)$ model, $n \geq 2$

Consider the model introduced in Section 1.2.5, with $n \geq 2$. To each spin configuration $\sigma = (\sigma_1, \ldots, \sigma_N) \in (\mathcal{S}^{n-1})^N$ we associate the weight

$$\mu_{N,h}(\sigma) = e^{\frac{\beta}{2N}\sum_{jk=1}^N \sigma_j\cdot\sigma_k} e^{h\sum_{j=1}^N \sigma_j\cdot\hat{e}},$$

where $\beta = \frac{1}{T}$ is the inverse temperature, $h \geq 0$ is the intensity of the magnetic field and $\hat{e} \in \mathcal{S}^{n-1}$ is the direction of the magnetic field. The average of a function $f(\sigma)$ is defined by

$$\mathbb{E}_N^{h,\beta}[f] := \tfrac{1}{Z_N(h)} \int_{(\mathcal{S}^{n-1})^N} d\sigma\ \mu_{N,h}(\sigma) f(\sigma),$$

where $d\sigma = \prod_{j=1}^N d\sigma_j$, $\int_{\mathcal{S}^{n-1}} d\sigma_j = 1$, and

$$Z_N(h) := \int_{(\mathcal{S}^{n-1})^N} d\sigma\ \mu_{N,h}(\sigma). \tag{3.6}$$

In particular, the magnetization is defined by

$$M_N(h) = M_N(h,\beta) := \tfrac{1}{N}\mathbb{E}_N^{h,\beta}\left[\sum_{j=1}^N \sigma_j\right] \in \mathbb{R}^n.$$

By construction $|M_N(h)| \leq 1$. Note that each spin σ_j can be decomposed as

$$\sigma_j = \sigma_j^e + \sigma_j^\perp$$

where $\sigma_j^\perp \cdot \hat{e} = 0$. The measure satisfies

$$\mu_{N,h}(\sigma^e, \sigma^\perp) = \mu_{N,h}(\sigma^e, -\sigma^\perp) \qquad \forall h \geq 0,$$

hence

$$M_N(h) \cdot \hat{e}^\perp = 0,$$

i.e.

$$M_N(h) = (M_N(h)\cdot\hat{e})\hat{e}.$$

Moreover, for $h = 0$ we have $M_N(0) = 0$ by the symmetry $\sigma \to -\sigma$. We show now $M_N(h) \cdot \hat{e} > 0$ for all $h > 0$. Indeed

$$\begin{aligned} M_N(h) \cdot \hat{e} &= \tfrac{1}{NZ_N(h)} \int_{(\mathcal{S}^{n-1})^N} d\sigma \; \mu_{N,h}(\sigma)(\textstyle\sum_j \sigma_j \cdot \hat{e}) = \tfrac{1}{NZ_N(h)} \tfrac{1}{2} \int_{(\mathcal{S}^{n-1})^N} d\sigma \; (\textstyle\sum_j \sigma_j \cdot \hat{e})(\mu_{N,h}(\sigma) - \mu_{N,h}(-\sigma)) \\ &= \tfrac{1}{NZ_N(h)} \int_{(\mathcal{S}^{n-1})^N} d\sigma \; e^{\frac{\beta}{2N} \sum_{jk=1}^N \sigma_j \sigma_k} (\textstyle\sum_j \sigma_j \cdot \hat{e}) \sinh\left(h \textstyle\sum_j \sigma_j \cdot \hat{e} \right). \end{aligned}$$

For $h > 0$ we have $x \sinh(xh) > 0$ $\forall x \in \mathbb{R} \setminus \{0\}$. Hence $M_N(h) \cdot \hat{e} > 0$. Finally the map $h \mapsto M_N(h) \cdot \hat{e}$ is continuous and hence $\lim_{h \to 0} M_N(h) = M_N(0) = 0$.

Question 1: does the limit $M(h) := \lim_{N \to \infty} M_N(h)$ exist?

Question 2: in case the limit exists do we have $\lim_{h \to 0} M(h) = 0$?

As in the case of the Ising model, the answer to the first question is yes. The answer to the second question depends on the value of β. Before we can state the result we need a multivariable version of the Laplace principle.

Proposition 3.3 (Laplace's principle (II)). *Let $H, g \in C^\infty(\mathbb{R}^n; \mathbb{R})$ be two given functions. Assume*

(a) H admits a unique global minimum in φ_m and $H''(\varphi_m) > 0$ as a quadratic form, where $H''(\varphi) \in \mathbb{R}^{n \times n}$ is the hessian matrix defined via $H''(\varphi)_{ij} = \partial_{\varphi_i} \partial_{\varphi_j} H(\varphi)$.

(b) $\inf_{\substack{\varphi, \text{local min} \\ \varphi \neq \varphi_m}} H(\varphi) - H(\varphi_m) > 0$

(c) $\exists N_0 > 0$ such that $\int_{\mathbb{R}^n} d\varphi \; e^{-N_0 H(\varphi)} < \infty$ and $\int_{\mathbb{R}^n} d\varphi \; e^{-N_0 H(\varphi)} |g(\varphi)| < \infty$.

Then for $N \to \infty$ we have

(i) $\int_{\mathbb{R}^n} d\varphi \; e^{-NH(\varphi)} = e^{-NH(\varphi_m)} \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det(NH''(\varphi_m))}} \left(1 + O\left(\frac{1}{N}\right)\right)$.

(ii) $\langle g \rangle := \frac{\int_{\mathbb{R}^n} d\varphi \; e^{-NH(\varphi)} g(\varphi)}{\int_{\mathbb{R}^n} d\varphi \; e^{-NH(\varphi)}} = g(\varphi_m) + O\left(\frac{1}{N}\right)$

If we have k global minima $\varphi_1, \ldots, \varphi_k$, under the same assumptions for each minimum, we obtain

(i)' $\int_{\mathbb{R}^n} e^{-NH(\varphi)} = e^{-NH_m} \sum_{j=1}^k \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det(NH''(\varphi_j))}} \left(1 + O\left(\frac{1}{N}\right)\right)$, where $H_m = \min_\varphi H(\varphi) = H(\varphi_j)$ $j = 1, \ldots, k$.

(ii)' $\langle g \rangle = \frac{\sum_{j=1}^k (\det H''(\varphi_j))^{-\frac{1}{2}} g(\varphi_j)}{\sum_{j=1}^k (\det H''(\varphi_j))^{-\frac{1}{2}}} + O\left(\frac{1}{N}\right)$

Proof. Works as in the scalar case (exercise) □

Theorem 3.4.

(i) For all $\beta > 0$ and $h \geq 0$ $\hat{e} \in \mathcal{S}^{n-1}$ we have

$$M_N(h) \cdot \hat{e} = \langle g \rangle_\varphi := \frac{\int_{\mathbb{R}^n} d\varphi \; e^{-NH(\varphi)} g(\varphi)}{\int_{\mathbb{R}^n} d\varphi \; e^{-NH(\varphi)}},$$

with $g(\varphi) := \frac{(\varphi - h)\cdot\hat{e}}{\beta}$ *and*

$$H(\varphi) := \frac{|\varphi - h\hat{e}|^2}{2\beta} - F(\varphi), \qquad F(\varphi) := \ln \int_{\mathcal{S}^{n-1}} dS\ e^{S\cdot\varphi}.$$

In particular $F(\varphi) = f(|\varphi|)$ *with*

$$f(r) := \ln \int_{\mathcal{S}^{n-1}} dS\ e^{S_1 r},$$

where S_1 *is the first component of the vector* S*. In the following we set* $r = r(\varphi) = |\varphi|$ *and*

$$\langle g(S)\rangle_r := \frac{\int_{\mathcal{S}^{n-1}} dS\ e^{S_1 r} g(S)}{\int_{\mathcal{S}^{n-1}} dS\ e^{S_1 r}},$$

(ii) For all $\beta > 0$ *and* $h \geq 0$ $\hat{e} \in \mathcal{S}^{n-1}$ *the limit* $M(h) := \lim_{N\to\infty} M_N(h)$ *exists.*
For $h = 0$ *the limit is* $M(0) = 0$*.*
For $h > 0$ *the limit is*

$$M(h) = f'\left(r_+(h,\beta)\right)\hat{e} = \langle S_1\rangle_{r_+(h,\beta)}\,\hat{e},$$

where $r_+(h,\beta) > 0$ *is the unique solution of*

$$u(r) = \frac{h}{\beta} \qquad \text{with} \quad u(r) := \frac{1}{\beta} - \frac{f'(r)}{r}.$$

The function $h \mapsto M(h)$ *is continuous on* $(0,\infty)$*. Moreover*

- *if* $\beta \leq n$ $\lim_{h\downarrow 0} M(h) = M(0) = 0$*,*
- *if* $\beta > n$ $\lim_{h\downarrow 0} M(h) = f'(r_\beta)\hat{e}$*, where* $r_\beta > 0$ *is the unique solution of* $u(r) = 0$*.*

Remark. For $\beta > n$ the measure μ_N exhibits spontaneous symmetry breaking as $N \to \infty$ since there are uncountably many limit measures, one for each direction $\hat{e}$.

Proof. Proof of Theorem 3.4 (i) We have $M_N(h)\cdot\hat{e} = \frac{1}{N}\partial_h \ln Z_N(h)$. We proved in Lemma 1.10 the identity

$$Z_N(h) = \left(\tfrac{N}{2\pi\beta}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} d\varphi\ e^{-N\left(\frac{|\varphi - h\hat{e}|^2}{2\beta} - F(\varphi)\right)}.$$

The result follows taking the derivative of this function in h.

□

To prove the second statement we need some properties of f, which are collected in the next theorem.

Theorem 3.5. *Let* $f\colon [0,\infty) \to \mathbb{R}$ *be defined via* $f(r) = \ln \int_{\mathcal{S}^{n-1}} dS\ e^{S_1 r}$*.*
The following statements hold:

(i) $f'(r) > 0\ \forall r > 0$, $f'(0) = 0$ *and* $\lim_{r\to\infty} f'(r) = 1$,

(ii) $f''(r) > 0\ \forall r \geq 0$, $f''(0) = \frac{1}{n}$ *and* $\lim_{r\to\infty} f''(r) = 0$,

(iii) $f'''(r) < 0 \ \forall r > 0$,

(iv) $0 < f''(r) < \frac{1}{n} \ \forall r > 0$.

Proof.
(i) We compute

$$f'(r) = \langle S_1 \rangle_r = \frac{\int_{\mathcal{S}^{n-1}} dS \ e^{S_1 r} S_1}{\int_{\mathcal{S}^{n-1}} dS \ e^{S_1 r}} = \frac{\int_{\mathcal{S}^{n-1}} dS \ S_1 \ \sinh(S_1 r)}{\int_{\mathcal{S}^{n-1}} dS \ \cosh(S_1 r)}.$$

It follows that $f'(r) > 0 \ \forall r >$. Moreover $f'(0) = \langle S_1 \rangle_0 = 0$ by symmetry.
To show $\lim_{r\to\infty} f'(r) = 1$ note that for all function $f(S_1)$ we have

$$\langle f(S_1) \rangle_r = \frac{\int_{-1}^{1} dS_1 (1 - S_1^2)^{\frac{n-3}{2}} e^{rS_1} f(S_1)}{\int_{-1}^{1} dS_1 (1 - S_1^2)^{\frac{n-3}{2}} e^{rS_1}} \tag{3.7}$$

and perform asymptotic analysis. Note that in this case we integrate on $(-1, 1)$ instead of $\mathbb{R}$.

(ii) We compute

$$f''(r) = \langle S_1^2 \rangle_r - \langle S_1 \rangle_r^2 = \langle (S_1 - \langle S_1 \rangle_r)^2 \rangle_r > 0.$$

Moreover

$$f''(0) = \langle S_1^2 \rangle_0 = \int_{\mathcal{S}^{n-1}} dS \ S_1^2 = \frac{1}{n} \int_{\mathcal{S}^{n-1}} dS \sum_{j=1}^{n} S_j^2 = \frac{1}{n}.$$

To compute the limit as $r \to \infty$ note that $\lim_{r\to\infty} \langle S_1 \rangle_r^2 = 1$ by (i). We argue

$$\langle S_1^2 \rangle_r = 1 - \langle 1 - S_1^2 \rangle_r = 1 - \frac{\langle S_1 \rangle_r}{r} \to_{r\to\infty} 1.$$

To prove the identity $\langle 1 - S_1^2 \rangle_r = \frac{\langle S_1 \rangle_r}{r}$ use the representation 3.7 and integrate by parts.

(iii) see Theorem D.2 in Appendix D of *Phase Transitions in Quantum Spin Systems with Isotropic and Nonisotropic Interactions* F. J. Dyson, E. H. Lieb, 2 and B. Simon, in *Journal of Statistical Physics*, Vol. 18, No. 4, 1978.

(iv) Follows directly from (iii) and the fact that $f''(0) = \frac{1}{n}$.

□

Proof. Proof of Theorem 3.4 (ii) The first and second derivative of H are given by

$$\partial_{\varphi_i} H(\varphi) = \left(\frac{1}{\beta} - \frac{f'(r)}{r} \right) \varphi_i - \frac{h}{\beta} \hat{e}_i, \qquad H''(\varphi) = \left(\frac{1}{\beta} - \frac{f'(r)}{r} \right) \mathrm{Id} - \left(f''(r) - \frac{f'(r)}{r} \right) |\varphi\rangle\langle\varphi|,$$

where

$$|\varphi\rangle\langle\varphi| = \frac{1}{|\varphi|^2} \varphi \otimes \varphi$$

is the projection on the direction $\frac{1}{|\varphi|}\varphi$. The matrix $H''(\varphi)$ has two eigenvalues. The first is

$$\lambda_1(\varphi) = \frac{1}{\beta} - f''(r),$$

with multiplicity 1. The corresponding eigenvector is φ. The second eigenvalue is

$$\lambda_2(\varphi) = \frac{1}{\beta} - \frac{f'(r)}{r},$$

with multiplicity $n-1$. All vectors in the space $\varphi^\perp$ are eigenveectors of $H''(\varphi)$ with eigenvalue $\lambda_2(\varphi)$. Note that, since $f'(0)=0$ we have

$$\lim_{r\downarrow 0}\frac{f'(r)}{r}=f''(0)=\frac{1}{n}, \tag{3.8}$$

and hence

$$\lambda_1(0)=\lambda_2(0)=\frac{1}{\beta}-\frac{1}{n}$$

[13: 25.11.2024]

[14: 29.11.2024]

Case 1: $h=0$. In this case we have $M_N(0)=0$ by symmetry. We try to recover this result via Proposition 3.3. We have

$$\partial_\varphi H(\varphi)=0 \quad\Leftrightarrow\quad \left(\frac{1}{\beta}-\frac{f'(r)}{r}\right)\varphi=0.$$

The value $\varphi=0$ is a solution. We check now if there is also a solution $\varphi\neq 0$. In this case $\varphi=r\omega$ where $r>0$ and $\omega\in\mathcal{S}^{n-1}$ and the critical point equation becomes

$$0=\left(\frac{r}{\beta}-f'(r)\right)\omega=u(r)\omega \quad\Leftrightarrow\quad u(r)=0.$$

We compute

$$u(0)=0,\qquad u'(r)=\frac{1}{\beta}-f'(r)=\lambda_1(r),\qquad u(r)=r\lambda_2(r).$$

Using $f''(r)<\frac{1}{n}\ \forall r>0$ and $f''(0)=\frac{1}{n}$ we deduce

$$u'(0)=\frac{1}{\beta}-\frac{1}{n},\qquad u'(r)>u'(0)=\frac{1}{\beta}-\frac{1}{n}\ \forall r>0.$$

• For $\beta\leq n$ we have $u(r)>0\ \forall r>0$ and hence H has a unique critical point in $\varphi=0$. It follows, since $H(\varphi)\to_{|\varphi|\to\infty}\infty$, that $\varphi=0$ is the unique global minimum. The hessian matrix at $\varphi=0$ is

$$H''(0)=\left(\frac{1}{\beta}-\frac{1}{n}\right)\mathrm{Id}\ >0\qquad \forall\beta<n.$$

By Proposition 3.3 it follows, for $\beta<n$,

$$\langle g\rangle_\varphi=g(0)+O(N^{-1})=O(N^{-1}).$$

• In the case $\beta>n$ $u'(0)<0$ and $\lim_{r\to\infty}u(r)=\frac{1}{\beta}>0$. Since $u'''(r)=-f'''(r)>0\ \forall r>0$, it follows that there exists a unique $r_\beta>0$ solution of $u(r)=0$. Therefore $\varphi=r_\beta\omega$ is a critical point $\forall\omega\in\mathcal{S}^{n-1}$. Note that, since $h=0$, H is rotation invariant: $H(r_\beta\omega)=\frac{r_\beta^2}{2\beta}-f(r_\beta)$ $\forall\omega\in\mathcal{S}^{n-1}$. The hessian matrix at $\varphi=0$ is

$$H''(0)=\left(\frac{1}{\beta}-\frac{1}{n}\right)\mathrm{Id}\ <0,$$

so $\varphi=0$ is a local maximum and hence $\{r_\beta\omega\}_{\omega\in\mathcal{S}^{n-1}}$ is a manifold of global minima. The eigenvalues at r_β are

$$\lambda_1(r_\beta)=u'(r_\beta)>0,\qquad \lambda_2(r_\beta)=\frac{u(r_\beta)}{r_\beta}=0.$$

So we can apply asymptotic analysis only in the radial direction.
Case 2: $h > 0$. We have

$$\partial_\varphi H(\varphi) = 0 \quad \Leftrightarrow \quad \left(\frac{1}{\beta} - \frac{f'(r)}{r}\right)\varphi = \frac{h}{\beta}\hat{e}.$$

The value $\varphi = 0$ is no longer a solution. The solution must be of the form $\varphi = \pm r\hat{e}$, for some $r > 0$. The critical point equation becomes

$$\pm u(r)\hat{e} = \frac{h}{\beta}\hat{e} \quad \Leftrightarrow \quad u(r) = \pm\frac{h}{\beta}.$$

• For $\beta \le n$ we have $u(r) > 0\ \forall r > 0$, so there is a unique solution $r_+(h,\beta) > 0$ of $u(r) = \frac{h}{\beta}$ and there is no solution for $u(r) = -\frac{h}{\beta}$. Hence there is a unique global minimum in $\varphi_m = r_+\hat{e}$. The eigenvalues at the minimum are

$$\lambda_1(r_+) = u'(r_+) > 0, \qquad \lambda_2(r_+) = \frac{u(r_+)}{r_+} = \frac{h}{r_+\beta} > 0,$$

therefore $H''(\varphi_m) > 0$ and Proposition 3.3 yields

$$\langle g\rangle_\varphi = g(\varphi_m) + O(N^{-1}).$$

We compute

$$g(\varphi_m) = \frac{1}{\beta}(\varphi_m - h\hat{e})\cdot\hat{e} = \frac{r_+ - h}{\beta} = f'(r_+) = \langle S_1\rangle_{r_+} \in (0,1).$$

This implies that the limit $N \to \infty$ is well defined and

$$M(h) = \lim_{N\to\infty} M_N(h) = f'(r_+).$$

The function $h \mapsto r_+(h,\beta)$ is continuous on $[0,\infty)$ and

$$\lim_{h\downarrow 0} r_+(h,\beta) = 0,$$

hence

$$\lim_{h\downarrow 0} M(h) = 0.$$

• For $\beta > n$ we have seen that u' is monotone increasing,

$$u(r) < 0 \quad \forall 0 < r < r_\beta, \qquad \text{and} \quad u(r) > 0 \quad \forall r > r_\beta.$$

Let $u_m := \min_{r>0} u(r) < 0$. We distinguish three cases.
For $h > \beta u_m$ there is a unique solution $r_+(h,\beta) > r_\beta$ of $u(r) = \frac{h}{\beta}$ and there is no solution for $u(r) = -\frac{h}{\beta}$. Hence there is a unique global minimum in $\varphi_m = r_+\hat{e}$. The eigenvalues at the minimum are

$$\lambda_1(r_+) = u'(r_+) > 0, \qquad \lambda_2(r_+) = \frac{u(r_+)}{r_+} = \frac{h}{r_+\beta} > 0,$$

therefore $H''(\varphi_m) > 0$ and Proposition 3.3 yields

$$\langle g\rangle_\varphi = g(\varphi_m) + O(N^{-1}) = f'(r_+) + O(N^{-1}).$$

For $0 < h < \beta u_m$ there is a unique solution $r_+(h,\beta) > r_\beta$ of $u(r) = \frac{h}{\beta}$ and there are two solutions $0 < r_0(h,\beta) < r_-(h,\beta) < r_\beta$ for $u(r) = -\frac{h}{\beta}$. Hence there are three critical points:

$$\varphi_+ = r_+\hat{e}, \quad \varphi_- = -r_-\hat{e}, \quad \varphi_0 = -r_0\hat{e}.$$

The eigenvalues at the three points are

$$\begin{aligned}
\lambda_1(r_+) &= u'(r_+) > 0, & \lambda_2(r_+) &= \frac{u(r_+)}{r_+} > 0, \\
\lambda_1(r_-) &= u'(r_-) > 0, & \lambda_2(r_-) &= \frac{u(r_-)}{r_-} < 0 \\
\lambda_1(r_0) &= u'(r_0) < 0, & \lambda_2(r_0) &= \frac{u(r_0)}{r_0} < 0,
\end{aligned}$$

hence there is a unique (global) minimum in $\varphi_m = r_+\hat{e}$ and $H''(\varphi_m) > 0$. Proposition 3.3 yields

$$\langle g \rangle_\varphi = g(\varphi_m) + O(N^{-1}) = f'(r_+) + O(N^{-1}).$$

For $h = \beta u_m$ there is a unique solution $r_+(h,\beta) > r_\beta$ of $u(r) = \frac{h}{\beta}$ and a unique solution $0 < r_-(h,\beta) < r_\beta$ for $u(r) = -\frac{h}{\beta}$. Hence there are two critical points:

$$\varphi_+ = r_+\hat{e}, \quad \varphi_0 = -r_-\hat{e}.$$

The eigenvalues at the two points are

$$\begin{aligned}
\lambda_1(r_+) &= u'(r_+) > 0, & \lambda_2(r_+) &= \frac{u(r_+)}{r_+} > 0, \\
\lambda_1(r_-) &= u'(r_-) = 0, & \lambda_2(r_-) &= \frac{u(r_-)}{r_-} < 0
\end{aligned}$$

hence there is a unique (global) minimum in $\varphi_m = r_+\hat{e}$ and $H''(\varphi_m) > 0$. Proposition 3.3 yields

$$\langle g \rangle_\varphi = g(\varphi_m) + O(N^{-1}) = f'(r_+) + O(N^{-1}).$$

This implies that the limit $N \to \infty$ is well defined for all $\beta > n, h > 0$ and

$$M(h) = \lim_{N\to\infty} M_N(h) = f'(r_+).$$

The function $h \mapsto r_+(h,\beta)$ is continuous on $[0,\infty)$ and

$$\lim_{h\downarrow 0} r_+(h,\beta) = r_\beta > 0,$$

hence

$$\lim_{h\downarrow 0} M(h) = r_\beta\hat{e} \neq 0.$$

□

[14: 29.11.2024]

[15: 02.12.2024]

3.4 Asymptotic analysis of complex integrals

Theorem 3.6. *Let U be an open set with $\mathbb{R} \subset U \subset \mathbb{C}$, $f, g\colon U \to \mathbb{C}$ two analytic functions and $\gamma\colon \mathbb{R} \to U$ a smooth path. Assume the following assumptions hold.*

(a) $e^{-Nf}, e^{-Nf}g, e^{-Nf\circ\gamma}, e^{-Nf\circ\gamma}g\circ\gamma \in L^1(\mathbb{R};\mathbb{C})\ \forall N > 0$ *and*

$$\int_{\mathbb{R}} dx\, e^{-Nf(x)} = \int_{\mathbb{R}} dx\, \gamma'(x)\, e^{-Nf(\gamma(x))}, \qquad \int_{\mathbb{R}} dx\, g(x) e^{-Nf(x)} = \int_{\mathbb{R}} dx\, \gamma'(x)\, g(\gamma(x)) e^{-Nf(\gamma(x))}.$$

(b) *The function $x \mapsto H(x) :=\ \text{Re}\ f(\gamma(x))$ admits $q \geq 1$ global minimum points $x_1, \ldots, x_q$ with $H''(x_j) > 0\ \forall j = 1, \ldots, q$ and*

$$\inf_{\substack{x \text{ local minimum} \\ x \neq x_1, \ldots, x_q}} [H(x) - H_m] > 0, \qquad \text{where} \qquad H_m = H(x_j)\ \forall j = 1, \ldots, q.$$

(c) *The point $z_j := \gamma(x_j)$ is a critical point of f (i.e. $f'(z_j) = 0$) $\forall j = 1, \ldots, q$.*

Then as $N \to \infty$ we have, setting $f_\gamma := f \circ \gamma$ and $g_\gamma := g \circ \gamma$,

$$(i) \qquad \sqrt{N} \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi}}\, e^{-Nf(x)} = e^{-NH_m} \left[\sum_{j=1}^{q} e^{-iN\text{Im} f(z_j)} \frac{\gamma'(x_j)}{\sqrt{(f\circ\gamma)''(x_j)}} + O\left(\frac{1}{N}\right) \right]$$

$$(ii) \qquad \sqrt{N} \int_{\mathbb{R}} dx\, g(x) e^{-Nf(x)} = e^{-NH_m} \Big[\sum_{j=1}^{q} e^{-iN\text{Im} f(z_j)} \frac{g(z_j)\gamma'(x_j)}{\sqrt{(f\circ\gamma)''(x_j)}} \cdot$$

$$\cdot \left(g(z_j) + \frac{1}{2N} \left[\frac{g_\gamma''(x_j)}{f_\gamma''(x_j)} - \frac{g_\gamma'(x_j) f_\gamma^{(3)}(x_j)}{f_\gamma''(x_j)^2} - \frac{g_\gamma(x_j) f_\gamma^{(4)}(x_j)}{4 f_\gamma''(x_j)^2} \right] + O\left(\frac{1}{N}\right) \right) \Big],$$

where the square denotes the principal root.

Remark 1 Since $\text{Re} f(z_j) = H(x_j) = H_m\ \forall j = 1, \ldots, q$ we have

$$e^{-NH_m} \sum_{j=1}^{q} e^{-iN\text{Im} f(z_j)} \frac{\gamma'(x_j)}{\sqrt{(f\circ\gamma)''(x_j)}} = \sum_{j=1}^{q} e^{-Nf(z_j)} \frac{\gamma'(x_j)}{\sqrt{(f\circ\gamma)''(x_j)}}$$

This sum may even vanish since the phases $e^{-iN\text{Im} f(z_j)}$ are strongly oscillating.

Remark 2 The assumption (c) ensures

$$(f\circ\gamma)''(x_j) = f'(z_j)\gamma''(x_j) + f''(z_j)\gamma'(x_j)^2 = f''(z_j)\gamma'(x_j)^2.$$

The assumption $H''(x_j) > 0$ then ensures that $\text{Re} f''(z_j)\gamma'(x_j)^2 > 0$. Note that, if $\sqrt{f''(z_j)\gamma'(x_j)^2} = \sqrt{f''(z_j)}\gamma'(x_j)$ holds, we get

$$\frac{\gamma'(x_j)}{\sqrt{(f\circ\gamma)''(x_j)}} = \frac{1}{\sqrt{(f''(z_j))}}.$$

This happens for example when $\text{Re}\gamma'(x_j) > 0$ and $\text{Re} f''(z_j) > 0$ hold.

Proof.
We only sketch the argument for (i). (ii) works on the same way. As in the proof of Prop. 3.1, we decompose $\mathbb{R} = \cup_{j=1}^{q+1} I_j$, where $I_j := \{|x - x_j| < \varepsilon_N\}$ for $j = 1, \ldots, q$ and $I_{q+1} := \mathbb{R} \setminus \cup_{j=1}^{q} I_j$. We set $\varepsilon_N = N^{\alpha - \frac{1}{2}}$ with $0 < \alpha < \frac{1}{10}$ so that $N\varepsilon_N^5 = o(N^{-1})$.

• The region I_{q+1} is far from all minima hence we bound the absolute value of the integral

$$e^{NH_m} \left| \sqrt{N} \int_{I_{q+1}} \frac{dx}{\sqrt{2\pi}} \gamma'(x)\ e^{-Nf(\gamma(x))} \right| \leq \sqrt{N} \int_{I_{q+1}} \frac{dx}{\sqrt{2\pi}}\ |\gamma'(x)|\ e^{-N(H(x) - H_m)}.$$

Using $H(x) - H_m \geq \frac{\varepsilon_N^2}{4} \min_{j=1,\ldots,q} H''(x_j) = \varepsilon_N^2 C$ with $C > 0$, the last integral is bounded by

$$\sqrt{N} e^{-N\varepsilon_N^2 \frac{C}{2}} \int_{\mathbb{R}} \frac{dx}{\sqrt{2\pi}} |\gamma'(x)|\ e^{-(H(x) - H_m)} = O\left(e^{-N^{2\alpha}c}\right),$$

for some constant $c > 0$.

• In the region I_j, $j = 1, \ldots, q$, we replace $f \circ \gamma$ and γ' by the corresponding Taylor expansions up to a finite order

$$Nf(\gamma(x)) - N\text{Re}\ f(z_j) = iN\text{Im}\ f(z_j) + \sum_{m=2}^{4} \frac{1}{m!} (f \circ \gamma)^{(m)}(x_j)\, N(x - x_j)^m + O(N\varepsilon_N^5),$$
$$\gamma'(x) = \gamma'(x_j) + \gamma''(x_j)(x - x_j) + O\left(\frac{1}{N}\right).$$

The corrections of order 1, $1/\sqrt{N}$ and $1/N$ are extracted explicitely, the remaining terms are estimated in absolute value. □

As an example of application we consider the average of $\det(E - H)$ where $H \in \mathbb{C}_{herm}^{N \times N}$ is a random matrix in the GUE ensemble. We have proved in Lemma 2.23 the identity

$$\langle \det(E - H) \rangle_N = \tfrac{\sqrt{N}}{\sqrt{2\pi}} \int_{\mathbb{R}} db\ e^{-N\frac{b^2}{2}} (E - ib)^N =: I_N(E). \tag{3.9}$$

By the symmetry $b \to -b$ we have

$$I_N(-E) = (-1)^N I_N(E).$$

Moreover, for $E = 0$ we have

$$I_N(0) = (-i)^N \tfrac{\sqrt{N}}{\sqrt{2\pi}} \int_{\mathbb{R}} db\ e^{-N\frac{b^2}{2}} b^N = \begin{cases} 0 & \text{if } N \text{ odd} \\ (N-1)!!\ N^{-\frac{N}{2}} & \text{if } N \text{ even.} \end{cases} \tag{3.10}$$

Using Stirling's approximation formula we get

$$I_N(0) = O\left(e^{-\frac{N}{2}}\right) \to_{N \to \infty} 0.$$

It remains to study the case $E > 0$. We will need the following preliminary lemma.

Lemma 3.7. *Let $F \colon \mathbb{C} \to \mathbb{C}$ be defined via*

$$F(z) := e^{-\frac{N}{2}z^2} (E - iz)^N.$$

- *For all $c \in \mathbb{R}$ it holds:* $\int_{\mathbb{R}} dx\ F(x) = \int_{\mathbb{R}+ic} dz\ F(z) = \int_{\mathbb{R}} dx\ F(x+ic)$.
- *Fix $c \in \mathbb{R}$ and define $H(x) := -\frac{1}{N} \ln |F(x+ic)|$. Then*

$$
\begin{aligned}
H(x) &= \frac{x^2 - c^2}{2} - \frac{1}{2} \ln \left[(E+c)^2 + x^2 \right] \\
H'(x) &= \frac{x}{(E+c)^2 + x^2} \left[x^2 - (1 - (E+c)^2) \right] \\
H''(x) &= 1 + \frac{x^2 - (E+c)^2}{\left[x^2 + (E+c)^2 \right]^2}
\end{aligned}
$$

Proof. Exercise. For (i) use the fact that F is analytic on $\mathbb{C}$ and Cauchy theorem. (ii) follows by direct computation.

□

Theorem 3.8. *Let $E > 0$ and consider the integral $I_N(E)$ defined in* (3.9) *above.*

(i) For $E > 2$ we define $E_\pm := \frac{E}{2} \pm \sqrt{\frac{E^2}{4} - 1}$. We have

$$0 < E_- < 1 < E_+ < E, \qquad E - E_+ = E_-, \qquad E_+ E_- = 1,$$

and, as $N \to \infty$,

$$I_N(E) = e^{\frac{N}{2} E_-^2} \frac{E_+^N}{\sqrt{1 - E_-^2}} \left[1 + O\left(\tfrac{1}{N} \right) \right].$$

(ii) For $0 < E < 2$ we define $\mathcal{E}_\pm := \frac{E}{2} \pm i\sqrt{1 - \frac{E^2}{4}}$. We have

$$|\mathcal{E}_\pm| = 1, \qquad \mathcal{E}_- = \overline{\mathcal{E}_+}, \qquad E - \mathcal{E}_+ = \mathcal{E}_-, \qquad \mathcal{E}_+ \mathcal{E}_- = 1,$$

and, as $N \to \infty$,

$$I_N(E) = e^{\frac{N}{2} \mathcal{E}_-^2} \frac{\mathcal{E}_+^N}{\sqrt{1 - \mathcal{E}_-^2}} + e^{\frac{N}{2} \mathcal{E}_+^2} \frac{\mathcal{E}_-^N}{\sqrt{1 - \mathcal{E}_+^2}} + e^{N \mathrm{Re} \mathcal{E}_-^2} O\left(\tfrac{1}{N} \right).$$

Remark Note that $\mathrm{Re}\mathcal{E}_-^2 = \mathrm{Re}\mathcal{E}_+^2 = -(1 - \frac{E^2}{2}) < 0$ for $0 < E < \sqrt{2}$ and hence

$$I_N(E) = O\left(e^{-\frac{N}{2}(1 - \frac{E^2}{2}))} \right) \to_{N \to \infty} 0$$

for $0 \leq E < \sqrt{2}$. On the contrary, for $E > 2$ the integral diverges exponentially as $N \to \infty$.

Proof.
We write $F(z) = e^{-Nf(z)}$ with

$$f(z) = \frac{z^2}{2} - \ln(E - iz).$$

We start by looking for the critical points of f. We compute

$$f'(z) = z + \frac{i}{E - iz}.$$

Hence $f'(z) = 0$ iff

$$iz = \frac{E}{2} \pm \sqrt{\frac{E^2}{4} - 1}.$$

Case 1: $E > 2$. In this case the critical points are

$$z_c = -iE_\pm.$$

Via a complex translation we can cross only one of the two points. Setting $c = -E_-$ (resp. $-E_+$) the path crosses only $-iE_-$ (resp $-iE_+$.)

• Set $c = -iE_-$. We argue

$$E + c = E - E_- = E_+ > 1 \Rightarrow 1 - (E + c)^2 = 1 - E_+^2 < 0,$$

and hence, using Lemma 3.7, the function $H(x) = -\ln |F(x - iE_-)|$ a unique (global) minimum in $x = 0$. Hence the unique global minimum is at the critical point $z_1 = -iE_-$. We compute, using Lemma 3.7 again,

$$H''(0) = 1 - E_-^2 > 0$$

since $0 < E_- < 1$. Thm. 3.6 now yields the result.

• Note that setting $c = -iE_+$ does not work. Indeed, in this case we get $E+c = E-E_+ = E_- < 1$ and hence H has two global minima in $x = \pm\sqrt{1 - E_-^2}$, while 0 is now a local maximum. So we cannot apply Thm. 3.6.

Case 2: $0 < E < 2$. In this case the critical points are

$$z_c = -i\mathcal{E}_\pm,$$

and the complex translated path $\mathbb{R} - i\frac{E}{2}$ crosses both points. Setting $c = -\frac{E}{2}$, we see that the function H admits two global minima in $x = \pm\sqrt{1 - \frac{E^2}{4}}$ (wich correspond to the two critical points for f) and a local maximum in $x = 0$. Moreover

$$H''(\pm\sqrt{1 - \frac{E^2}{4}}) = 2\left(1 - \frac{E^2}{4}\right) > 0.$$

Thm. 3.6 now yields the result.

□

[15: 02.12.2024]

[16: 06.12.2024]

4 Supermathematics

4.1 Supervectors and supermatrices

Recall the definition of Grassmann algebra and generators Def 2.1 and 2.14.

Definition 4.1 (Complex conjugate)**.** *Consider a Grassmann algebra $\mathcal{G} = \mathcal{G}_{\mathbb{K}}[\mathcal{V}]$ of even dimension and let*

$$\{\overline{\psi}_1, \psi_1, \dots, \overline{\psi}_N, \psi_N\}$$

be a set of generators.

(i) We define the complex conjugate operation on the generators via

$$\psi_j^c := \overline{\psi}_j, \qquad \overline{\psi}_j^c := -\psi_j, \qquad j = 1, \dots, N.$$

The complex conjugate of a product of generators is defined via

$$\left[\prod_{i\in I} \psi_i \prod_{j\in J} \overline{\psi}_j\right]^c := \prod_{i\in I} \psi_i^c \prod_{j\in J} \overline{\psi}_j^c = \prod_{i\in I} \overline{\psi}_i (-1)^{|J|} \prod_{j\in j} \psi_j.$$

For any element $v \in \mathcal{G}$, we have the unique decomposition

$$v = \sum_{I,J\subset\{1,\dots,N\}} v_{I,J}\ \psi^I \overline{\psi}^J.$$

We define complex conjugate operation on v via

$$v^c := \sum_{I,J\subset\{1,\dots,N\}} \overline{v_{I,J}}\ \left[\psi^I \overline{\psi}^J\right]^c = \sum_{I,J\subset\{1,\dots,N\}} \overline{v_{I,J}}\ (-1)^{|J|}\ \overline{\psi}^I \psi^J,$$

where $\overline{v_{I,J}}$ is the standard complex conjugare in $\mathbb{C}$.

(ii) An element $v \in \mathcal{G}$ is called real *if $v^c = v$.*

Lemma 4.2. *Consider a Grassmann algebra $\mathcal{G} = \mathcal{G}_{\mathbb{K}}[\mathcal{V}]$ of even dimension and let*

$$\{\overline{\psi}_1, \psi_1, \dots, \overline{\psi}_N, \psi_N\}$$

be the set of generators we use to define the complex conjugate. The following statements hold.

(i) $(\psi_j^c)^c = -\psi_j$ and $(\overline{\psi}_j^c)^c = -\overline{\psi}_j$ $\forall j = 1, \dots, N$.

(ii) We have

$$\begin{cases} \forall v \in \mathcal{G}^{\text{even}} & v^c \in \mathcal{G}^{\text{even}} \text{ and } (v^c)^c = v \\ \forall v \in \mathcal{G}^{\text{odd}} & v^c \in \mathcal{G}^{\text{odd}} \text{ and } (v^c)^c = -v. \end{cases}$$

(iii) For all $v, v' \in \mathcal{G}^{\text{odd}}$ we have

$$(v^c v)^c = v^c v, \qquad (v^c v' + v'^c v)^c = (v^c v' + v'^c v).$$

(iv) Let

$$v = \sum_{I,J\subset\{1,\dots,N\}} v_{I,J}\ \psi^I \overline{\psi}^J \in \mathcal{G}.$$

Then v is real iff $v_{JI} = \overline{v_{IJ}}(-1)^{|J|(1+|I|)}$ $\forall I, J \subset \{1, \dots, N\}$.

Proof.
(i) We compute $(\psi_j^c)^c = \overline{\psi}_j^c = -\psi_j$ and $(\overline{\psi}_j^c)^c = -\psi_j^c = -\overline{\psi}_j$ $\forall j = 1, \dots, N$.

(ii) The statement follows from $((\psi^I \overline{\psi}^J)^c)^c = (-1)^{|J|} (\overline{\psi}^I \psi^J)^c = (-1)^{|I|+|J|} (\psi^I \overline{\psi}^J)$.

(iii) Let $v, v' \in \mathcal{G}^{\text{odd}}$. We compute, using (ii),

$$(v^c v)^c = -v v^c = v^c v, \qquad (v^c v' + v'^c v)^c = (-v v'^c - v' v^c) = (v^c v' + v'^c v).$$

(iv) The statement follows from

$$v^c = \sum_{I,J \subset \{1,\dots,N\}} \overline{v_{I,J}}\ (-1)^{|J|}\ \overline{\psi}^I \psi^J = \sum_{J,I \subset \{1,\dots,N\}} \overline{v_{I,J}}\ (-1)^{|J|(1+|I|)}\ \psi^J \overline{\psi}^I.$$

□

Definition 4.3. *Let* $\{\overline{\psi}_1, \psi_1, \dots, \overline{\psi}_N, \psi_N, \overline{\xi}_1, \xi_1, \dots, \overline{\xi}_{N'}, \xi_{N'}\}$ *a set of generators.*
We consider the three Grassmann algebras
$\mathcal{G} = \mathcal{G}_{\mathbb{K}}[\overline{\psi}_1, \psi_1, \dots, \overline{\psi}_N, \psi_N,]$
$\mathcal{G}' = \mathcal{G}_{\mathbb{K}}[\overline{\psi}_1, \psi_1, \dots, \overline{\psi}_N, \psi_N, \overline{\xi}_1, \xi_1, \dots, \overline{\xi}_{N'}, \xi_{N'}]$
$\mathcal{A} = \mathcal{G}_{\mathbb{K}}[\overline{\xi}_1, \xi_1, \dots, \overline{\xi}_{N'}, \xi_{N'}]$.
Fix $m, n \in \mathbb{N}$.

(i) A $(m|n)$ *graded vector (or supervector) on* $\mathcal{G}$ *is a vector with* m *components in* $\mathcal{G}^{\text{even}}$ *and* n *components in* $\mathcal{G}^{\text{odd}}$ *:*

$$\Phi = \begin{pmatrix} \Phi_b \\ \Phi_f \end{pmatrix} = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_m \\ \chi_1 \\ \vdots \\ \chi_n \end{pmatrix} \in (\mathcal{G}^{\text{even}})^m \times (\mathcal{G}^{\text{odd}})^n =: \mathcal{G}^{m|n}.$$

We call $\varphi = \Phi_b$ *the bosonic component and* $\chi = \Phi_f$ *the fermionic component of* Φ.
We define

$$\overline{\Phi} := \begin{pmatrix} \varphi^c \\ \chi^c \end{pmatrix}, \qquad \Phi^t := \left(\varphi^t, \chi^t\right), \qquad \Phi^* := \overline{\Phi}^t.$$

(ii) A linear transformation $L \colon \mathcal{G}^{m|n} \to \mathcal{G}'^{m'|n'}$ *must have the form* $L(\Phi) = M\Phi$ *where*

$$M = \begin{pmatrix} M_{bb} & M_{bf} \\ M_{fb} & M_{ff} \end{pmatrix} = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix} \qquad a \in (\mathcal{A}^{\text{even}})^{m' \times m}, b \in (\mathcal{A}^{\text{even}})^{n' \times n}, \sigma \in (\mathcal{A}^{\text{odd}})^{m' \times n}, \rho \in (\mathcal{A}^{\text{odd}})^{n' \times m}.$$

M *is called a supermatrix. We write* $M \in \mathcal{A}^{(m'|n') \times (m|n)}$. *We call* $a = M_{bb}$ *the boson-boson block,* $b = M_{ff}$ *the fermion-fermion block,* $\sigma = M_{bf}$ *the boson-fermion block and* $\rho = M_{fb}$ *the fermion-boson block.*

(iii) For $M \in \mathcal{A}^{(m|n) \times (m|n)}$ *we define the analog of trace, determinant, transpose and adjoint as follows.*

(a) The supertrace is defined by $\operatorname{Str} M := \operatorname{tr} a - \operatorname{tr} b$.

(b) Assuming b *is invertible, the superdeterminant is defined by*

$$\operatorname{Sdet} M := \frac{\det(a - \sigma b^{-1}\rho)}{\det b}.$$

Assuming both a *and* b *are invertible, we also have*

$$\operatorname{Sdet} M := \frac{\det a}{\det b - \rho a^{-1}\sigma}.$$

(c) The transpose/adjoint is defined by

$$M^t = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix}^t := \begin{pmatrix} a^t & \rho^t \\ -\sigma^t & b^t \end{pmatrix}, \qquad M^* := \begin{pmatrix} a^* & \rho^* \\ -\sigma^* & b^* \end{pmatrix} = \begin{pmatrix} (a^c)^t & (\rho^c)^t \\ -(\sigma^c)^t & (b^c)^t \end{pmatrix}.$$

Remark. By construction $\Phi^*\Phi$ is a real element in $\mathcal{G}^{\text{even}}$. This follows from

$$\Phi^*\Phi = \begin{pmatrix} \varphi^c \\ \chi^c \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \sum_{j=1}^m \varphi_j^c \varphi_j + \sum_{k=1}^n \chi_k^c \chi_k.$$

In the same way $\Phi^*\Phi' + \Phi'^*\Phi$ is a real element in $\mathcal{G}^{\text{even}}$ for all $\Phi, \Phi' \in \mathcal{G}^{m|n}$.

Lemma 4.4.

(i) Str *is uniquely defined by the following two requirements:*

(a) the Str *is a linear combination of elements from the diagonal of* M *and*

(b) $\operatorname{Str} \Phi \otimes \Phi^* = \Phi^*\Phi \;\forall \Phi \in \mathcal{A}^{m|n}$.

Moreover we have

$$\operatorname{Str} M_1 M_2 = \operatorname{Str} M_2 M_1 \qquad \forall M_1, M_2 \in \mathcal{A}^{(m|n)\times(m|n)}. \tag{4.1}$$

(ii) Sdet *is uniquely defined by the following two requirements:*

(a) $\operatorname{Sdet} M \in \mathcal{A}^{\text{even}}$,

(b) $\operatorname{Sdet}(M_1 M_2) = \operatorname{Sdet} M_1 \operatorname{Sdet} M_2 \;\forall M_1, M_2 \in \mathcal{A}^{(m|n)\times(m|n)}$ *and*

(c) $\ln \operatorname{Sdet}(M) = \operatorname{Str} \ln(M)$ *for all* M *such that with* a, b *hermitian and positive definite (see also Remark 1 below).*

Moreover we have

$$\operatorname{Sdet} M^t = \operatorname{Sdet} M \qquad \forall M \in \mathcal{A}^{(m|n)\times(m|n)}.$$

(iii) Setting $(\Phi, M\Phi) := \Phi^* M \Phi$, M^t *is uniquely defined by the requirement*

$$(\Phi, M\Phi) = (M^*\Phi, \Phi) \qquad \forall \Phi \in \mathcal{G}^{m|n}.$$

Remark 1 Note that $\operatorname{Sdet} M \in \mathcal{A}^{\text{even}}$, hence $\ln \operatorname{Sdet} M$ can be defined as in Example 4 at the end of Section 2.2.
The function $\ln M$ takes values on supermatrices and is defined via

$$\ln M := -\sum_{k\geq 1} \frac{(-1)^k}{k} (M-1)^k \in \mathcal{A}^{(m|n)\times(m|n)}.$$

Remark 2 Note that $(M^*)^* = M$ but $(M^t)^t \neq M$. Indeed

$$(M^*)^* = \begin{pmatrix} a^* & \rho^* \\ -\sigma^* & b^* \end{pmatrix}^* = \begin{pmatrix} (a^*)^* & -(\sigma^*)^* \\ -(\rho^*)^* & (b^*)^* \end{pmatrix} = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix},$$

where we used $(a^*)^*_{ij} = (a^c_{ij})^c = a_{ij}$ since $a_{ij} \in \mathcal{A}^{\text{even}}$ (same for $(b^*)^*$) and $-(\sigma^*)^*_{ij} = -(\sigma^c_{ij})^c = \sigma_{ij}$ since $\sigma_{ij} \in \mathcal{A}^{\text{odd}}$ (same for $-(\rho^*)^*$).
On the contrary

$$(M^t)^t = \begin{pmatrix} a^t & \rho^t \\ -\sigma^t & b^t \end{pmatrix}^t = \begin{pmatrix} a & -\sigma \\ -\rho & b \end{pmatrix} \neq M.$$

Proof of Lemma 4.4.
(i) We compute

$$\Phi \otimes \Phi^* = \begin{pmatrix} \varphi \otimes \varphi^* & \varphi \otimes \chi^* \\ \chi \otimes \varphi^* & \chi \otimes \chi^* \end{pmatrix}.$$

To satisfy (a) we must have

$$\mathrm{Str}\, \Phi \otimes \Phi^* = \sum_{j=1}^n \alpha_j \varphi_j \varphi_j^c + \sum_{j=1}^n \beta_j \chi_j \chi_j^c$$

for some given parameters $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{C}$. We also have

$$\Phi^*\Phi = \sum_{j=1}^n \varphi_j^c \varphi_j + \sum_{k=1}^m \chi_k^c \chi_k = \sum_{j=1}^n \varphi_j \varphi_j^c - \sum_{k=1}^m \chi_j \chi_j^c.$$

Therefore we have $\mathrm{Str}\, \Phi \otimes \Phi^* = \Phi^*\Phi$ for all Φ iff $\alpha_j = 1\ \forall j = 1, \ldots, n$ and $\beta_k = -1\ \forall k = 1, \ldots, m$.
(4.1) follows by direct computation (exercise).

(ii) We distinguish three cases.

Case 1. Consider $M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and a, b are both hermitian and positive definite.
Then, using also (2.9),

$$\ln \mathrm{Sdet}\, M = \mathrm{Str}\, \ln M = \mathrm{Str}\, \begin{pmatrix} \ln a & 0 \\ 0 & \ln b \end{pmatrix} = \mathrm{tr}\, \ln a - \mathrm{tr}\, \ln b = \ln \frac{\det a}{\det b}.$$

Therefore we obtain

$$\mathrm{Sdet}\, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \frac{\det a}{\det b}.$$

Case 2. We consider the case $M = \begin{pmatrix} 1 & \sigma \\ \rho & 1 \end{pmatrix} = 1 + X$ with $X = \begin{pmatrix} 0 & \sigma \\ \rho & 0 \end{pmatrix}$. We compute

$$X^{2k} = \begin{pmatrix} (\sigma\rho)^k & 0 \\ 0 & (\rho\sigma)^k \end{pmatrix}, \qquad X^{2k+1} = \begin{pmatrix} 0 & (\sigma\rho)^k \sigma \\ (\rho\sigma)^k \rho & 0 \end{pmatrix} \quad k \geq 0. \tag{4.2}$$

Using

$$\ln \mathrm{Sdet}\, (1 + X) = \mathrm{Str}\, \ln(1 + X) = -\sum_{k \geq 0} \frac{(-1)^k}{k} \mathrm{Str}\, X^k,$$

we argue

$$\begin{aligned}\operatorname{Str} X^{2k} &= \operatorname{tr}(\sigma\rho)^k - \operatorname{tr}(\rho\sigma)^k = 2\operatorname{tr}(\sigma\rho)^k = -2\operatorname{tr}(\rho\sigma)^k\\ \operatorname{Str} X^{2k+1} &= 0,\end{aligned}$$

hence

$$\begin{aligned}\ln \operatorname{Sdet}(1+X) &= -\sum_{k\geq 0}\frac{1}{2k}\operatorname{Str} X^{2k} = -\sum_{k\geq 0}\frac{1}{k}\operatorname{tr}(\sigma\rho)^k = \operatorname{tr}\,\ln(1-\sigma\rho) = \ln\det(1-\sigma\rho)\\ &= \sum_{k\geq 0}\frac{1}{k}\operatorname{tr}(\rho\sigma)^k = -\operatorname{tr}\,\ln(1-\rho\sigma) = -\ln\det(1-\rho\sigma).\end{aligned}$$

Therefore we obtain

$$\operatorname{Sdet}\begin{pmatrix}1 & \sigma\\ \rho & 1\end{pmatrix} = \det(1-\sigma\rho) = \frac{1}{\det(1-\rho\sigma)}.$$

Case 3. Finally consider the more general case $\begin{pmatrix}a & \sigma\\ \rho & b\end{pmatrix}$ with a, b hermitian and positive definite. We argue

$$M = \begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\begin{pmatrix}1 & a^{-1}\sigma\\ b^{-1}\rho & 1\end{pmatrix} = \begin{pmatrix}1 & \sigma b^{-1}\\ \rho a^{-1} & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix} \tag{4.3}$$

hence

$$\operatorname{Sdet} M = \operatorname{Sdet}\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\operatorname{Sdet}(1+X) = \operatorname{Sdet}\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\operatorname{Sdet}(1+Y),$$

with

$$X = \begin{pmatrix}0 & a^{-1}\sigma\\ b^{-1}\rho & 0\end{pmatrix}, \qquad Y = \begin{pmatrix}0 & \sigma b^{-1}\\ \rho a^{-1} & 0\end{pmatrix}.$$

The formula now follows from the previous special cases.
Using the formulas for the Sdet one can show that $\operatorname{Sdet} M^t = \operatorname{Sdet} M$ (exercise).

[16: 06.12.2024]

[17: 09.12.2024]

(iii) We compute

$$\begin{aligned}\Phi^* M\Phi &= \varphi^* a\varphi + \varphi^*\sigma\chi + \chi^*\rho\varphi + \chi^* b\chi\\ &= (a^*\varphi)^*\varphi - (\sigma^*\varphi)^*\chi + (\rho^*\chi)^*\varphi + (b^*\chi)^*\chi.\end{aligned}$$

The result follows. □

Lemma 4.5 (Inverse of a supermatrix)**.** *Let* $M = \begin{pmatrix}a & \sigma\\ \rho & b\end{pmatrix} \in \mathcal{A}^{(m|n)\times m|n}$ *be a supermatrix with* a, b *invertible. Then* M *is invertible and*

$$M^{-1} = \begin{pmatrix}(M^{-1})_{bb} & (M^{-1})_{bf}\\ (M^{-1})_{fb} & (M^{-1})_{ff}\end{pmatrix} = \begin{pmatrix}(a-\sigma b^{-1}\rho)^{-1} & -(a-\sigma b^{-1}\rho)^{-1}\sigma b^{-1}\\ -(b-\rho a^{-1}\sigma)^{-1}\rho a^{-1} & (b-\rho a^{-1}\sigma)^{-1}\end{pmatrix}$$

Moreover we have the relations

$$\begin{aligned}(1-\sigma b^{-1}\rho a^{-1})^{-1}\sigma b^{-1} &= \sigma b^{-1}(1-\rho a^{-1}\sigma b^{-1})\\ (b-\rho a^{-1}\sigma)^{-1} &= b^{-1} + b^{-1}\rho(a-\sigma b^{-1}\rho)^{-1}\sigma b^{-1},\end{aligned}$$

and hence we can express M^{-1} *in terms of* $(a-\sigma b^{-1}\rho)^{-1}$ *or* $(b-\rho a^{-1}\sigma)^{-1}$ *only.*

Proof.

Case 1. Consider $M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. In this case

$$M^{-1} = \begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix},$$

where the inverse of a (resp. b) was defined in Example 2 of Section 2.2.

Case 2. Consider $M = \begin{pmatrix} 1 & \sigma \\ \rho & 1 \end{pmatrix} = 1 + X$. In this case we define

$$M^{-1} := \sum_{k \geq 0} (-1)^k X^k.$$

The sum above is finite and satisfies $M^{-1}M = MM^{-1} = 1$. Using the exact formulas in (4.2) we compute

$$M^{-1} = \sum_{k\geq 0} X^{2k} - \sum_{k\geq 0} X^{2k+1} = \begin{pmatrix} (1-\sigma\rho)^{-1} & -(1-\sigma\rho)\sigma \\ -(1-\rho\sigma)^{-1}\rho & (1-\rho\sigma)^{-1} \end{pmatrix}.$$

Moreover

$$(1-\sigma\rho)^{-1}\sigma = \sum_{k\geq 0} (\sigma\rho)^k \sigma = \sum_{k\geq 0} \sigma(\rho\sigma)^k = \sigma(1-\rho\sigma)^{-1}.$$

Similarly we argue $(1-\rho\sigma)^{-1}\rho = \rho(1-\rho\sigma)^{-1}$ and

$$(1-\rho\sigma)^{-1} = 1 + \sum_{k\geq 1} (\rho\sigma)^k = 1 + \rho \sum_{k\geq 0} (\sigma\rho)^k \sigma = 1 + \rho(1-\sigma\rho)^{-1}\sigma.$$

Case 3. Consider $M = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix}$. The result follows from (4.3) together with Case 1 and 2. □

Theorem 4.6 (Gaussian integral)**.** *Consider the supermatrix* $M = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix} \in \mathcal{A}^{(m|n)\times(m|n)}$ *and the supervector* $\Phi = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$ *with* $\varphi \in \mathbb{C}^m$ *and* $\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} \in (\mathcal{G}^{\text{odd}})^n$. *Set* $\psi_j^c = \overline{\psi_j}$, *hence* $\Phi^* = (\varphi^*, \psi^*) = (\overline{\varphi}_1, \dots, \overline{\varphi}_m, \overline{\psi}_1, \dots \overline{\psi}_n)$. *We define*

$$d\Phi^* d\Phi := \prod_{j=1}^m \frac{d\overline{\varphi}_j d\varphi_j}{2\pi} \prod_{k=1}^n d\overline{\psi}_k d\psi_k = \left(\frac{d\overline{\varphi} d\varphi}{2\pi}\right)^m \left(d\overline{\psi} d\psi\right)^n.$$

Assume $\operatorname{Re}[\operatorname{body}(a)] > 0$. *Then we have*

$$\int d\Phi^* d\Phi \; e^{-\Phi^* M \Phi} = \frac{\det(b - \rho a^{-1}\sigma)}{\det a}.$$

If in addition b is invertible we have

$$\int d\Phi^* d\Phi \; e^{-\Phi^* M \Phi} = \frac{1}{\operatorname{Sdet} M}.$$

Notation. Assume $\mathrm{Re}[\mathrm{body}(a)] > 0$ and b is invertible. Then M is invertible. The normalized Gaussian measure on Φ with mean zero and covariance M^{-1} is defined as follows:

$$d\mu_{M^{-1}}(\Phi^*, \Phi) := (\mathrm{Sdet}\, M)\ d\Phi^* d\Phi\ e^{-\Phi^* M \Phi}. \tag{4.4}$$

Proof.
We show that the integral above is well defined.

$$\begin{aligned}\int d\Phi^* d\Phi\ e^{-\Phi^* M\Phi} &= \int \left(\frac{d\overline{\varphi} d\varphi}{2\pi}\right)^m \ \left(d\overline{\psi} d\psi\right)^n e^{-\varphi^* a\varphi} e^{-\varphi^*\sigma\psi - \psi^*\rho\varphi - \psi^* b\psi} \\ &= \sum_{I,J\subset\{1,\ldots,n\}} \int \left(\frac{d\overline{\varphi} d\varphi}{2\pi}\right)^m \left(d\overline{\psi} d\psi\right)^n v_{IJ}(\varphi)\psi^I \overline{\psi}^J,\end{aligned}$$

where

$$v_{IJ}(\varphi) = e^{-\varphi^* \mathrm{body}(a)\varphi} P(\varphi, \overline{\varphi}).$$

This function is integrable since $\mathrm{Re}[\mathrm{body}(a)] > 0$.
To prove the statement we can proceed in two ways.

Proof 1. We integrate the φ variables first.

$$\begin{aligned}&\int_{\mathbb{C}^m} \left(\frac{d\overline{\varphi} d\varphi}{2\pi}\right)^m e^{-\varphi^* a\varphi} e^{-\varphi^*\sigma\psi - \psi^*\rho\varphi} = \int_{\mathbb{C}^m} \left(\frac{d\overline{\varphi} d\varphi}{2\pi}\right)^m e^{-(\overline{\varphi}, a\varphi)} e^{-(\overline{\varphi}, v)} e^{-(\overline{w}, \varphi)} \\ &= \frac{1}{\det a} e^{(\overline{w}, a^{-1}v)} = \frac{1}{\det a} e^{\psi^*\rho a^{-1}\sigma\psi}.\end{aligned}$$

Integrating now the ψ variables we obtain

$$\begin{aligned}\int d\Phi^* d\Phi\ e^{-\Phi^* M\Phi} &= \frac{1}{\det a} \int \left(d\overline{\psi} d\psi\right)^n e^{-\psi^* b\psi} e^{\psi^*\rho a^{-1}\sigma\psi} \\ &= \frac{1}{\det a} \int \left(d\overline{\psi} d\psi\right)^n e^{-\psi^*(b - \rho a^{-1}\sigma)\psi} = \frac{\det(b - \rho a^{-1}\sigma)}{\det a}.\end{aligned}$$

Proof 2. In the case also b is invertible, we can integrate the ψ variables first.

$$\begin{aligned}&\int \left(d\overline{\psi} d\psi\right)^n e^{-\psi^* b\psi} e^{-\varphi^*\sigma\psi - \psi^*\rho\varphi} = \int \left(d\overline{\psi} d\psi\right)^n e^{-(\overline{\psi}, b\chi)} e^{-(\overline{\psi}, \alpha)} e^{-(\overline{\beta}, \psi)} \\ &= \det b\ e^{(\overline{\beta}, b^{-1}\alpha)} = \det b\ e^{\varphi^*\sigma b^{-1}\rho\varphi}.\end{aligned}$$

Integrating now the φ variables we obtain

$$\begin{aligned}\int d\Phi^* d\Phi\ e^{-\Phi^* M\Phi} &= \det b \int_{\mathbb{C}^m} \left(\frac{d\overline{\varphi} d\varphi}{2\pi}\right)^m e^{-(\overline{\varphi}, a\varphi)} e^{\varphi^*\sigma b^{-1}\rho\varphi} = \det b \int_{\mathbb{C}^m} \left(\frac{d\overline{\varphi} d\varphi}{2\pi}\right)^m e^{-(\overline{\varphi}, (a - \sigma b^{-1}\rho)\varphi)} \\ &= \frac{\det b}{\det(a - \sigma b^{-1}\rho)} = \frac{1}{\mathrm{Sdet}\, M} = \frac{\det(b - \rho a^{-1}\sigma)}{\det a}.\end{aligned}$$

□

Theorem 4.7. *Consider the supermatrix* $M = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix} \in \mathcal{A}^{(m|n)\times(m|n)}$ *and* $\Phi', \tilde{\Phi} \in \mathcal{A}^{m|n}$ *two parameter supervectors. Assume* $\mathrm{Re}[\mathrm{body}(a)] > 0$ *and* b *is invertible and define* $C := M^{-1}$. *Then, for all* $v, w \in \mathbb{C}$ *we have*

$$\int d\mu_C(\Phi, \Phi^*) e^{z\Phi^*\Phi' + w\tilde{\Phi}^*\Phi} = e^{zw\ \tilde{\Phi}^* C \Phi'}. \tag{4.5}$$

In particular:

(i) (Laplace transpose) $\int d\mu_C(\Phi, \Phi^*) e^{\Phi^*\Phi' + \tilde{\Phi}^*\Phi} = e^{\tilde{\Phi}^* C\Phi'}$,

(ii) (Fourier transform) $\int d\mu_C(\Phi, \Phi^*) e^{i(\Phi^*\Phi' + \tilde{\Phi}^*\Phi)} = e^{-\tilde{\Phi}^* C\Phi'}$,

(iii) (second moment) $\int d\mu_C(\Phi, \Phi^*) \Phi_\alpha \Phi^*_\beta = C_{\alpha\beta}$.

Proof. To prove (4.5) complete the square and translate φ and ψ. To prove (iii) derive the Laplace transform (exercise). ☐

Remarks Assume now $m = n$ so that we have the same number of bosonic and fermionic components.

(i) If $\sigma = \rho = 0$ and $a = b$ we obtain

$$\mathrm{Sdet} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \frac{\det a}{\det a} = 1$$

and

$$\int d\Phi^* d\Phi\ e^{-\Phi^* M \Phi} = \int_{\mathbb{C}^n} \left(\frac{d\overline{\varphi} d\varphi}{2\pi}\right)^n e^{-(\overline{\varphi}, a\varphi)} \int \left(d\overline{\psi} d\psi\right)^n e^{-(\overline{\psi}, a\psi)} = \frac{\det a}{\det a} = 1.$$

(ii) For any supermatrix $M \in \mathcal{A}^{(n|n)\times(n|n)}$ it holds (exercise)

$$\mathrm{Sdet}\,(\lambda M) = \mathrm{Sdet}\, M \qquad \forall \lambda \in \mathbb{K}. \tag{4.6}$$

[17: 09.12.2024]

[18: 13.12.2024]

4.2 Dual representation for the averaged resolvent: GUE case

Recall that a random matrix in the GUE ensemble is a hermitian matrix $H \in \mathbb{C}^{N\times N}_{herm}$ with probability measure proportional to $dH\ e^{-\frac{N}{2}\mathrm{tr}\, H^2}$ (cf. Section 1.2.6).

Facts about GUE . Since $H^* = H$ the matrix is diagonalizable with real random eigenvalues $\lambda_1, \ldots, \lambda_N$. The corresponding joint probability distribution can be computed explicitely (see the book of Mehta)

$$d\rho_{GUE}(\lambda_1, \ldots, \lambda_N) = C_N \prod_{j=1}^N d\lambda_j \prod_{j<k} |\lambda_j - \lambda_k|^2 e^{-\frac{N}{2}\sum_{j=1}^N \lambda_j^2},$$

where C_N is the normalization constant. A random matrix ensemble whose eigenvalue statistics (in the appropriate scaling limit) coincides with the one for the GUE ensemble is said to belong to the Wigner-Dyson $\beta = 2$ universality class.
If the GUE ensemble is replaced by the GOE ensemble (i.e. $H \in \mathbb{R}^{N\times N}_{herm}$) we have instead

$$d\rho_{GOE}(\lambda_1, \dots, \lambda_N) = C_N \prod_{j=1}^{N} d\lambda_j \, \prod_{j<k} |\lambda_j - \lambda_k| \; e^{-\frac{N}{2}\sum_{j=1}^{N}\lambda_j^2},$$

A random matrix ensemble whose eigenvalue statistics (in the appropriate scaling limit) coincides with the one for the GOE ensemble is said to belong to the Wigner-Dyson $\beta = 1$ universality class.

Both measures above contain two competing effects: the factor $e^{-\frac{N}{2}\sum_{j=1}^{N}\lambda_j^2}$ is maximized when all λ_j are small (of order $\frac{1}{\sqrt{N}}$) while the term $\prod_{j<k}|\lambda_j - \lambda_k|^{\beta}$ (with $\beta = 1, 2$) is maximized when the distance between the eigenvalues is large (repulsive interaction). As a result the eigenvalues are approximately uniformly distributed on the interval $(-2, 2)$.

Information of the position of the spectrum for a hermitial random matrix can be inferred from the *averaged integrated density of states*

$$\mathbb{E}\left[\frac{\sum_{\lambda\in\sigma(H)} \mathbf{1}_{\lambda<E}}{N}\right] = \int_{-\infty}^{E} d\rho_N(E).$$

In the case of GUE one can show $d\rho_N(E) = dE \; \rho_N(E)$, where

$$\rho_N(E) := \frac{1}{N}\sum_{j=1}^{N}\int_{\mathbb{R}^N}(\prod_{k=1}^{N} d\lambda_k) \; \rho(\lambda_1, \dots, \lambda_N) \; \delta(\lambda_j - E) = \mathbb{E}\left[\frac{1}{N}\sum_{j=1}^{N}\delta(\lambda_j - E)\right], \tag{4.7}$$

is called the averaged density of states. We argue

$$\begin{aligned}\mathbb{E}\left[\frac{1}{N}\sum_{j=1}^{N}\delta(\lambda_j - E)\right] &= \frac{1}{\pi}\lim_{\varepsilon\to 0+}\mathbb{E}\left[\frac{1}{N}\sum_{j=1}^{N}\frac{\varepsilon}{(E-\lambda_j)^2+\varepsilon^2}\right] \qquad (4.8)\\ &= -\frac{1}{N\pi}\lim_{\varepsilon\to 0+}\operatorname{Im}\mathbb{E}\left[\sum_{j=1}^{N}(E+i\varepsilon-\lambda_j)^{-1}\right] = -\frac{1}{N\pi}\lim_{\varepsilon\to 0+}\operatorname{Im}\mathbb{E}\left[\operatorname{tr}(E+i\varepsilon-H)^{-1}\right].\end{aligned}$$

Therefore we need to study the average of the resolvent. The following theorem gives a dual representation for this average.

Theorem 4.8. *For $E \in \mathbb{R}$ and $\varepsilon > 0$ we define $z = E + i\varepsilon$. We denote by R the $(1|1) \times (1|1)$ supermatrix*

$$R := \begin{pmatrix} a & \overline{\rho} \\ \rho & ib \end{pmatrix}, \quad \text{with } a, b \in \mathbb{R}, \; \overline{\rho}, \rho \text{ Grassmann.}$$

We also define

$$dR := \frac{da\,db}{2\pi} d\overline{\rho} d\rho.$$

With this notation

$$(i)\quad \mathbb{E}\left[(E+i\varepsilon-H)^{-1}_{j_0 i_0}\right] = \delta_{i_0 j_0}\int dR\; e^{-\frac{N}{2}\operatorname{Str} R^2} a \tfrac{1}{(\operatorname{Sdet} z-R)^N} \tag{4.9}$$

$$= \delta_{i_0 j_0}\tfrac{N}{2\pi}\int da\,db\; e^{-\frac{N}{2}(a^2+b^2)} a \tfrac{(z-ib)^N}{(z-a)^N}\left[1-\tfrac{1}{(z-a)(z-ib)}\right], \tag{4.10}$$

$$(ii)\quad 1 = \int dR\; e^{-\frac{N}{2}\operatorname{Str} R^2}\tfrac{1}{(\operatorname{Sdet} z-R)^N} = \tfrac{N}{2\pi}\int da\,db\; e^{-\frac{N}{2}(a^2+b^2)}\tfrac{(z-ib)^N}{(z-a)^N}\left[1-\tfrac{1}{(z-a)(z-ib)}\right].$$

Proof. We proved in Proposition 2.24 the identity

$$\mathbb{E}\left[(E+i\varepsilon-H)^{-1}_{i_0 j_0}\right] = (-i)\int d\Phi^* d\Phi\; \varphi_{i_0}\overline{\varphi}_{j_0}\; e^{iz\sum_{j=1}^N(\overline{\varphi}_j\varphi_j+\overline{\psi}_j\psi_j)} e^{-\frac{1}{2N}\sum_{jk=1}^N(\overline{\varphi}_j\varphi_k+\overline{\psi}_j\psi_k)(\overline{\varphi}_k\varphi_j+\overline{\psi}_k\psi_j)},$$

where we defined

$$d\Phi^* d\Phi := \left(\frac{d\overline{\varphi}_j d\varphi_j}{2\pi}\right)^N \left(d\overline{\psi} d\psi\right)^N.$$

To write the formula above in a more compact way we define

$$\Phi_j := \begin{pmatrix}\varphi_j\\ \psi_j\end{pmatrix}, \qquad \Phi_j^* := \left(\overline{\varphi}_j, \overline{\psi}_j\right).$$

Then

$$\mathbb{E}\left[(E+i\varepsilon-H)^{-1}_{i_0 j_0}\right] = (-i)\int d\Phi^* d\Phi\;\varphi_{i_0}\overline{\varphi}_{j_0}\; e^{iz\sum_{j=1}^N \Phi_j^*\Phi_j} e^{-\frac{1}{2N}\sum_{j,k=1}^N(\Phi_j^*\Phi_k)(\Phi_k^*\Phi_j)}.$$

We reorganize this integral as follows.

$$\sum_{j,k=1}^N(\Phi_j^*\Phi_k)(\Phi_k^*\Phi_j) = \sum_{j,k=1}^N \operatorname{Str}(\Phi_k\otimes\Phi_k^*)(\Phi_j\otimes\Phi_j^*) = \operatorname{Str}(\sum_{j=1}^N\Phi_j\otimes\Phi_j^*)^2 = \operatorname{Str} M^2,$$

where

$$M = \begin{pmatrix} A & \Sigma\\ \Gamma & B\end{pmatrix} = \begin{pmatrix}\sum_{j=1}^N|\varphi_j|^2 & \sum_{j=1}^N\varphi_j\overline{\psi}_j\\ \sum_{j=1}^N\psi_j\overline{\varphi}_j & \sum_{j=1}^N\psi_j\overline{\psi}_j\end{pmatrix}.$$

Note that $\Sigma=\Gamma^c$ and $A^c=A$, $B^c=B$, and hence $M^*=M$. We compute

$$\operatorname{Str} M^2 = A^2-B^2+2\Gamma^c\Gamma.$$

Using (1.16) we write

$$e^{-\frac{N}{2}A^2} = \frac{\sqrt{N}}{\sqrt{2\pi}}\int_{\mathbb{R}} da\; e^{-\frac{N}{2}a^2}e^{-iaA} = \frac{\sqrt{N}}{\sqrt{2\pi}}\int_{\mathbb{R}} da\; e^{-\frac{N}{2}a^2}e^{-i\sum_{j=1}^N\overline{\varphi}_j a\varphi_j}$$

Using Theorem 2.21 we write

$$e^{\frac{N}{2}B^2} = \frac{\sqrt{N}}{\sqrt{2\pi}}\int_{\mathbb{R}} db\; e^{-\frac{N}{2}b^2}e^{bB} = \frac{\sqrt{N}}{\sqrt{2\pi}}\int_{\mathbb{R}} db\; e^{-\frac{N}{2}b^2}e^{-i\sum_{j=1}^N\overline{\psi}_j(ib)\psi_j}$$

Using Theorem 2.13

$$e^{-N\Gamma^c\Gamma} = \frac{1}{N}\int d\overline{\rho}d\rho\; e^{-N\overline{\rho}\rho}e^{-i(\overline{\rho}\Gamma+\Gamma^c\rho)} = \frac{1}{N}\int d\overline{\rho}d\rho\; e^{-N\overline{\rho}\rho}e^{-i\sum_{j=1}^N(\overline{\varphi}_j\overline{\rho}\psi_j+\overline{\psi}_j\rho\varphi_j)}.$$

Putting these identities together and using

$$a^2+b^2+2\overline{\rho}\rho=\operatorname{Str}R^2,\quad \overline{\varphi}_j a\varphi_j+\overline{\psi}_j(ib)\psi_j+\overline{\varphi}_j\overline{\rho}\psi_j+\overline{\psi}_j\rho\varphi_j=\Phi_j^*R\Phi_j,$$

we obtain

$$e^{-\frac{N}{2}\operatorname{Str}M^2}=\int dR\; e^{-\frac{N}{2}\operatorname{Str}R^2}e^{-i\sum_{j=1}^N\Phi_j^*R\Phi_j},$$

and hence

$$\begin{aligned}\mathbb{E}\left[(E+i\varepsilon-H)^{-1}_{i_0j_0}\right]&=(-i)\int d\Phi^*d\Phi\int dR\;\;\varphi_{i_0}\overline{\varphi}_{j_0}\;e^{-\frac{N}{2}\operatorname{Str}R^2}e^{i\sum_{j=1}^N\Phi_j^*(z-R)\Phi_j}\\&=\sum_{IJ\subset\{1,\dots,N\},k,k'\in\{0,1\}}\int_{\mathbb{C}^N}(d\overline{\varphi}d\varphi)^N(d\overline{\psi}d\psi)^N\int_{\mathbb{R}^2}da\,db\;d\overline{\rho}d\rho\;v_{IJ}(\overline{\varphi},\varphi,a,b)\psi^I\overline{\psi}^J\rho^k\overline{\rho}^{k'},\end{aligned}$$

where

$$v_{IJ}(\overline{\varphi},\varphi,a,b)=e^{-\varepsilon\sum_j|\varphi_j|^2}e^{i(E-a)\sum_j|\varphi_j|^2}e^{-\frac{N}{2}(a^2+b^2)}P_{I,J,k,k'}(\overline{\varphi},\varphi,b),$$

and $P_{I,J,k,k'}(\overline{\varphi},\varphi,b)$ is a polynome. Therefore $v_{IJ}\in L^1(\mathbb{C}^N\times\mathbb{R}^2)$ for all $\varepsilon>0$ and we can exchange the integration oder:

$$\begin{aligned}\mathbb{E}\left[(E+i\varepsilon-H)^{-1}_{i_0j_0}\right]&=(-i)\int dR\;e^{-\frac{N}{2}\operatorname{Str}R^2}\int d\Phi^*d\Phi\;\varphi_{i_0}\overline{\varphi}_{j_0}\;e^{i\sum_{j=1}^N\Phi_j^*(z-R)\Phi_j}\\&=\delta_{i_0j_0}(-i)\int dR\;e^{-\frac{N}{2}\operatorname{Str}R^2}\left(\int d\Phi_1^*d\Phi_1e^{i\Phi_1^*(z-R)\Phi_1}\right)^{N-1}\int d\Phi_1^*d\Phi_1|\varphi_1|^2e^{i\Phi_1^*(z-R)\Phi_1}.\end{aligned}$$

In particular this shows that $\mathbb{E}\left[(E+i\varepsilon-H)^{-1}_{j_0j_0}\right]$ is independent from j_0. We argue

$$\mathbb{E}\left[(E+i\varepsilon-H)^{-1}_{i_0j_0}\right]=\delta_{i_0j_0}\mathbb{E}\left[(E+i\varepsilon-H)^{-1}_{j_0j_0}\right]=\delta_{i_0j_0}\frac{1}{N}\mathbb{E}\left[\operatorname{tr}(E+i\varepsilon-H)^{-1}\right].$$

Moreover

$$\begin{aligned}\frac{1}{N}\mathbb{E}\left[\operatorname{tr}(E+i\varepsilon-H)^{-1}\right]&=\frac{(-i)}{N}\int d\Phi^*d\Phi\int dR\;e^{-\frac{N}{2}\operatorname{Str}R^2}\left(\sum_{j=1}^N|\varphi_j|^2\right)\;e^{i\sum_{j=1}^N\Phi_j^*(z-R)\Phi_j}\\&=\frac{1}{N}\int d\Phi^*d\Phi\int dR\;e^{-\frac{N}{2}\operatorname{Str}R^2}\partial_a e^{i\sum_{j=1}^N\Phi_j^*(z-R)\Phi_j}\\&=\int d\Phi^*d\Phi\int dR\;e^{-\frac{N}{2}\operatorname{Str}R^2}a\;e^{i\sum_{j=1}^N\Phi_j^*(z-R)\Phi_j}\end{aligned}$$

where we applied integration by parts with respect to the variable a. Exchanging the integrals we obtain

$$\begin{aligned}\mathbb{E}\left[(E+i\varepsilon-H)^{-1}_{i_0j_0}\right]&=\delta_{i_0j_0}\int dR\;e^{-\frac{N}{2}\operatorname{Str}R^2}a\;\left(\int d\Phi_1^*d\Phi_1e^{i\Phi_1^*(z-R)\Phi_1}\right)^N\\&=\delta_{i_0j_0}\int dR\;e^{-\frac{N}{2}\operatorname{Str}R^2}a\;\frac{1}{(\operatorname{Sdet}(-i)(z-R))^N}=\delta_{i_0j_0}\int dR\;e^{-\frac{N}{2}\operatorname{Str}R^2}a\;\frac{1}{(\operatorname{Sdet}z-R)^N},\end{aligned}$$

where in the last two steps we used (4.5) and (4.6). This completes the proof of (4.9). To prove (4.10) we integrate in $\overline{\rho},\rho$. We compute, using also $(\overline{\rho}\rho)^2=0$,

$$\operatorname{Str}R^2=a^2-b^2+2\overline{\rho}\rho,\qquad \operatorname{Sdet}(z-R)^{-1}=\frac{(z-ib)}{(z-a)}\frac{1}{1-\overline{\rho}\rho\frac{1}{(z-ib)(z-a)}}=\frac{(z-ib)}{(z-a)}e^{\overline{\rho}\rho\frac{1}{(z-ib)(z-a)}}.$$

Then (4.10) follows from

$$\int d\bar{\rho}d\rho\ e^{-N\bar{\rho}\rho(1-\frac{1}{(z-ib)(z-a)})} = N\left(1-\frac{1}{(z-ib)(z-a)}\right).$$

This concludes the proof of (i). The proof of (ii) works in the same way. □

[18: 13.12.2024]

[19: 16.12.2024]

4.3 Dual representation for the averaged resolvent: band matrix case

To set up the model, let $\Lambda = \Lambda_L := [-L, L]^d \cap \mathbb{Z}^d$ be a finite cube in $\mathbb{Z}^d$. We consider a random matrix $H \in \mathbb{C}^{\Lambda\times\Lambda}_{herm}$ whose matrix elements are independent *not identically distributed.* Precisely, the probability distribution is given by

$$dP(H) \propto dH\ e^{-\frac{N}{2}\sum_{ij\in\Lambda}\frac{|H_{ij}|^2}{J_{ij}}}$$

where

$$J_{ij} = f(|i-j|) \simeq \begin{cases} 0 & \text{if } |i-j| > W \\ c > 0 & \text{if } |i-j| \leq W \end{cases}$$

Therefore, the matrix elements are non-zero only in a band of width W centered around the diagonal. The parameter W is called the *band-width.* These models arise in condensed matter physics in the context of disordered conductors (cf. for example the review by Spencer *Random banded and sparse matrices*).

Heuristics. We define $N = |\Lambda|$ and consider two extreme cases.

• If $W = 1$ the matrix is diagonal hence the eigenvalues are i.i.d. random variables

$$d\rho(\lambda_1, \dots, \lambda_N) \propto \prod_j d\lambda_j e^{-\frac{1}{2c}\lambda_j^2}.$$

In this case the eigenvalue spacings satisfy Poisson statistics in the limit $N \to \infty$. For example, the probability there is no eigenvalue in the interval $(E, E + \frac{s}{N})$ is proportional to $(1 - c'\frac{s}{N})^N \to_{N\to\infty} e^{-sc'}$. In particular the probability converges to 1 when $s \to 0$.

• If $W > N$ the matrix is in the GUE ensemble whose eigenvalue distribution is

$$d\rho(\lambda_1, \dots, \lambda_N) \propto [\prod_j d\lambda_j]\prod_{i<j}(\lambda_i - \lambda_j)^2\prod_j e^{-\frac{1}{2c}\lambda_j^2},$$

In this case the eigenvalue spacings satisfy Wigner-Dyson statistics in the limit $N \to \infty$. For example, the probability there is no eigenvalue in the interval $(E, E + \frac{s}{N})$ is proportional to $se^{-s^2c'}$ in the limit $N \to \infty$ (see the book of Mehta). In particular it converges to 0 when $s \to 0$ (level repulsion).

In the general case we expect a phase transition between Poisson and Wigner-Dyson statistics depending on W and the dimension d of the lattice in the limit $N \to \infty$. Precisely:

(i) for $d = 1$ the model exhibits Poisson statistics for $W < N^{\frac{1}{2}}$ and Wigner statistics for $W > N^{\frac{1}{2}}$ (proved);

(ii) for $d = 2$ the model exhibits Poisson statistics for $W < \ln N$ and Wigner statistics for $W > \ln N$ (conjecture);

(iii) for $d \geq 3$ there eixists a $W_0 > 0$ independent of N such that the model exhibits Poisson statistics for $W < W_0$ and Wigner statistics for $W > W_0$ (conjecture).

Here we show a dual representation for the averaged resolvent.

Theorem 4.9. *Let*

$$\mathbb{E}[f(H)] := \frac{1}{Z} \int_{\mathbb{C}^{\Lambda\times\Lambda}_{herm}} dH \, e^{-\frac{1}{2}\sum_{jk} \frac{|H_{jk}|^2}{J_{jk}}} f(H),$$

where we assumed that $J_{kj} = J_{jk} > 0 \; \forall jk$, and $Z > 0$ is the normalization constant. Assume in addition that $J > 0$ as a quadratic form.
For each lattice point $j \in \Lambda$, we introduce the $(1|1) \times (1|1)$ supermatrix R_j

$$R_j := \begin{pmatrix} a_j & \bar{\rho}_j \\ \rho_j & ib_j \end{pmatrix}, \quad \text{with } a_j, b_j \in \mathbb{R},\ \bar{\rho}_j, \rho_j \text{ Grassmann.}$$

We also define

$$dR := \prod_j dR_j = \left(\frac{da\, db}{2\pi}\right)^{\Lambda} (d\bar{\rho} d\rho)^{\Lambda}, \quad (R, J^{-1}R) := \sum_{jk} (J^{-1})_{jk} R_j R_k.$$

With this notation the following identities hold for all $z = E + i\varepsilon$, with $E \in \mathbb{R}$ and $\varepsilon > 0$.

(i)
$$\mathbb{E}\left[(z - H)^{-1}_{j_0 i_0}\right] = \delta_{i_0 j_0} \int dR \; e^{-\frac{1}{2}\mathrm{Str}\,(R, J^{-1}R)} (J^{-1}a)_{j_0} \prod_j \tfrac{1}{\mathrm{Sdet}\, z - R_j} \tag{4.11}$$

$$= \delta_{i_0 j_0} \int_{\mathbb{R}^{2\Lambda}} \left(\frac{da\, db}{2\pi}\right)^{\Lambda} e^{-\frac{1}{2}((a, J^{-1}a) + (b, J^{-1}b))} (J^{-1}a)_{j_0} \left[\prod_j \tfrac{(z - ib_j)}{(z - a_j)}\right] \det\left(J^{-1} - D\right), \tag{4.12}$$

where $D = \mathrm{diag}\,\{D_j\}_{j\in\Lambda}$ and

$$D_j := 1 - \frac{1}{(z - a_j)(z - ib_j)}.$$

(ii) We also have

$$1 = \int dR \; e^{-\frac{1}{2}\mathrm{Str}\,(R, J^{-1}R)} \prod_j \tfrac{1}{\mathrm{Sdet}\, z - R_j}$$

$$= \int_{\mathbb{R}^{2\Lambda}} \left(\frac{da\, db}{2\pi}\right)^{\Lambda} e^{-\frac{1}{2}((a, J^{-1}a) + (b, J^{-1}b))} \left[\prod_j \tfrac{(z - ib_j)}{(z - a_j)}\right] \det\left(J^{-1} - D\right).$$

Proof. Exercise. Works as in the case of GUE. □

4.4 Asymptotic analysis of the dual represenations

Theorem 4.10. *Let U be an open set with $\mathbb{R}^n \subset U \subset \mathbb{C}^n$, $F, g\colon U \to \mathbb{C}$ two functions analytic in each variable separately and $\gamma\colon \mathbb{R}^n \to U$ a smooth path. Assume the following assumptions hold.*

(a) $e^{-NF}, e^{-NF}g, e^{-NF\circ\gamma}, e^{-NF\circ\gamma} g\circ\gamma \in L^1(\mathbb{R}^n;\mathbb{C})$ $\forall N > 0$ *and*

$$\int_{\mathbb{R}^n} dx\; e^{-NF(x)} = \int_{\mathbb{R}^n} dx\;\; \det(\partial\gamma(x))\; e^{-NF(\gamma(x))},$$
$$\int_{\mathbb{R}^n} dx\; g(x) e^{-NF(x)} = \int_{\mathbb{R}^n} dx\;\; \det(\partial\gamma(x))\; g(\gamma(x)) e^{-NF(\gamma(x))}.$$

(b) The function $x \mapsto H(x) :=$ Re $F(\gamma(x))$ admits $q \geq 1$ global minimum points $x_1, \ldots, x_q$ with $H''(x_j) \in \mathbb{R}^{n\times n}$ strictly positive $\forall j = 1, \ldots, q$ and

$$\inf_{\substack{x \text{ local minimum}\\ x\neq x_1,\ldots,x_q}} [H(x) - H_m] > 0, \qquad \text{where} \qquad H_m = H(x_j)\; \forall j = 1, \ldots, q.$$

(c) The point $z_j := \gamma(x_j)$ is a critical point of f (i.e. $\partial f(z_j) \in \mathbb{C}^n$ vanishes) $\forall j = 1, \ldots, q$.

Then as $N \to \infty$ we have

(i) $\frac{N^{\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} dx\; e^{-NF(x)} = e^{-NH_m}\left[\sum_{j=1}^q e^{-iN\mathrm{Im}F(z_j)} \frac{\det\partial\gamma(x_j)}{\sqrt{\det(f\circ\gamma)''(x_j)}} + O\left(\frac{1}{N}\right)\right],$

(ii) $\frac{N^{\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} dx\; g(x) e^{-NF(x)} = e^{-NH_m}\left[\sum_{j=1}^q e^{-iN\mathrm{Im}F(z_j)} \frac{g(z_j)\det\partial\gamma(x_j)}{\sqrt{\det(f\circ\gamma)''(x_j)}} + O\left(\frac{1}{N}\right),\right]$

where the square denotes the principal root.

Proof. Exercise (works as in the scalar case)

□

Application 1: averaged DOS for GUE. Recall the definition of the averaged density of states (DOS) in (4.7) and its equivalent formulation (4.8). Using the dual representation from Theorem 4.8 we can write

$$\rho_N(E) = -\frac{1}{\pi}\lim_{\varepsilon\to 0+} \mathrm{Im}\; I_N(E + i\varepsilon),$$

where we defined

$$I_N(E + i\varepsilon) := \frac{N}{2\pi}\int da\, db\; e^{-\frac{N}{2}(a^2+b^2)} \frac{(E + i\varepsilon - ib)^N}{(E + i\varepsilon - a)^N}\left[1 - \tfrac{1}{(E+i\varepsilon-a)(E+i\varepsilon-ib)}\right]\; a, \tag{4.13}$$

Our goal is to study this integral as $N \to \infty$. Note that

$$\overline{I_N(E + i\varepsilon)} = I_N(E - i\varepsilon) = I_N(-E + i\varepsilon),$$

and therefore it is sufficient to consider the case $E \geq 0$.

[19: 16.12.2024]

[20: 20.12.2024]

We introduce, for $k=0,1$, the two integrals

$$I_{N,a}^{(k)}(E+i\varepsilon) := \frac{\sqrt{N}}{\sqrt{2\pi}} \int da\ e^{-\frac{N}{2}a^2} \frac{1}{(E+i\varepsilon-a)^{N+k}} a$$
$$I_{N,b}^{(k)}(E+i\varepsilon) := \frac{\sqrt{N}}{\sqrt{2\pi}} \int db\ e^{-\frac{N}{2}b^2} (E+i\varepsilon-ib)^{N-k}.$$

With this notation, we can reformulate $I_N(E+i\varepsilon)$ as

$$I_N(E+i\varepsilon) = I_{N,a}^{(0)}(E+i\varepsilon) I_{N,b}^{(0)}(E+i\varepsilon) - I_{N,a}^{(1)}(E+i\varepsilon) I_{N,b}^{(1)}(E+i\varepsilon).$$

We define

$$I_{N,a}^{(k)} = I_{N,a}^{(k)}(E) := \lim_{\varepsilon\to 0+} I_{N,a}^{(k)}(E+i\varepsilon), \qquad I_{N,b}^{(k)} = I_{N,b}^{(k)}(E) = \lim_{\varepsilon\to 0+} I_{N,b}^{(k)}(E+i\varepsilon).$$

The asymptotics of $I_{N,b}^{(0)}$ was studied in Theorem 3.8 (for $E>0$) and equation (3.10) (for $E=0$). In the last case a direct computation also gives

$$I_{N,b}^{(k)}(0) = (-i)^{N-k} \tfrac{\sqrt{N}}{\sqrt{2\pi}} \int_{\mathbb{R}} db\ e^{-N\frac{b^2}{2}} b^{N-k} = \begin{cases} 0 & \text{if } N-k \text{ odd} \\ (N-k-1)!!\ N^{-\frac{N-k}{2}} & \text{if } N-k \text{ even,} \end{cases}.$$

For $E\in(0,\infty)$, with $E\neq 2$, using the same arguments as in Theorem 3.8, we obtain the following.

- For $E>2$ we define $E_\pm := \frac{E}{2} \pm \sqrt{\frac{E^2}{4}-1}$. We have

  $$0<E_-<1<E_+<E, \qquad E-E_+ = E_-, \qquad E_+E_- = 1,$$

  and, as $N\to\infty$,

  $$I_{N,b}^{(k)}(E) = e^{\frac{N}{2}E_-^2} \frac{E_+^{N-k}}{\sqrt{1-E_-^2}} \left[1+O\left(\tfrac{1}{N}\right)\right]. \tag{4.14}$$

- For $0<E<2$ we define $\mathcal{E}_\pm := \frac{E}{2} \pm i\sqrt{1-\frac{E^2}{4}}$. We have

  $$|\mathcal{E}_\pm| = 1, \qquad \mathcal{E}_- = \overline{\mathcal{E}_+}, \qquad E-\mathcal{E}_+ = \mathcal{E}_-, \qquad \mathcal{E}_+\mathcal{E}_- = 1,$$

  and, as $N\to\infty$,

  $$I_{N,b}^{(k)}(E) = e^{\frac{N}{2}\mathcal{E}_-^2} \frac{\mathcal{E}_+^{N-k}}{\sqrt{1-\mathcal{E}_-^2}} + e^{\frac{N}{2}\mathcal{E}_+^2} \frac{\mathcal{E}_-^{N-k}}{\sqrt{1-\mathcal{E}_+^2}} + e^{N\mathrm{Re}\mathcal{E}_-^2} O\left(\tfrac{1}{N}\right). \tag{4.15}$$

The next result gives the asymptotic behavior of $I_{N,a}^{(k)}(E)$ for $E\geq 0$, $E\neq 2$.

Theorem 4.11.

(i) For $E\in\mathbb{R}$ we consider two complex paths:

$$\gamma_R := (-\infty, E-R) \cup C_R \cup (E+R,\infty), \qquad C_R := \{E-Re^{-i\theta}\}_{\theta\in[0,\pi]}, R>0 \tag{4.16}$$
$$\mathbb{R}-ic := \{x-ic\}_{x\in\mathbb{R}},\ c>0. \tag{4.17}$$

Then, for all $R>0$ and $c>0$ we have

$$\lim_{\varepsilon\to 0+} I_{N,a}^{(k)} = \frac{\sqrt{N}}{\sqrt{2\pi}} \int_{\gamma_R} da\ e^{-\frac{N}{2}a^2} \frac{1}{(E-a)^{N+k}} a$$
$$= \frac{\sqrt{N}}{\sqrt{2\pi}} \int_{\mathbb{R}-ic} da\ e^{-\frac{N}{2}a^2} \frac{1}{(E-a)^{N+k}} a.$$

(ii) For $E > 2$ we have, as $N \to \infty$,

$$I^{(k)}_{N,a}(E) = e^{-\frac{N}{2}E_-^2} E_- \frac{E_-^{N+k}}{\sqrt{1-E_-^2}} \left[1 + O\left(\tfrac{1}{N}\right)\right]. \tag{4.18}$$

(iii) For $0 < E < 2$ we have, as $N \to \infty$,

$$I^{(k)}_{N,a}(E) = e^{-\frac{N}{2}\mathcal{E}_-^2} \mathcal{E}_- \frac{\mathcal{E}_-^{N+k}}{\sqrt{1-\mathcal{E}_-^2}} + e^{N \mathrm{Re}\mathcal{E}_-^2} O\left(\tfrac{1}{N}\right). \tag{4.19}$$

Proof.
(i) exercise: use the fact that the function $z \mapsto e^{-\frac{N}{2}z^2}(E+i\varepsilon-z)^{-N-k}$ is analytic on $\mathbb{C}\setminus\{E+i\varepsilon\}$.

(ii) Fix $E > 2$. We study the function

$$f(a) := \frac{a^2}{2} + \ln(E-a).$$

We compute

$$f'(a) = a - \frac{1}{E-a} = \frac{a^2 - Ea + 1}{a - E}, \qquad f''(a) = 1 - \frac{1}{(E-a)^2}.$$

The critical points are real: $a = E_\pm$. If we consider f as a function on $\mathbb{R}$, f has a local minimum in E_-, a local maximum in E_+ and converges to $-\infty$ as $a \to E$. To ensure our integration path reaches a critical point we use the contour γ_R. Our goal is to choose R such that $\mathrm{Re} f$ along the path γ_R has a unique global minimum in $a = E_-$. Therefore we study

$$H(\theta) := \mathrm{Re}\ f(E - Re^{-i\theta}) = \ln R + \frac{E^2 - 2RE\cos\theta + R^2\cos 2\theta}{2}.$$

We compute

$$H'(\theta) = R\sin\theta\,(E - 2R\cos\theta).$$

Since $R\sin\theta > 0$ for all $\theta \in (0,\pi)$ we only need to study $E - 2R\cos\theta$. This function is strictly positive on $(0,\pi)$ if $E - 2R > 0$, i.e. $R < \frac{E}{2}$. In this case we obtain $H'(\theta) > 0$ on $(0,\pi)$ and hence $H(\theta) \geq H(0) = \mathrm{Re}\, f(E-R)$.
Since $E_- < \frac{E}{2}$ we can set $R := E_-$. With this choice $H(\theta) \geq f(E_+)$ which is a local max for f on $\mathbb{R}$, and hence $\mathrm{Re}\, f$ along γ_{E_-} admits a unique global minimum in E_-. Moreover $f''(E_-) = 1 - E_-^2 > 0$. The result now follows from Theorem 4.10.

(iii) Fix $0 \leq E < 2$. The critical points are complex: $a = \mathcal{E}_\pm$. Only $\mathcal{E}_-$ lies in $\mathbb{C}_-$. In this case it is more convenient to use the contour $\mathbb{R} - ic$ with $c := \sqrt{1 - \frac{E^2}{4}}$. Therefore we study

$$H(a) := \mathrm{Re}\ f(a - i\sqrt{1 - \frac{E^2}{4}}).$$

For $E^2 < \frac{32}{9}$ this function admits a unique global minimum in $a = \frac{E}{2}$ and $H''(\frac{E}{2}) = \mathrm{Re}\ f''(\mathcal{E}_-) = \mathrm{Re}\ (1 - \mathcal{E}_-^2) > 0$ (exercise). In this case the result follows from Theorem 4.10.

For $\frac{32}{9} < E^2 < 2$ the point $\frac{E}{2}$ becomes a local minimum while the global minimum moves to $a = E$. This happens because we are approaching the real axis and hence the pole. To solve the problem we can perform a contour rotation $a - i\sqrt{1-\frac{E^2}{4}} \to ae^{-i\theta} - i\sqrt{1-\frac{E^2}{4}}$ (for details see Disertori: *Density of states for GUE through supersymmetric approach*).

□

Corollary 4.12.

(i) $I_N(E) := \lim_{\varepsilon\to 0+} I_N(E+i\varepsilon)$ *is well defined.*

(ii) For $E > 2$ *we have, as* $N \to \infty$,

$$I_N(E) = E_- + O\left(\frac{1}{N}\right),$$

(iii) For $0 \le E < 2$ *we have, as* $N \to \infty$,

$$I_N(E) = \mathcal{E}_- + O\left(\frac{1}{N}\right).$$

Proof. To prove (i) we multiply the corresponding formulas for $I^{(k)}_{N,a}$ and $I^{(k)}_{N,b}$.

(ii) We compute

$$I^{(k)}_{N,a}(E)I^{(k)}_{N,b}(E) = E_- \tfrac{E_+^{N-k}E_-^{N+k}}{(1-E_-^2)} + O\left(\tfrac{1}{N}\right) = E_- \tfrac{E_-^{2k}}{(1-E_-^2)} + O\left(\tfrac{1}{N}\right).$$

Hence

$$I_N(E) = I^{(0)}_{N,a}(E)I^{(0)}_{N,b}(E) - I^{(1)}_{N,a}(E)I^{(1)}_{N,b}(E) = E_- \tfrac{1-E_-^2}{1-E_-^2} + O\left(\tfrac{1}{N}\right) = E_- + O\left(\tfrac{1}{N}\right).$$

(iii) We compute

$$\begin{aligned} I^{(k)}_{N,a}(E)I^{(k)}_{N,b}(E) &= \tfrac{\mathcal{E}_-}{\sqrt{1-\mathcal{E}_-^2}} e^{-\frac{N}{2}\mathcal{E}_-^2}\mathcal{E}_-^{N+k}\left[e^{\frac{N}{2}\mathcal{E}_-^2}\tfrac{\mathcal{E}_+^{N-k}}{\sqrt{1-\mathcal{E}_-^2}} + e^{\frac{N}{2}\mathcal{E}_+^2}\tfrac{\mathcal{E}_-^{N-k}}{\sqrt{1-\mathcal{E}_+^2}}\right] + O\left(\tfrac{1}{N}\right) \\ &= \mathcal{E}_-\left[\tfrac{\mathcal{E}_-^{2k}}{1-\mathcal{E}_-^2} + e^{\frac{N}{2}(\mathcal{E}_+^2-\mathcal{E}_-^2)}\tfrac{\mathcal{E}_-^{2N}}{\sqrt{1-\mathcal{E}_+^2}\sqrt{1-\mathcal{E}_-^2}}\right] + O\left(\tfrac{1}{N}\right). \end{aligned}$$

Hence

$$I_N(E) = I^{(0)}_{N,a}(E)I^{(0)}_{N,b}(E) - I^{(1)}_{N,a}(E)I^{(1)}_{N,b}(E) = \mathcal{E}_- \tfrac{1-\mathcal{E}_-^2}{1-\mathcal{E}_-^2} + O\left(\tfrac{1}{N}\right) = \mathcal{E}_- + O\left(\tfrac{1}{N}\right).$$

□

Remark 1

$(iii) \Rightarrow \rho_N(E) = \frac{1}{\pi}\sqrt{1-\frac{E^2}{4}} + O\left(\frac{1}{N}\right)$ for $E \in (-2,2)$.
The limit is called *semi-circle law.*

$(ii) \Rightarrow \rho_N(E) = 0 + O\left(\frac{1}{N}\right)$ for $|E| > 2$. More precise extimates show

$$\rho_N(E) \propto e^{-Ng(E)},$$

where the function $g(E)$ is known explicitely.

Remark 2 $\lim_{E\to 2-}\sqrt{1-\frac{E^2}{4}}=0$. Setting $E=2-\frac{t}{N^{2\beta}}$ with $t>0$ $\beta>0$, we get

$$\sqrt{1-\frac{E^2}{4}}=O\left(\frac{1}{N^{\beta}}\right).$$

A more careful asymptotic analysis of the integral gives, for $0<\beta<\frac{1}{3}$,

$$\rho_N\left(2-\frac{t}{N^{2\beta}}\right)=\frac{1}{\pi}\sqrt{1-\frac{E^2}{4}}+O\left(\frac{1}{N^{1-2\beta}}\right).$$

For $0<\beta<\frac{1}{3}$ the first term is still dominant. For $\beta\geq\frac{1}{3}$ the first term is as small as the correction so we have to change the formulation.
Setting $E=2-\frac{t}{N^{\frac{2}{3}}}$ with $t\in\mathbb{R}$ one can prove (see Disertori: *Density of states for GUE through supersymmetric approach*).

$$\rho_N\left(2-\frac{t}{N^{\frac{2}{3}}}\right)=O\left(\frac{1}{N^{\frac{1}{3}}}\right)=\frac{1}{N^{\frac{1}{3}}}F(t)+O\left(\frac{1}{N^{\frac{2}{3}}}\right),$$

where $F(t)$ can be explicitely written in terms of Airy functions and $F(0)\neq 0$.

Application 2: averaged DOS for band matrices. Recall the definition of random band matrix $H\in\mathbb{C}^{\Lambda\times\Lambda}_{herm}$, with $\Lambda=[-L,L]^d\cap\mathbb{Z}^d$, in Section 4.3, and consider the covariance

$$J_{jk}:=(-W^2\Delta+1)^{-1}_{jk}$$

where $-\Delta$ is the lattice Laplacian with periodic boundary conditions at the boundary of Λ. This matrix satisfies $J_{jk}>0$ $\forall j,k$ and $J>0$ as a quadratic form. Moreover

$$0<J_{jk}=f(|j-k|)\leq P_W(|j-k|)e^{-\frac{|j-k|}{W}},$$

where $P_W(|j-k|)$ is a prefactor (with at most polynomial growth) depending on the dimension. For example in $d=3$ $P_W(|j-k|)\propto\frac{1}{W^2(1+|j-k|)}$.
We want to study

$$\rho_\Lambda(E):=-\frac{1}{\pi}\lim_{\varepsilon\to 0+}\frac{1}{|\Lambda|}\mathrm{Im}\,\mathbb{E}\left[\mathrm{tr}\,(E+i\varepsilon-H)^{-1}\right]$$

where, using Theorem 4.9,

$$\begin{aligned}
&\frac{1}{|\Lambda|}\mathbb{E}\left[\mathrm{tr}\,(E+i\varepsilon-H)^{-1}\right]\\
&=\int_{\mathbb{R}^{2\Lambda}}\left(\frac{da\,db}{2\pi}\right)^{\Lambda}e^{-\frac{1}{2}((a,J^{-1}a)+(b,J^{-1}b))}\frac{\sum_{j_0}(J^{-1}a)_{j_0}}{|\Lambda|}\left[\prod_j\frac{(E+i\varepsilon-ib_j)}{(E+i\varepsilon-a_j)}\right]\det\left(J^{-1}-D\right)\\
&=\frac{1}{|\Lambda|}\sum_{j_0}\int_{\mathbb{R}^{2\Lambda}}\left(\frac{da\,db}{2\pi}\right)^{\Lambda}e^{-\frac{1}{2}((a,J^{-1}a)+(b,J^{-1}b))}a_{j_0}\left[\prod_j\frac{(E+i\varepsilon-ib_j)}{(E+i\varepsilon-a_j)}\right]\det\left(J^{-1}-D\right)\\
&=\int_{\mathbb{R}^{2\Lambda}}\left(\frac{da\,db}{2\pi}\right)^{\Lambda}e^{-\frac{1}{2}((a,J^{-1}a)+(b,J^{-1}b))}a_0\left[\prod_j\frac{(E+i\varepsilon-ib_j)}{(E+i\varepsilon-a_j)}\right]\det\left(J^{-1}-D\right),
\end{aligned}$$

where

$$D = \operatorname{diag}\{D_j\}_{j\in\Lambda}, \qquad D_j := 1 - \frac{1}{(E+i\varepsilon-a_j)(E+i\varepsilon-ib_j)},$$

in the third line we used

$$\sum_{j_0}(J^{-1}a)_{j_0} = W^2(1,-\Delta a) + \sum_{j_0} a_{j_0}$$

together with $-\Delta\, 1 = 0$ and in the last line we used translation invariance. Note that

$$e^{-\frac{1}{2}(a,J^{-1}a)} = e^{-\frac{W^2}{2}\sum_{|j-k|=1}(a_j-a_k)^2} e^{-\frac{1}{2}\sum_j a_j^2}.$$

Hence, if we assume $W \gg 1$, this function is very small whenever $|a_j - a_k| > \frac{1}{\sqrt{W}}$ for some nearest neighbor pair j,k. Therefore we expect the integration measure to concentrate near the configuration $a_j = a$ and $b_j = b$ $\forall j$. Fixing Λ we compute

$$\begin{aligned}
&\lim_{W\to\infty}\int_{\mathbb{R}^{2\Lambda}}\left(\frac{da\,db}{2\pi}\right)^{\Lambda} e^{-\frac{1}{2}((a,J^{-1}a)+(b,J^{-1}b))} a_0 \left[\prod_j \tfrac{(E+i\varepsilon-ib_j)}{(E+i\varepsilon-a_j)}\right]\det\left(J^{-1}-D\right)\\
&= \int_{\mathbb{R}}\frac{da\,db}{2\pi} e^{-\frac{|\Lambda|}{2}(a^2+b^2)} a \left(\tfrac{(E+i\varepsilon-ib)}{(E+i\varepsilon-a)}\right)^{|\Lambda|}\left(1-\frac{1}{(E+i\varepsilon-a)(E+i\varepsilon-ib)}\right).
\end{aligned}$$

The last integral is the dual representation we obtained in the GUE case. We studied the corresponding asymptotic as $N = |\Lambda| \to \infty$ in the previous section. Our problem is to prove a similar result fixing W (it may depend on N) and taking the limit $N = |\Lambda| \to \infty$ first. This is done via a combination of cluster expansion (to reduce the main computation to a finite volume Λ_0 independent of Λ) plus saddle analysis in the finite volume Λ_0.

[20: 20.12.2024]

[21: 07.01.2025]

5 Supersymmetry

5.1 Linear transformations on complex supervectors

Remember that $C_0(\mathbb{R};\mathbb{C})$ is the set of continuous functions vanishing at infinity.

Lemma 5.1. *Let $f \in C^1(\mathbb{R};\mathbb{C}) \cap C_0(\mathbb{R};\mathbb{C})$ with $f, f' \in L^1(\mathbb{R})$ and Let $\psi, \overline{\psi}$ be Grassmann variables. Then*

$$\int \frac{d\overline{\varphi}d\varphi}{2\pi} d\overline{\psi}d\psi\ f(\overline{\varphi}\varphi + \overline{\psi}\psi) = f(0).$$

Proof. We prove first that the integral above is well defined. Since $(\overline{\psi}\psi)^2 = 0$ we have

$$f(\overline{\varphi}\varphi + \overline{\psi}\psi) = f(\overline{\varphi}\varphi) + f'(\overline{\varphi}\varphi)\overline{\psi}\psi. \tag{5.1}$$

Therefore we only need to check that

$$\int_{\mathbb{C}} d\overline{\varphi}d\varphi |f^{(k)}(\overline{\varphi}\varphi)| < \infty \qquad k = 0,1.$$

We argue, using $d\overline{\varphi}d\varphi = 2dx\,dy$ with $\varphi = x + iy$,

$$\int_{\mathbb{C}} d\overline{\varphi}d\varphi |f^{(k)}(\overline{\varphi}\varphi)| = 2\pi\int_0^\infty 2r dr\ |f^{(k)}(r^2)| = 2\pi\int_0^\infty du\ |f^{(k)}(u)| < \infty.$$

To prove the lemma we argue

$$\int \frac{d\overline{\varphi}d\varphi}{2\pi} d\overline{\psi}d\psi \; f(\overline{\varphi}\varphi + \overline{\psi}\psi) = \int_{\mathbb{C}} \frac{d\overline{\varphi}d\varphi}{2\pi} \partial_{\overline{\psi}}\partial_{\psi} \left[f(\overline{\varphi}\varphi) + f'(\overline{\varphi}\varphi)\overline{\psi}\psi\right]$$
$$= -\int_{\mathbb{C}} \frac{d\overline{\varphi}d\varphi}{2\pi} f'(\overline{\varphi}\varphi) = -\int_0^\infty 2rdr \; f'(r^2) = f(0).$$

□

Remark. The following two examples show that the argument above works only if we have *perfect grading*, i.e. the same number of bosonic and fermionic variables.

Example 1: 4 fermionic and 2 bosonics variables. Assume $f \in C^2$ $f, f', f'' \in L^1(\mathbb{R})$ and $f' \in C_0(\mathbb{R};\mathbb{C})$ Let $\psi_1, \overline{\psi}_1, \psi_2, \overline{\psi}_2$ be four Grassmann variables. We compute

$$f(\overline{\varphi}\varphi + \overline{\psi}_1\psi_1 + \overline{\psi}_2\psi_2) = f(\overline{\varphi}\varphi) + f'(\overline{\varphi}\varphi)\;(\overline{\psi}_1\psi_1 + \overline{\psi}_2\psi_2) + f''(\overline{\varphi}\varphi)\;\overline{\psi}_1\psi_1\,\overline{\psi}_2\psi_2.$$

Hence

$$\int_{\mathbb{C}} \frac{d\overline{\varphi}d\varphi}{2\pi} \partial_{\overline{\psi}_1}\partial_{\psi_1}\partial_{\overline{\psi}_2}\partial_{\psi_2} f(\overline{\varphi}\varphi + \overline{\psi}_1\psi_1 + \overline{\psi}_2\psi_2) = \int \frac{d\overline{\varphi}d\varphi}{2\pi} f''(\overline{\varphi}\varphi) = f'(0) \neq f(0).$$

Example 2: 2 fermionic and 4 bosonic variables. Assume $f(x), xf'(x) \in L^1$ and $xf \in C_0(\mathbb{R};\mathbb{C})$. We compute

$$\int_{\mathbb{C}^2} \frac{d\overline{\varphi_1}d\varphi_1 d\overline{\varphi_2}d\varphi_2}{(2\pi)^2} \partial_{\overline{\psi}}\partial_{\psi} f(\sum_{j=1}^{2} \overline{\varphi}_j\varphi_j + \overline{\psi}\psi) = -\int_{\mathbb{C}^2} \frac{d\overline{\varphi_1}d\varphi_1 d\overline{\varphi_2}d\varphi_2}{(2\pi)^2} f'(\overline{\varphi_1}\varphi_1 + \overline{\varphi_2}\varphi_2)$$
$$= -\frac{4}{(2\pi)^2}|\mathcal{S}^3| \int_0^\infty dr\; r^3 f'(r^2)) = -\frac{2}{(2\pi)^2}|\mathcal{S}^3| \int_0^\infty du\; uf'(u) = \frac{2}{(2\pi)^2}|\mathcal{S}^3| \int_0^\infty du\; f(u) \neq f(0).$$

One-parameter groups of transformations Remember that a linear transformation on $\Phi \in \mathcal{G}^{m|n}$ is represented by a supermatrix. For a given $X \in \mathcal{A}^{(m|n)\times(m|n)}$ we consider the function

$$\begin{array}{llll} \mathcal{R} & : \mathbb{R} & \to \mathcal{A}^{(m|n)\times(m|n)} \\ & t & \mapsto \mathcal{R}(t) := e^{tX}. \end{array}$$

This function defines an abelian group. Moreove $\mathcal{R}$ is smooth with $\mathcal{R}'(0) = X$.

Definition 5.2. *Let $X \in \mathcal{A}^{(m|n)\times(m|n)}$ be a given supermatrix.*

(i) $X = \mathcal{R}'(0)$ *is called the* (infinitesimal) generator *of the group $t \mapsto \mathcal{R}(t)$.*

(ii) The infinitesimal generator $X = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix}$ is called

(a) even if $\sigma = 0 = \rho$,

(b) odd if $a = 0 = b$.

(iii) *Let* $\varphi_1, \dots, \varphi_m \in \mathbb{C}$, $\psi_1, \overline{\psi}_1, \dots, \psi_n, \overline{\psi}_n$ *a family of generators for* $\mathcal{G}$ *and remember*

$$\Phi = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \qquad \overline{\Phi} = \begin{pmatrix} \overline{\varphi} \\ \overline{\psi} \end{pmatrix}, \qquad \overline{\Phi}^* = (\overline{\varphi},\ \overline{\psi}.).$$

The group $\mathcal{R}$ *acts on* Φ *via*

$$\Phi(t) := e^{tX}\Phi, \qquad \Phi^*(t) := \Phi^* e^{tX^*}.$$

Lemma 5.3. *Consider the group* $t \mapsto \mathcal{R}(t) = e^{tX}$

(i) *We have*

$$\frac{d}{dt}\Phi(t)_{|t=0} = X\Phi =: d\Phi = \begin{pmatrix} d\varphi \\ d\psi \end{pmatrix}, \qquad \frac{d}{dt}\overline{\Phi}(t)_{|t=0} = \overline{X}\,\overline{\Phi} =: d\overline{\Phi} = \begin{pmatrix} d\overline{\varphi} \\ d\overline{\psi} \end{pmatrix}.$$

(ii) *For any function* $F(\Phi, \overline{\Phi})$ *regular enough we have*

$$\frac{d}{dt}F(\Phi(t), \overline{\Phi}(t))_{|t=0} = \sum_{j=1}^{m}\left(d\varphi_j\partial_{\varphi_j} + d\overline{\varphi}_j\partial_{\overline{\varphi}_j}\right)F + \sum_{l=1}^{n}\left(d\psi_l\partial_{\psi_l} + d\overline{\psi}_l\partial_{\overline{\psi}_l}\right)F.$$

(iii) *Assume now* X *is odd.*

(a) $\mathcal{R}(t)$ *leaves the function* $\Phi^*\Phi$ *invariant* $\Leftrightarrow \mathcal{R}(t)$ *is unitary i.e.* $\mathcal{R}(t)^*\mathcal{R}(t) = 1\ \forall t \in \mathbb{R}$ $\Leftrightarrow X^* = -X \Leftrightarrow$

$$X = \begin{pmatrix} 0 & \alpha \\ \alpha^* & 0 \end{pmatrix}$$

(b) *Assume* $\mathcal{R}(t) = e^{tX}$ *with* X *as above. Then*

$$\frac{d}{dt}F(\Phi(t), \overline{\Phi}(t))_{|t=0} = \sum_{j=1}^{m}\sum_{l=1}^{n}(\alpha_{jl}D_{jl} + \overline{\alpha}_{jl}\overline{D}_{jl})\ F,$$

where we defined

$$D_{jl} := \psi_l\partial_{\varphi_j} - \overline{\varphi}_j\partial_{\overline{\psi}_l}, \qquad \overline{D}_{jl} := \overline{\psi}_l\partial_{\overline{\varphi}_j} + \varphi_j\partial_{\psi_l}. \tag{5.2}$$

Proof.
$(i)(ii)$ exercise

$(iii)(a)$ We argue $\Phi^*(t)\Phi(t) = \Phi^*\Phi\ \forall t \Leftrightarrow X^* = -X$. Since

$$\begin{pmatrix} 0 & \sigma \\ \rho & 0 \end{pmatrix}^t = \begin{pmatrix} 0 & \rho^* \\ -\sigma^* & 0 \end{pmatrix},$$

we obtain $\sigma = -\rho^*$ and $\rho = \sigma^*$. The first condition follows from the second. The result follows setting $\sigma = \alpha$.

$(iii)(b)$ The result follows from

$$d\varphi_j = (\alpha\psi)_j = \sum_{l=1}^{n}\alpha_{jl}\psi_l, \qquad d\overline{\varphi}_j = (\overline{\alpha}\overline{\psi})_j = \sum_{l=1}^{n}\overline{\alpha}_{jl}\overline{\psi}_l$$

$$d\psi_l = (\alpha^*\varphi)_l = \sum_{j=1}^{m}\overline{\alpha}_{jl}\ \varphi_j, \qquad d\overline{\psi}_l = (-\alpha^t\overline{\varphi})_l = -\sum_{j=1}^{m}\alpha_{jl}\overline{\varphi}_j$$

□

In the following we concentrate on the case $m = n$ (perfect grading) and $\rho_{jl} = \delta_{jl}\rho_j$, hence

$$D_j := D_{jj} = \psi_j \partial_{\varphi_j} - \overline{\varphi}_j \partial_{\overline{\psi}_j}, \qquad \overline{D}_j := \overline{D}_{jl} := \overline{\psi}_j \partial_{\overline{\varphi}_j} + \varphi_j \partial_{\psi_j}. \tag{5.3}$$

In this case we can reorganize the supervector as $\Phi = \{\Phi_j\}_{j=1,\dots,n}$ with

$$\Phi_j = \begin{pmatrix} \varphi_j \\ \psi_j \end{pmatrix}, \qquad \overline{\Phi}_j = \begin{pmatrix} \overline{\varphi}_j \\ \overline{\psi}_j \end{pmatrix}, \qquad \overline{\Phi}_j^* = (\overline{\varphi}_j,\ \overline{\psi}_j),$$

and

$$(\mathcal{R}(t)\Phi)_j = R_j(t)\Phi_j, \qquad R_j(t) = e^{tX_j}, \quad X_j = \begin{pmatrix} 0 & \alpha_j \\ \overline{\alpha}_j & 0 \end{pmatrix} \in \mathcal{A}^{(1|1)\times(1|1)}.$$

Definition 5.4. *Consider the unitary rotation $\mathcal{R}(t) = \{R_j(t)\}_{j=1}^n$ introduced above.*

(i) $\mathcal{R}(t)$ is called a local rotation. The action of the infinitesimal transformation on a function is given by

$$\frac{d}{dt} F(\Phi(t), \overline{\Phi}(t))_{|t=0} = \sum_{j=1}^m \sum_{l=1}^n (\alpha_j D_j + \overline{\alpha}_j \overline{D}_j) F,$$

(ii) If $R_j = R = e^{tX}$ with $X = \begin{pmatrix} 0 & \alpha \\ \overline{\alpha} & 0 \end{pmatrix}$ for all j, we call $\mathcal{R}(t)$ a global rotation. Then the action of the infinitesimal transformation on a function is given by

$$\frac{d}{dt} F(\Phi(t), \overline{\Phi}(t))_{|t=0} = \alpha DF + \overline{\alpha}\overline{D}F, \qquad D := \sum_{j=1}^n D_j, \quad \overline{D} := \sum_j \overline{D}_j.$$

Proposition 5.5 (properties of D_j)**.**

(i) Consider $\Phi_j(t) := e^{tX_j}$ with X_j defined above. We have $\Phi_j(t)^*\Phi_j(t) = \Phi_j^*\Phi_j$ $\forall t$ and $D_j\, \Phi_j^*\Phi_j = 0 = \overline{D}_j\, \Phi_j^*\Phi_j$.

(ii) Consider a global rotation $\Phi_j(t) := e^{tX}$ with X defined above. We have $\Phi_j^*(t)\Phi_k)(t) = \Phi_j^*\Phi_k$ $\forall t$, $\forall j, k = 1, \dots n$ and $D\, \Phi_j^*\Phi_k = 0 = \overline{D}\, \Phi_j^*\Phi_k$.

(iii) Let $F(\Phi, \overline{\Phi})$ be a function

$$F(\Phi, \overline{\Phi}) = \sum_{IJ \subset \{1,\dots,n\}} v_{IJ}(\overline{\varphi}, \varphi)\psi^I \overline{\psi}^J = \sum_{IJ \subset \{1,\dots,n\}} g_{IJ}(x, y)\psi^I \overline{\psi}^J,$$

where $\varphi_j = x_j + iy_j$. Assume $g_{IJ} \in C^1(\mathbb{R}^{2n}) \cap C_0(\mathbb{R}^{2n})$ and $\partial_x g_{IJ}, \partial_y g_{IJ} \in L^1(\mathbb{R}^{2n})$. Then

$$\int d\Phi_j^* d\Phi_j\ D_j F = 0 \qquad \forall j = 1, \dots, n.$$

Proof.
$(i)(ii)$ exercise

(iii) We have

$$\int d\Phi_j^* d\Phi_j\ D_j F = \int_{\mathbb{C}} \frac{d\overline{\varphi} d\varphi}{2\pi} \partial_{\overline{\psi}_j} \partial_{\psi_j}\ D_j F.$$

We compute

$$\partial_{\overline{\psi}_j}\partial_{\psi_j}\ D_jF = \partial_{\overline{\psi}_j}\partial_{\psi_j}\ (\psi_j\partial_{\varphi_j} - \overline{\varphi}_j\partial_{\overline{\psi}_j})F = \partial_{\varphi_j}\partial_{\overline{\psi}_j}\partial_{\psi_j}\ (\psi_jF).$$

The result now follows from

$$\int_{\mathbb{C}} d\overline{\varphi}d\varphi\partial_{\varphi_j}v_{IJ}(\varphi,\overline{\varphi}) = \int_{\mathbb{R}^2} 2dx_jdy_j\frac{1}{2}\left(\partial_{x_j} - \partial_{y_j}\right)v_{IJ}(\varphi(x,y),\overline{\varphi}(x,y)) = 0.$$

□

Remark. In (iii) we need g_{IJ} differentiable to ensure D_jF is well defined, $\partial_x g_{IJ}, \partial_y g_{IJ} \in L^1(\mathbb{R}^{2n})$ to ensure $\partial_{\varphi_j}\partial_{\overline{\psi}_j}D_jF \in L^1(\mathbb{R}^{2n})$ and $g_{IJ} \in C_0(\mathbb{R}^{2n})$ to ensure that $\int d\Phi_j^*d\Phi_j\ D_jF = 0$. Note that we do not really need $\partial_x g_{IJ}, \partial_y g_{IJ}$ to be continuous.

[21: 07.01.2025]

[22: 10.01.2025]

Theorem 5.6 (localization theorem I)**.** *Let $F(\Phi,\overline{\Phi})$ be a function*

$$F(\Phi,\overline{\Phi}) = \sum_{IJ\subset\{1,\dots,n\}} v_{IJ}(\overline{\varphi},\varphi)\psi^I\overline{\psi}^J = \sum_{IJ\subset\{1,\dots,n\}} g_{IJ}(x,y)\psi^I\overline{\psi}^J,$$

where $\varphi_j = x_j + iy_j$. Assume

(a) $g_{IJ} \in C^1(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n}) \cap L^1(\mathbb{R}^{2n})$, $|\varphi|^2 g_{IJ} \in L^1(\mathbb{R}^{2n})$, $|\varphi|g_{IJ} \in C_0(\mathbb{R}^{2n})$ and $\partial(\varphi g_{IJ}) \in L^1(\mathbb{R}^{2n})$,

(b) $DF = 0$.

Then

$$\int (d\Phi^*d\Phi)^n\ F = F(0), \tag{5.4}$$

where we defined

$$(d\Phi^*d\Phi)^n := \prod_{j=1}^n \frac{d\overline{\varphi}_jd\varphi_j}{2\pi}d\overline{\psi}_jd\psi_j.$$

Example Consider a function $f\colon \mathbb{C}^{n^2} \to \mathbb{C}$ such that

- f is analytic in each variable z_{ij} separately.
- The function $h(x,y) := \prod_{ij}\partial_{z_{ij}}^{n_{ij}} f(z)_{z_{ij}=\overline{\varphi}_i\varphi_j}$ with $\varphi = x + iy$, satisfies the assumptions (a) of the theorem above.

Then $DF\Big(\{\Phi_i^*\Phi_j\}_{ij=1}^n\Big) = 0$ and

$$\int (d\Phi^*d\Phi)^n\ f\Big(\{\Phi_i^*\Phi_j\}_{ij=1}^n\Big) = f(0).$$

Proof of Theorem 5.6. We define, for $t \geq 0$,

$$I(t) := \int (d\Phi^*d\Phi)^n\ e^{-t\Phi^*\Phi}F(\Phi,\overline{\Phi}).$$

For $t = 0$ we recover the integral we want to study.

• We show that $I'(t) = 0$. We compute

$$I'(t) = -\int (d\Phi^* d\Phi)^n \;\; e^{-t\Phi^*\Phi}\Phi^*\Phi \; F(\Phi, \overline{\Phi}).$$

This integral is well defined also for $t = 0$ since $|\varphi|^2 g_{IJ} \in L^1(\mathbb{R}^{2n})$. Note that

$$\Phi_j^*\Phi_j = D_j(-\varphi_j\overline{\psi}_j), \qquad \text{hence} \quad \Phi^*\Phi = \sum_j \Phi_j^*\Phi_j = D\lambda, \quad \lambda := -\sum_j \varphi_j\overline{\psi}_j.$$

Therefore

$$I'(t) = -\int (d\Phi^* d\Phi)^n \; (D\lambda) \; e^{-t\Phi^*\Phi} F(\Phi, \overline{\Phi}) = -\int (d\Phi^* d\Phi)^n \; D\left(\lambda e^{-t\Phi^*\Phi} F(\Phi, \overline{\Phi})\right),$$

where we used $D\left(e^{-t\Phi^*\Phi} F(\Phi, \overline{\Phi})\right) = 0$. The result now follows from Prop 5.5.

• Since $I'(t) = 0$ we have

$$I = I(0) = \lim_{t\to\infty} I(t).$$

We compute, by scaling,

$$I(t) = \int (d\Phi^* d\Phi)^n \; e^{-t\Phi^*\Phi} F(\Phi, \overline{\Phi}) = \int (d\Phi^* d\Phi)^n \; e^{-\Phi^*\Phi} F(\frac{1}{\sqrt{t}}\Phi, \frac{1}{\sqrt{t}}\overline{\Phi}).$$

Note that $d\Phi^* d\Phi$ is scale invariant because we have perfect grading. We have

$$F(\frac{1}{\sqrt{t}}\Phi, \frac{1}{\sqrt{t}}\overline{\Phi}) = \sum_{IJ\subset\{1,\ldots,n\}} \frac{1}{\sqrt{t}^{|I|+|J|}} v_{IJ}\left(\frac{1}{\sqrt{t}}\varphi, \frac{1}{\sqrt{t}}\overline{\varphi}\right)\psi^I\overline{\psi}^J.$$

By the assumptions on F the function v_{IJ} is bounded, hence

$$C_{IJ} := \sup_{t\geq 0}\int (d\overline{\varphi} d\varphi)^n |v_{IJ}\left(\frac{1}{\sqrt{t}}\varphi, \frac{1}{\sqrt{t}}\overline{\varphi}\right)| \; e^{-\overline{\varphi}\varphi} \leq \|v_{IJ}\|_{L^\infty}\int (d\overline{\varphi} d\varphi)^n \; e^{-\overline{\varphi}\varphi} < \infty \qquad \forall I, J,$$

and therefore all contributions with $|I| + |J| \geq 1$ vanish in the limit $t \to \infty$. Using

$$\prod_j \partial_{\overline{\psi}_j}\partial_{\psi_j} e^{-\overline{\psi}\psi} = 1,$$

and

$$\lim_{t\to\infty} v_{IJ}\left(\frac{1}{\sqrt{t}}\varphi, \frac{1}{\sqrt{t}}\overline{\varphi}\right) = v_{IJ}(0,0) \text{ pointwise} \qquad \forall I, J,$$

we obtain by dominated convergence

$$\lim_{t\to\infty} I(t) = \lim_{t\to\infty}\int_{\mathbb{C}^n}\left(\frac{d\overline{\varphi}d\varphi}{2\pi}\right)^n e^{-\overline{\varphi}\varphi} F\Big(\frac{1}{\sqrt{t}}\varphi, \frac{1}{\sqrt{t}}\overline{\varphi}, 0, 0\Big) = F(0)\int_{\mathbb{C}^n}\left(\frac{d\overline{\varphi}d\varphi}{2\pi}\right)^n e^{-\overline{\varphi}\varphi} = F(0).$$

□

Remark. We need $g_{IJ} \in L^1(\mathbb{R}^{2n})$ to ensure the starting integral is well defined, g_{IJ} differentiable to ensure $D_j F$ is well defined and $\partial(\varphi g_{IJ}) \in L^1(\mathbb{R}^{2n})$ to ensure the integral $I'(t)$ is well defined for all $t \geq 0$. . Finally we need $|\varphi| g_{IJ} \in C_0(\mathbb{R}^{2n})(\mathbb{R}^{2n})$ to ensure $\int D_j\left(\lambda e^{-t\Phi^*\Phi}F\right) = 0$. Note that we do not really need $\partial_x g_{IJ}, \partial_y g_{IJ}$ to be continuous.

5.2 Linear transformations on real supervectors

We consider now the real Grassmann algebra $\mathcal{G} = \mathcal{G}_{\mathbb{R}}[\psi_1, \ldots, \psi_N]$.

Definition 5.7. *For $m, n \geq 1$ we consider the real vector space $\mathcal{G}^{m|2n}$ of all supervectors Φ of the form*

$$\Phi = \begin{pmatrix} x_1 \\ \vdots \\ x_m \\ \xi_1 \\ \eta_1 \\ \vdots \\ \xi_n \\ \eta_n \end{pmatrix} \qquad \text{with } x_1, \ldots x_m \in \mathcal{G}^{\text{even}} \text{ and } \xi_1\eta_1, \ldots \xi_n\eta_n \in \mathcal{G}^{\text{odd}}.$$

We endow this space with the bilinear form

$$\begin{array}{lll} \cdot : & \mathcal{G}^{m|2n} \times \mathcal{G}^{m|2n} & \to \mathcal{G}^{\text{even}} \\ & (\Phi, \Phi') & \to \Phi \cdot \Phi' := \sum_{j=1}^m x_j x_j' + \sum_{l=1}^n (\xi_l' \eta_l + \xi_l \eta_l') \end{array}$$

Lemma 5.8.

(i) $\cdot$ *is symmetric* $\Phi \cdot \Phi' = \Phi' \cdot \Phi$.

(ii) It holds $\Phi \cdot \Phi = \sum_{j=1}^m x_j^2 + 2\sum_{l=1}^n \xi_l \eta_l$. *Moreover* $\text{body}(\Phi \cdot \Phi) \geq 0$ *and*

$$\text{body}(\Phi \cdot \Phi) = 0 \Leftrightarrow \ \text{body}(x_j) = 0 \forall j = 1, \ldots, m.$$

Proof.
(i) This follows from $\xi_l' \eta_l + \xi_l \eta_l' = \xi_l \eta_l' + \xi_l' \eta_l$.

(ii) The statement follows from $\text{body}(\Phi \cdot \Phi) = \sum_{j=1}^m \text{body}(x_j)^2$.

□

Remark. Note that $\Phi \cdot \Phi = 0$ does not imply $\Phi_j = 0$.

We restrict now the the case $m = 2n$ (perfect grading). In this case we can reformulate $\Phi \in \mathcal{G}^{2n|2n}$ as $\Phi = \{\Phi_j\}_{j=1}^n$ with

$$\Phi = \begin{pmatrix} x_j \\ y_j \\ \xi_j \\ \eta_j \end{pmatrix} \in \mathcal{G}^{2|2}.$$

The corresponding bilinear form is

$$\Phi_j \cdot \Phi_k := x_j x_k + y_j y_k + \xi_j \eta_k + \xi_k \eta_j.$$

We can reformulate $\Phi = \begin{pmatrix} x \\ y \\ \xi \\ \eta \end{pmatrix} \in \mathcal{G}^{2|2} = \mathcal{G}_{\mathbb{R}}^{2|2}$ in terms of two complex supervectors $\Phi_c, \overline{\Phi}_c \in \mathcal{G}'^{1|1}$, where $\mathcal{G}' = \mathcal{G}'_{\mathbb{C}}$ is a complex Grassmann algebra, as follows. Setting

$$\begin{aligned} \varphi &:= x + iy, \qquad & \overline{\varphi} &:= x - iy \\ \psi &:= \xi + \eta, \qquad & \overline{\psi} &:= \xi - \eta, \end{aligned}$$

we define

$$\Phi_c := \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \qquad \Phi_c^* := \begin{pmatrix} \overline{\varphi} & \overline{\psi} \end{pmatrix}.$$

We compute

$$\begin{aligned} \Phi_c^*\Phi_c &= \overline{\varphi}\varphi + \overline{\psi}\psi = x^2 + y^2 + 2\xi\eta = \Phi \cdot \Phi \\ \Phi_c^*\Phi_c' + \Phi_c'^*\Phi_c &= \overline{\varphi}\varphi' + \overline{\varphi}'\varphi + \overline{\psi}\psi' + \overline{\psi}'\psi = 2(xx' + yy' + \xi\eta' + \xi'\eta) = 2\,\Phi \cdot \Phi'. \end{aligned} \tag{5.5}$$

The inverse transformation is defined via

$$\begin{aligned} x &:= \frac{\varphi + \overline{\varphi}}{2}, \qquad & iy &:= \frac{\varphi - \overline{\varphi}}{2} \\ \xi &:= \frac{\psi + \overline{\psi}}{2}, \qquad & \eta &:= \frac{\psi - \overline{\psi}}{2}. \end{aligned}$$

Remember that, setting $\Phi_c(t) := e^{tX}\Phi_c$ with $X = \begin{pmatrix} 0 & \alpha \\ \overline{\alpha} & 0 \end{pmatrix} \in \mathcal{A}^{(1|1)\times(1|1)}$, we have

$$\frac{d}{dt}\Phi_c^*(t)\Phi_c(t) = 0$$

and

$$d\Phi_c = X\Phi_c = \begin{pmatrix} d\varphi \\ d\psi \end{pmatrix} = \begin{pmatrix} \alpha\psi \\ \overline{\alpha}\varphi \end{pmatrix}, \qquad d\overline{\Phi}_c = \begin{pmatrix} d\overline{\varphi} \\ d\overline{\psi} \end{pmatrix} = \begin{pmatrix} \overline{\alpha}\overline{\psi} \\ -\alpha\overline{\varphi} \end{pmatrix}.$$

Definition 5.9. *We define*

$$t \mapsto \Phi(t) = \begin{pmatrix} x(t) \\ y(t) \\ \xi(t) \\ \eta(t) \end{pmatrix}$$

via

$$\begin{aligned} x(t) &:= \frac{\varphi(t) + \overline{\varphi}(t)}{2}, \qquad & iy(t) &:= \frac{\varphi(t) - \overline{\varphi}(t)}{2} \\ \xi(t) &:= \frac{\psi(t) + \overline{\psi}(t)}{2}, \qquad & \eta(t) &:= \frac{\psi(t) - \overline{\psi}(t)}{2}. \end{aligned}$$

Proposition 5.10. *Set* $\beta := \frac{\alpha+\overline{\alpha}}{2}$ $\tilde{\beta} := \frac{\alpha-\overline{\alpha}}{2}$. *The following statements hold.*

(i) It holds

$$\frac{d}{dt}\Phi(t)_{|t=0} = \begin{pmatrix} dx \\ dy \\ d\xi \\ d\eta \end{pmatrix} = \begin{pmatrix} \beta\xi + \tilde{\beta}\eta \\ -i\left(\tilde{\beta}\xi + \beta\eta\right) \\ -\tilde{\beta}x + \beta iy \\ \beta x - \tilde{\beta}iy \end{pmatrix}.$$

(ii) $\frac{d}{dt}\Phi(t) \cdot \Phi(t) = 0\ \forall t$.

(iii) For any function $F(\Phi)$ *regular enough we have* $\frac{d}{dt}F(\Phi(t))_{|t=0} = (\beta D + \tilde{\beta}\tilde{D})F$, *with*

$$\begin{aligned} D &:= \xi\partial_x - i\eta\partial_y + iy\partial_\xi + x\partial_\eta \\ \tilde{D} &= \eta\partial_x - i\xi\partial_y - x\partial_\xi - iy\partial_\eta. \end{aligned}$$

(iv) $\int_{\mathbb{R}^2} dxdy\partial_\xi\partial_\eta DF = \int_{\mathbb{R}^2} dxdy\partial_\xi\partial_\eta\tilde{D}F = 0$

Proof. exercise ∎

Remark. We can generalize the results above to n supervectors $\Phi_1,\dots\Phi_n \in \mathcal{G}_{\mathbb{R}}^{2|2}$ as follows (exercise).

(i) It holds

$$\frac{d}{dt}\Phi_j(t)_{|t=0} = \begin{pmatrix} dx_j \\ dy_j \\ d\xi_j \\ d\eta_j \end{pmatrix} = \begin{pmatrix} \beta_j\xi_j + \tilde{\beta}_j\eta_j \\ -i\left(\tilde{\beta}_j\xi_j + \beta_j\eta_j\right) \\ -\tilde{\beta}_j x_j + \beta_j i y_j \\ \beta_j x_j - \tilde{\beta}_j i y_j \end{pmatrix}, \quad j = 1,\dots,n.$$

(ii) $\frac{d}{dt}\Phi_j(t)\cdot\Phi_j(t) = 0$ $\forall t$ and $\forall j = 1,\dots,n$. Moreover, if $\beta_j = \beta$ and $\tilde{\beta}_j = \tilde{\beta}$ $\forall j$ we also have

$$\frac{d}{dt}\Phi_j(t)\cdot\Phi_k(t) = 0 \qquad \forall t \in \mathbb{R},\ \forall j,k = 1,\dots,n.$$

(iii) For any function $F(\Phi_1,\dots,\Phi_n)$ regular enough we have

$$\frac{d}{dt}F(\Phi(t))_{|t=0} = \sum_{j=1}^{n}(\beta_j D_j + \tilde{\beta}_j\tilde{D}_j)F,$$

with

$$\begin{aligned} D_j &:= \xi_j\partial_{x_j} - i\eta_j\partial_{y_j} + iy_j\partial_{\xi_j} + x_j\partial_{\eta_j} \\ \tilde{D}_j &= \eta_j\partial_{x_j} - i\xi_j\partial_{y_j} - x_j\partial_{\xi_j} - iy_j\partial_{\eta_j}. \end{aligned}$$

Moreover, if $\beta_j = \beta$ and $\tilde{\beta}_j = \tilde{\beta}$ $\forall j$ we also have

$$\frac{d}{dt}F(\Phi(t))_{|t=0} = \beta\Big(\sum_{j=1}^{n} D_j\Big)F + \tilde{\beta}\Big(\sum_{j=1}^{n}\tilde{D}_j\Big)F.$$

(iv) $\int_{\mathbb{R}^2} dx_j dy_j \partial_{\xi_j}\partial_{\eta_j} D_j F = \int_{\mathbb{R}^2} dx_j dy_j \partial_{\xi_j}\partial_{\eta_j}\tilde{D}_j F = 0.$

Theorem 5.11 (localization theorem II)**.** *Let $F(\Phi) = F(\Phi_1,\dots,\Phi_n)$ be a function*

$$F(\Phi) = \sum_{IJ\subset\{1,\dots,n\}} v_{IJ}(x,y)\xi^I\eta^J.$$

Assume

(a) $v_{IJ} \in C^1(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n}) \cap L^1(\mathbb{R}^{2n})$, $|(x,y)|^2 g_{IJ} \in L^1(\mathbb{R}^{2n})$, $|(x,y)|g_{IJ} \in C_0(\mathbb{R}^{2n})$, $\partial_{x_k}(x_j g_{IJ}) \in L^1(\mathbb{R}^{2n})$, $\partial_{y_k}(x_j g_{IJ}) \in L^1(\mathbb{R}^{2n})$, $\partial_{x_k}(y_j g_{IJ}) \in L^1(\mathbb{R}^{2n})$, $\partial_{y_k}(y_j g_{IJ}) \in L^1(\mathbb{R}^{2n})$,

(b) $DF = \big(\sum_{j=1}^{n} D_j\big)F = 0.$

Then

$$\int (d\Phi)^n\ F = F(0), \tag{5.6}$$

where we defined

$$(d\Phi)^n := \prod_{j=1}^{n}\frac{dx_j dy_j}{2\pi}d\xi_j d\eta_j.$$

Proof. exercise ☐

[22: 10.01.2025]

[23: 14.01.2025]

6 The $H^{2|2}$ model

6.1 Motivation: random walks with memory

Definition 6.1. *Let $\Lambda \subset \mathbb{Z}^d$ a finite or infinite set and $E_\Lambda = \{e = \{i,j\} | i \neq j \in \Lambda\}$ the set of unordered pairs (undirected edges).*

(i) Let $W = \{W_e\}_{e \in E_\Lambda} \subset [0,\infty)^{E_\Lambda}$ be a family of non-negative weights such that $W_i := \sum_{j \in \Lambda \setminus \{i\}} W_{ij}$ satisfies $0 < W_i < \infty$ $\forall i \in \Lambda$. We define for each $i \neq j$ $p_{ij} := \frac{W_{ij}}{W_i}$. In particular $p_{ij} \in [0,1]$ and $\sum_{j \in \Lambda \setminus \{i\}} p_{ij} = 1$ $\forall i \in \Lambda$.

The random walk (RW) on Λ in the environment W is a countable family of random variables $X = \{X_n\}_{n \in \mathbb{N}} \in \Lambda^{\mathbb{N}}$ where n is a discrete time and X_n can be seen as the position of the random walker at time n. The time evolution is defined as follows. Setting $X_0 = i_0$ the starting point, the probability the random walker follows the trajectory $i_0, \dots, i_n$ up to time n is

$$\mathbb{P}^W_{i_0}\left(X_n = i_n, X_{n-1} = i_{n-1}, \dots, X_0 = i_0\right) := p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{n-1} i_n}.$$

The random walk is simple if $W_{ij} = W \mathbf{1}_{|i-j|=1}$. In this case we have $p_{ij} = \frac{1}{2d} \mathbf{1}_{|i-j|=1}$, for $\Lambda = \mathbb{Z}^d$.

(ii) Let $d\mu(W)$ be a probability measure on $[0,\infty)^{E_\Lambda}$ such that $0 < W_i < \infty$ $\forall i \in \Lambda$ $\mu-$almost surely. The random walk in random environment (RWRE) on Λ starting at i_0 with mixing measure μ is a countable family of random variables $X = \{X_n\}_{n \in \mathbb{N}} \in \Lambda^{\mathbb{N}}$ such that

$$\mathbb{P}_{i_0}\left(X_j = i_j \forall j = 1, \dots n\right) := \int_{[0,\infty)^{E_\Lambda}} d\mu(W)\ \mathbb{P}^W_{i_0}\left(X_j = i_j \forall j = 1, \dots n\right).$$

(iii) Let $a = \{a_e\}_{e \in E_\Lambda} \subset [0,\infty)^{E_\Lambda}$ such that $0 < \sum_{j \in \Lambda \setminus \{i\}} a_{ij} < \infty$ $\forall i \in \Lambda$. We introduce $n-$ depedent weights

$$W_e(n) := \begin{cases} 0 & \text{if } a_e = 0 \\ a_e + T_{e(n)} & \text{if } a_e > 0 \end{cases}$$

where $T_e(n)$ is the number of s the walker has crossed e (in any direction) up to time n. Setting $W_i(n) := \sum_{j \in \Lambda \setminus \{i\}} W_{ij}(n)$ we define the time-dependent crossing probabilities

$$p_{ij}(n) := \frac{W_{ij}(n)}{W_i(n)}.$$

The linearly edge-reinforced random walk (ERRW) is a countable family of random variables $X = \{X_n\}_{n \in \mathbb{N}} \in \Lambda^{\mathbb{N}}$ such that

$$\mathbb{P}^{ERRW}_{i_0}\left(X_j = i_j \forall j = 1, \dots n\right) := p_{i_0 i_1}(1) p_{i_1 i_2}(2) \cdots p_{i_{n-1} i_n}(n-1).$$

For more random walk models with reinforcement see also the review by G. Kozma *Reinforced random walk* https://arxiv.org/abs/1208.0364.

Remark 1 The behavior of a random walk in a random environment is determined by the mixing measure $d\mu$.
For example, if the measure is supported near $W_{ij} = W \mathbf{1}_{|i-j|=1}$ then X behaves as a simple random walk.

Remark 2 The random walk has no memory: the conditional probability

$$\mathbb{P}_{i_0}^{W}\left(X_{n+1}=j\,|\,X_n=i,X_{n-1}=i_{n-1},\dots,X_0=i_0\right) = \frac{\mathbb{P}_{i_0}^{W}\left(X_{n+1}=j,X_n=i,X_{n-1}=i_{n-1},\dots,X_0=i_0\right)}{\mathbb{P}_{i_0}^{W}\left(X_n=i,X_{n-1}=i_{n-1},X_{n-1}=i_{n-1},\dots,X_0=i_0\right)} = p_{i_n i_{n+1}}$$

does not depend on $\{X_j\}_{j=0}^{n-1}$. The random walk in a random environment and the edge-reinforced random walk are both history-dependent. In particular the ERRW is more likely to cross edges it has already crossed in the past (reinforcement towards the past).

Theorem 6.2. *Let $E_\Lambda^a := \{e\in E_\Lambda|\ a_e>0\}$.*
Assume $\Lambda\subset\mathbb{Z}^d$ is finite and (Λ,E_Λ^a) is a connected graph.
Then the ERRW is a RW in a random environment. Precisely

$$\mathbb{P}_{i_0}^{ERRW}\left(X_j=i_j\forall j=1,\dots n\right)=\int d\mu(\omega)\ \mathbb{P}_{i_0}^{\omega}\left(X_j=i_j\forall j=1,\dots n\right),$$

where the random weights are parametrized as $\omega_{ij}=\mathbf{1}_{\{i,j\}\in E_\Lambda^a}W_{ij}e^{u_i+u_j}$, where $W=\{W_e\}_{e\in E_\mu^a}\in[0,\infty)^{E_\mu^a}$, $u=\{u_j\}_{j\in\Lambda}\in\mathbb{R}^\Lambda$, and we defined

$$d\mu(\omega):=\prod_{e\in E_\Lambda^a}d\gamma_{a_e}(W_e)\ d\mu_{\Lambda,i_0}^W(u),$$

where

$$d\gamma_{a_e}(W_e):=\frac{dW_e}{\Gamma(a_e)}W_e^{-1+a_e}e^{-W_e}\mathbf{1}_{W_e>0}$$

$$d\mu_{\Lambda,i_0}^W(u):=d\delta(u_{i_0})\prod_{j\in\Lambda\setminus\{i_0\}}\frac{du_j}{\sqrt{2\pi}}e^{-u_j}e^{-F_\Lambda^W(\nabla u)}\sqrt{\det_{i_0i_0}\left(A_\Lambda(u)\right)},$$

$$F_\Lambda^W(\nabla u):=\sum_{\{ij\}\in E_\Lambda^a}W_{ij}(\cosh(u_i-u_j)-1)$$

and $A_\Lambda(u)\in\mathbb{R}^{\Lambda\times\Lambda}$ is defined via

$$A_\Lambda(u)_{ij}:=-\mathbf{1}_{\{ij\}\in E_\Lambda^a}W_{ij}e^{u_i+u_j}+\mathbf{1}_{i=j}\sum_{k\in\Lambda\setminus\{i\}}W_{ik}e^{u_i+u_k}.$$

Finally $\det_{i_0i_0}A$ is the determinant of the matrix obtained by removing the row and column i_0 from A.

Proof. See C. Sabot, P. Tarrés: *Edge-reinforced random walk, vertex-reinforced jump process and the supersymmetric hyperbolic sigma model* J. Eur. Math. Soc. 17, 2353–2378. □

Remark. The weighted Laplacian $-\Delta_\Lambda^\omega$ on Λ is defined by

$$(-\Delta_\Lambda^\omega)_{ij}=-\mathbf{1}_{i\neq j}\omega_{ij}+\mathbf{1}_{i=j}\sum_{k\in\Lambda\setminus\{i\}}\omega_{ik}.$$

Hence $A_\Lambda(u)=-\Delta_\Lambda^\omega$ with weights $\omega_{ij}=\mathbf{1}_{\{ij\}\in E_\Lambda^a}W_{ij}e^{u_i+u_j}$. The matrix restricted to $\Lambda_{i_0}:=\Lambda\setminus\{i_0\}$ can be written as

$$A_\Lambda(u)_{|\Lambda_{i_0}\times\Lambda_{i_0}}=-\Delta_{\Lambda_{i_0}}^\omega+\hat{m}\in\mathbb{R}^{\Lambda_{i_0}\times\Lambda_{i_0}},$$

with

$$\hat{m}_{ij} = \delta_{ij} W_{ii_0} e^{u_i}.$$

Note that, since $W_e > 0$ γ_{a_e}−almost surely we have $E^a_\Lambda = E^W_\Lambda$ and the graph (Λ, E^W_Λ) is connected almost surely. We compute for $\varphi \in \mathbb{R}^{\Lambda \setminus \{i_0\}}$,

$$(\varphi, (-\Delta^\omega_{\Lambda_{i_0}} + \hat{m})\varphi) = \sum_{\{ij\} \in E^W_{\Lambda \setminus \{i_0\}}} W_{ij} e^{u_i + u_j} (\varphi_i - \varphi_j)^2 + \sum_{j \in \Lambda \setminus \{i_0\}} W_{ii_0} e^{u_i} \varphi_i^2 \geq 0.$$

Therefore $\det_{i_0 i_0} A_\Lambda(u) \geq 0$. We show now that the determinant is strictly positive. Indeed $(\varphi, (-\Delta^\omega_{\Lambda_{i_0}} + \hat{m})\varphi) = 0$ iff

- $\varphi_i = \varphi_j$ $\forall i \neq j \in \Lambda_{i_0}$ such that $W_{ij} > 0$, and
- $\varphi_j = 0$ $\forall j \in \Lambda_{i_0}$ such that $W_{ji_0} > 0$.

Since (Λ, E^W_Λ) is a connected graph almost surely, the above conditions hold only for $\varphi = 0$. Hence $\det_{i_0 i_0} A_\Lambda(u) > 0$.

Theorem 6.3. *Let $\Lambda \subset \mathbb{Z}^d$ be a finite set.*
Let $W = \{W_e\}_{e \in E_\Lambda} \in [0,\infty)^{E_\Lambda}$ and $\varepsilon = \{\varepsilon_j\}_{j \in \Lambda} \in [0,\infty)^\Lambda$ be two family of weights such that (Λ, E^W_Λ) is a connected graph and $\sum_{j \in \Lambda} \varepsilon_j > 0$.
We define

$$F^W_\Lambda(\nabla u) := \sum_{\{ij\} \in E^W_\Lambda} W_{ij} (\cosh(u_i - u_j) - 1) \tag{6.1}$$

$$M^\varepsilon_\Lambda(u) := \sum_{j \in \Lambda} \varepsilon_j (\cosh(u_j) - 1) \tag{6.2}$$

$$D_\Lambda = D_\Lambda(u) := -\Delta^{\omega(u)}_\Lambda + \hat{\varepsilon} e^{\hat{u}} \in \mathbb{R}^{\Lambda \times \Lambda} \tag{6.3}$$

where

$$\omega(u)_{ij} = W_{ij} e^{u_i + u_j}, \qquad \hat{\varepsilon}_{ij} = \delta_{ij} \varepsilon_j, \qquad e^{\hat{u}}_{ij} = \delta_{ij} e^{u_j}.$$

We consider the measure on $\mathbb{R}^\Lambda$

$$d\rho^{W,\varepsilon}_\Lambda(u) := \prod_{j \in \Lambda} \frac{du_j}{\sqrt{2\pi}} e^{-u_j} e^{-F^W_\Lambda(\nabla u)} e^{-M^\varepsilon_\Lambda(u)} \sqrt{\det D_\Lambda(u)}. \tag{6.4}$$

The following statements hold.

(i) $\int_{\mathbb{R}^\Lambda} d\rho^{W,\varepsilon}_\Lambda(u) = 1$ $\forall W, \varepsilon$ *such that* (Λ, E^W_Λ) *is a connected graph and* $\sum_{j \in \Lambda} \varepsilon_j > 0$.

(ii) $d\mu^W_{\Lambda, i_0}(u) = d\delta(u_{i_0}) d\rho^{W,\varepsilon}_{\Lambda \setminus \{i_0\}}(u)$ *with* $\varepsilon_j := W_{ji_0}$ *for all* $j \in \Lambda \setminus \{i_0\}$.

Proof.
(ii) follows by replacing $u_{i_0} = 0$ is $F^W_\Lambda(\nabla u)$ and $M^\varepsilon_\Lambda(u)$ and remarking that $\det_{i_0 i_0} A_\Lambda(u) = \det D_{\Lambda \setminus \{i_0\}}$.
To prove (i) we need to reformulate $\int_{\mathbb{R}^\Lambda} d\rho^{W,\varepsilon}_\Lambda(u)$ as an integral over bosonic and fermionic variables and then use the localization theorem 5.11. This will be done in the next section.

□

[23: 14.01.2025]

[24: 17.01.2025]

6.2 A hyperbolic nonlinear sigma model

Remember that, in the classical $O(N)$ model, the spin S_j at point $j \in \Lambda$ is an element in the Hilbert space $(\mathbb{R}^N, \cdot)$, where $\cdot$ is the euclidean scalar product. The spin must satisfy in addition the nonlinear constraint $S_j \cdot S_j = 1$. The corresponding energy functional is

$$H(S_\Lambda) = \frac{1}{2}\sum_{jk} J_{jk}|S_j - S_k|^2 + \sum_j h_j \cdot S_j = -\sum_{jk} J_{jk} S_j \cdot S_k + \sum_j h_j \cdot S_j + const.$$

We replace now $S \in \mathbb{R}^N$ with $v \in \mathcal{G}^{3|2}$ where $\mathcal{G}$ is a *real* Grassmann algebra i.e

$$v = \begin{pmatrix} x \\ y \\ z \\ \xi \\ \eta \end{pmatrix} \qquad x, y, z \in \mathcal{G}^{\text{even}} \ \xi, \eta \in \mathcal{G}^{\text{odd}}. \tag{6.5}$$

The euclidean scalar product is replaced by

$$(v, v') := xx' + yy' - zz' + \xi\eta' + \xi'\eta. \tag{6.6}$$

This a a Grassmann extension of the classical Minkowski inner product in 2 space dimensions. In particular we have

$$(v, v) = x^2 + y^2 - z^2 + 2\xi\eta.$$

Note that $\text{body}((v, v))$ may also be negative. The nonlinear contraint $S \cdot S = 1$ is replaced by $(v, v) = -1$. This holds iff

$$z^2 = 1 + x^2 + y^2 + 2\xi\eta.$$

Since $\text{body}(1 + x^2 + y^2 + 2\xi\eta) \geq 1$ we can define the square root of the above expression. Hence

$$(v, v) = -1 \qquad \Leftrightarrow \qquad z = \pm\sqrt{1 + x^2 + y^2 + 2\xi\eta}.$$

There are therefore 2 even (x, y) and 2 odd (ξ, η) independent variables (perfect grading). Note that, without the Grassmann part we would obtain the standard hyperbolic plane H^2.
For all v, v' satisfying $(v, v) = (v', v') = -1$ we compute

$$(v - v', v - v') = -2(v, v') - (v, v) - (v', v') = -2(1 + (v, v')). \tag{6.7}$$

Definition 6.4. *We say that the vector $v \in \mathcal{G}^{3|2}$ belongs to the hyperbolic space $H^{2|2}$ if*

$$(v, v) = -1, \qquad \textit{and} \qquad \text{body}(z) > 0.$$

Lemma 6.5.

(i) $v \in H^{2|2}$ *iff* $z = \sqrt{1 + x^2 + y^2 + 2\xi\eta}$.

(ii) *For all* $v, v' \in H^{2|2}$ *it holds*

- *(a)* $\text{body}((v - v', v - v')) \geq 0$ *and*
- *(b)* $\text{body}((v - v', v - v')) = 0$ *iff* $\text{body}(x) = \text{body}(x')$ *and* $\text{body}(y) = \text{body}(y')$.

Proof. We abbreviate $x_0 := \mathrm{body}(x)$, $y_0 := \mathrm{body}(y)$, $z_0 := \mathrm{body}(z)$.

(i) $(v,v) = -1$ implies $z = \pm\sqrt{1+x^2+y^2+2\xi\eta}$. The result now follows from the constraint $z_0 = \mathrm{body}(z) > 0$.

$(ii)(a)$ Using (6.6) and (6.7) we have

$$\mathrm{body}((v-v', v-v')) = -2\mathrm{body}(1+(v,v')) = -2(1+\mathrm{body}((v,v'))) = -2(1+x_0x_0'+y_0y_0'-z_0z_0')$$

We show now that $1+x_0x_0'+y_0y_0'-z_0z_0' \le 0$ and hence $\mathrm{body}((v-v', v-v')) \ge 0$. Indeed, setting $V_0 := (1, x_0, y_0)$ and $V_0' := (1, x_0', y_0')$ we argue

$$\begin{aligned} 1+x_0x_0'+y_0y_0'-z_0z_0' &= 1+x_0x_0'+y_0y_0' - \sqrt{1+x_0^2+y_0^2}\sqrt{1+{x_0'}^2+{y_0'}^2} \\ &= V_0 \cdot V_0' - |V_0|\,|V_0'| \le 0, \end{aligned}$$

where in the last line we used Cauchy-Schwarz.

$(ii)(b)$ We have $\mathrm{body}((v-v', v-v')) = 0$ iff $V_0 \cdot V_0' - |V_0|\,|V_0'| = 0$ iff there exists a $\lambda > 0$ such that $V_0 = \lambda V_0'$. Since the first components of V_0, V_0' coincide, we must have $V_0 = V_0'$. The result follows.

□

Definition 6.6. *Let $\Lambda \subset \mathbb{Z}^d$ be a finite set.*
Let $W = \{W_e\}_{e\in E_\Lambda} \in [0,\infty)^{E_\Lambda}$ and $\varepsilon = \{\varepsilon_j\}_{j\in\Lambda} \in [0,\infty)^\Lambda$ be two family of weights such that (Λ, E_Λ^W) is a connected graph and $\sum_{j\in\Lambda}\varepsilon_j > 0$.
With these parameters we define

$$d\nu_\Lambda^{W,\varepsilon}(v) := \prod_{j\in\Lambda} dv_j \; e^{-H_\Lambda^{W,\varepsilon}(v)} \tag{6.8}$$

where

$$dv_j := \frac{dx_j dy_j}{2\pi} d\xi_j d\eta_j \frac{1}{z_j} \qquad H_\Lambda^{W,\varepsilon}(v) := \frac{1}{2}\sum_{\{jk\}\in E_\Lambda^W} W_{jk}(v_j-v_k, v_j-v_k) + \sum_{j\in\Lambda}\varepsilon_j(z_j-1), \tag{6.9}$$

and $z_j = \sqrt{1+x_j^2+y_j^2+2\xi_j\eta_j}$. We use the notation

$$\langle f\rangle_{eucl} := \int d\nu_\Lambda^{W,\varepsilon}(v) f(v).$$

Theorem 6.7. *Under the same assumptions as above, the following statements hold.*

(i) $\prod_j \frac{1}{z_j} e^{-H_\Lambda^{W,\varepsilon}(v)}$ *is integrable.*

(ii) $\int d\nu_\Lambda^{W,\varepsilon}(v) = 1$

(iii) $\left\langle \prod_{i,j\in\Lambda}(1+(v_i-v_j, v_i-v_j))^{m_j}\right\rangle_{eucl} = 1$ $\forall i,j \in \Lambda$ *and* $m_{ij} \in \mathbb{R}$.

Proof.
(i) We integrate over $x_j, y_j \in \mathbb{R}$. Hence

$$\mathrm{body}(z_j) = \sqrt{1+x^2+y^2} \ge 1, \qquad \text{and} \qquad z_j = \mathrm{body}(z_j) + \frac{1}{\mathrm{body}(z_j)}\xi_j\eta_j.$$

Inserting this expansion in the formula

$$H_\Lambda^{W,\varepsilon}(v) = -\sum_{\{jk\}\in E_\Lambda^W} W_{jk}(1 + x_j x_k + y_j y_k - z_j z_k + \xi_j \eta_k + \xi_k \eta_j) + \sum_{j\in\Lambda} \varepsilon_j (z_j - 1),$$

we obtain

$$\prod_j \frac{1}{z_j} e^{-H_\Lambda^{W,\varepsilon}(v)} = \sum_{I,J\subset\Lambda} v_{IJ}(x,y)\ \xi^I \eta^J$$

where v_{IJ} is of the form

$$v_{IJ} = e^{\sum_{\{jk\}\in E_\Lambda^W} W_{jk}(1+x_jx_k+y_jy_k-\mathrm{body}(z_j)\mathrm{body}(z_k))} e^{-\sum_{j\in\Lambda}\varepsilon_j(\mathrm{body}(z_j)-1)} \frac{P(\mathrm{body}(z))}{\prod_j \mathrm{body}(z_j)^{n_j}}$$

for some polynome P in the variables z and some integers $n_j \geq 0$. Setting $\varphi_j = (x_j, y_j) \in \mathbb{R}^2$ we can reformulate this as

$$v_{IJ} = e^{\sum_{\{jk\}\in E_\Lambda^W} W_{jk}(1+\varphi_j\cdot\varphi_k-\sqrt{1+|\varphi_j|^2}\sqrt{1+|\varphi_k|^2})} e^{-\sum_{j\in\Lambda}\varepsilon_j(\sqrt{1+|\varphi_j|^2}-1)} \frac{P(\sqrt{1+|\varphi_j|^2})}{\prod_j (1+|\varphi_j|^2)^{\frac{n_j}{2}}}.$$

Since $1 + |\varphi_j|^2 \geq 1$ we have

$$\frac{|P(\sqrt{1+|\varphi_j|^2})|}{\prod_j (1+|\varphi_j|^2)^{\frac{n_j}{2}}} \leq C\ \prod_j (1 + |\varphi_j|^{m_j})$$

for some constant $C > 0$ and some powers $m_j \geq 0$. Let j_0 be a point such that $\varepsilon_{j_0} > 0$, which must exist since $\sum_j \varepsilon_j > 0$. Let T be a subset of E_Λ^W forming a spanning tree for Λ. T must exist since (Λ, E_Λ^W) is a connected graph. Using

$$1 + \varphi_j \cdot \varphi_k - \sqrt{1 + |\varphi_j|^2}\sqrt{1 + |\varphi_k|^2} \leq 0, \qquad \sqrt{1 + |\varphi_j|^2} - 1 \geq 0,$$

we can bound

$$|v_{IJ}| \leq\ e^{-\varepsilon_{j_0}(\sqrt{1+|\varphi_{j_0}|^2}-1\ )} C\ \prod_j (1 + |\varphi_j|^{m_j}) \prod_{\{jk\}\in T} e^{-W_{jk}(\sqrt{1+|\varphi_j|^2}\sqrt{1+|\varphi_k|^2}-\varphi_j\cdot\varphi_k-1)}.$$

For each $j \in \Lambda$ let γ_j be the unique path in the tree connecting j to j_0. We use this paths to endow Λ with a partial order as follows: for $i \neq j$ we say $i < j$ if $i \in \gamma_j$. For each $e \in T$ we write $e = (i_e, j_e)$ where $i_e < j_e$. With this notation

$$|v_{IJ}| \leq\ C\ e^{-\varepsilon_{j_0}(\sqrt{1+|\varphi_{j_0}|^2}-1\ )}(1 + |\varphi_{j_0}|^{m_{j_0}}) \prod_{e\in T} e^{-W_e(\sqrt{1+|\varphi_{i_e}|^2}\sqrt{1+|\varphi_{j_e}|^2}-\varphi_{i_e}\cdot\varphi_{j_e}-1)}(1 + |\varphi_{j_e}|^{m_{j_e}}).$$

Our goal is to use this bound to show

$$\int_{\mathbb{R}^{2\Lambda}} \prod_j d\varphi_j\ |v_{IJ}|\ < \infty.$$

We perform the integral over $\varphi_j \in \mathbb{R}^2$ starting from the maximal elements j in Λ (the leaves of the tree) and repeating recursively until we reach the minimal element j_0 (the root of the tree). The resul follows from the following two bounds.

$$I(\varphi') := \int_{\mathbb{R}^2} d\varphi\ e^{-W(\sqrt{1+|\varphi|^2}\sqrt{1+|\varphi'|^2}-\varphi\cdot\varphi'-1)}(1 + |\varphi|^m) \leq\ C_{W,m}\ (1 + |\varphi'|^{m+1})$$

$$I_0 := \int_{\mathbb{R}^2} d\varphi\ e^{-\varepsilon(\sqrt{1+|\varphi|^2}-1)}(1 + |\varphi|^m) \leq\ C'_{\varepsilon,m},$$

where $C_{W,m}, C'_{W,\varepsilon} > 0$ are some contants and $m \geq 0$ is an integer. To prove the first bound we argue

$$\sqrt{1+|\varphi|^2}\sqrt{1+|\varphi'|^2} - \varphi\cdot\varphi' - 1 \geq |\varphi|(\sqrt{1+|\varphi'|^2} - |\varphi'|) - 1,$$

hence

$$I(\varphi') \leq e^W \int_{\mathbb{R}^2} d\varphi\; e^{-W(\sqrt{1+|\varphi'|^2}-|\varphi'|)|\varphi|}(1+|\varphi|^m) = 2\pi e^W \int_0^\infty dr\; e^{-W(\sqrt{1+|\varphi'|^2}-|\varphi'|)r} r(1+r^m)$$
$$= 2\pi e^W \frac{1}{W(\sqrt{1+|\varphi'|^2}-|\varphi'|)}(C_1 + \frac{C_2}{W^m(\sqrt{1+|\varphi'|^2}-|\varphi'|)^m})$$

We have (exercise)

$$\sqrt{1+|\varphi'|^2} - |\varphi'| \geq \frac{1}{2}\frac{1}{1+|\varphi'|}.$$

Hence

$$I(\varphi') \leq C_W(1+|\varphi'|)^{m+1}.$$

To prove the second inequality we argue, using $\sqrt{1+|\varphi|^2} \geq |\varphi|$,

$$I_0 \leq e^\varepsilon \int_{\mathbb{R}^2} d\varphi\; e^{-\varepsilon|\varphi|}(1+|\varphi|^m) = C_{\varepsilon,m}.$$

(ii) Setting

$$\Phi_j := \begin{pmatrix} x_j \\ y_j \\ \xi_j \\ \eta_j \end{pmatrix}, \qquad \Phi\cdot\Phi' = xx' + yy' + \xi_j\eta'_j + \xi'_j\eta_j,$$

we have

$$z_j = \sqrt{1+\Phi_j\cdot\Phi_j}, \qquad (v_j, v_k) = \Phi_j\cdot\Phi_k - \sqrt{1+\Phi_j\cdot\Phi_j}\sqrt{1+\Phi_k\cdot\Phi_k},$$

and

$$dv_j = d\Phi_j \frac{1}{\sqrt{1+\Phi_j\cdot\Phi_j}}.$$

Therefore

$$\int d\nu_\Lambda^{W,\varepsilon}(v) = \int (d\Phi)^\Lambda f(\{\Phi_j\cdot\Phi_k\}_{jk}).$$

The function f satisfies the assumptions of Theorem 5.11. Hence

$$\int d\nu_\Lambda^{W,\varepsilon}(v) = f(0) = 1.$$

(iii) As in (ii) we use the localization theorem 5.11 The result follows from

$$(v_j, v_k) = \Phi_j\cdot\Phi_k - \sqrt{1+\Phi_j\cdot\Phi_j}\sqrt{1+\Phi_k\cdot\Phi_k} = -1 \qquad \text{for } \Phi = 0.$$

□

[24: 17.01.2025]

[25: 21.01.2025]

6.3 Horospherical coordinates

Lemma 6.8. *Let $f \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$ and ξ, η two odd elements in a real Grassmann algebra. Then $\forall g \in C^1(\mathbb{R})$ such that $(fg)' \in L^1(\mathbb{R})$ and $(fg) \in C_0(\mathbb{R})$ it holds*

$$\int_{\mathbb{R}} dx\, f(x) = \int_{\mathbb{R}} dx\, f(x + \xi\eta g(x))\,(1 + \xi\eta g'(x)).$$

Proof. We compute

$$f((x + \xi\eta g(x)) = f(x) + \xi\eta f'(x) g(x),$$

hence

$$f(x + \xi\eta g(x))\,(1 + \xi\eta g'(x)) = f(x) + \xi\eta\, \left(f'(x)g(x) + f(x)g'(x)\right) = f(x) + \xi\eta\, (fg)'(x)$$

Inserting this in the integral above we obtain

$$\int_{\mathbb{R}} dx\, f(x + \xi\eta g(x))\,(1 + \xi\eta g'(x)) = \int_{\mathbb{R}} dx\, f(x) + \xi\eta\, (fg)'(x) = \int_{\mathbb{R}} dx\, f(x) + \xi\eta \int_{\mathbb{R}} dx\, (fg)'(x) = \int_{\mathbb{R}} dx\, f(x).$$

□

Theorem 6.9. *Let $\mathcal{G}$ be a real Grassmann algebra. Consider the nonlinear map*

$$\begin{aligned} \mathcal{G}^{2|2} &\to \mathcal{G}^{2|2} \\ \tilde{\Phi} = \begin{pmatrix} u \\ s \\ \overline{\psi} \\ \psi \end{pmatrix} &\mapsto \Phi(\tilde{\Phi}) = \begin{pmatrix} x \\ y \\ \xi \\ \eta \end{pmatrix} \end{aligned}$$

defined by

$$\begin{cases} x & := \sinh u - e^u \left(\frac{s^2}{2} + \overline{\psi}\psi\right) \\ y & := e^u s \\ \xi & := e^u \overline{\psi} \\ \eta & := e^u \psi. \end{cases} \tag{6.10}$$

We define

$$z = z(\Phi) = \sqrt{1 + \Phi \cdot \Phi} = \sqrt{1 + x^2 + y^2 + 2\xi\eta}, \qquad z_0(\Phi) := \sqrt{1 + x^2 + y^2}.$$

The following statements hold.

(i) $z = z(\Phi(\tilde{\Phi})) = \cosh u + e^u \left(\frac{s^2}{2} + \overline{\psi}\psi\right)$. *The transformation is invertible with*

$$\begin{cases} u & := \ln(x + z) \\ s & := \frac{y}{x+z} \\ \overline{\psi} & := \frac{\xi}{x+z} \\ \psi & := \frac{\eta}{x+z}. \end{cases} \tag{6.11}$$

Moreover

$$\overline{\psi} = \frac{\xi}{x + z} = \frac{\xi}{x + z_0}, \qquad \psi = \frac{\eta}{x + z} = \frac{\eta}{x + z_0}.$$

(ii) For any function

$$F(\Phi) = v_0(x,y) + v_1(x,y)\xi + v_2(x,y)\eta + v_3(x,y)\xi\eta$$

with $v_j \in C^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2) \cap C_0(\mathbb{R}^2)\ \forall j = 0,\dots 3$, *it holds*

$$\int_{\mathbb{R}^2} dxdy\partial_\xi\partial_\eta\ F(\Phi) = \int_{\mathbb{R}^2} duds\partial_{\overline{\psi}}\partial_\psi\ \left[e^{-u}z(\Phi(\tilde{\Phi}))F(\Phi(\tilde{\Phi}))\right]. \tag{6.12}$$

Remark. There are three standard parametrizations for the hyperbolic plane $H^2 \cap \{z > 0\}$.

1. Euclidean coordinates: the independent variables are $x, y \in \mathbb{R}$ and $z = \sqrt{1 + x^2 + y^2}$.

2. Polar coordinates: the independent variables are $t \geq 0, \theta \in [0, 2\pi)$ and

$$x = \sinh t\,\cos\theta, \qquad y = \sinh t\,\sin\theta, \qquad z = \cosh t.$$

3. Horospherical coordinates: the independent variables are $u, s \in \mathbb{R}$ and

$$x = \sinh u - e^u\frac{s^2}{2}, \qquad y = se^u, \qquad z = \cosh u + e^u\frac{s^2}{2}. \tag{6.13}$$

Above we use a Grassmann extension of the horospherical coordinates.
The Jacobian of the coordinate change $(x, y) \to (u, s)$ is

$$\det\begin{pmatrix}\partial_u x & \partial_s x\\ \partial_u y & \partial_s y\end{pmatrix} = \begin{pmatrix}\cosh u - e^u\frac{s^2}{2} & -e^u s\\ se^u & e^u\end{pmatrix} = e^u(\cosh u + e^u\frac{s^2}{2}) = e^u z.$$

The coordinate change is invertible with (exercise)

$$u = \ln(x+z), \qquad s = \frac{y}{x+z}. \tag{6.14}$$

Note that u is well defined since $x + z = x + \sqrt{1+x^2+y^2} > 0$. Therefore, for any integrable function $f(x, y)$ we have

$$\int_{\mathbb{R}^2} dxdy\,f(x,y) = \int_{\mathbb{R}^2} duds\ \left[e^{-u}z\ f(x(u,s), y(u,s))\right]. \tag{6.15}$$

Proof of Theorem 6.9.
(i) exercise

(ii) Remember that we want to end up with $x = \sinh u - e^u\frac{s^2}{2} - e^u\overline{\psi}\psi$. Note that

$$e^u\overline{\psi}\psi = \xi\eta e^{-u} = \frac{\xi\eta}{x+z} = \frac{\xi\eta}{x + z_0(x,y)}$$

since $\xi^2 = 0$. For any function $f \in C^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2) \cap C_0(\mathbb{R}^2)$ and any fixed $y \in \mathbb{R}$, we have, using Lemma 6.8,

$$\begin{aligned}\int_{\mathbb{R}} dx f(x) &= \int_{\mathbb{R}} dx f\left(x - \frac{\xi\eta}{x+z_0(x,y)}\right)\left(1 - \xi\eta\partial_x\frac{1}{x+z_0(x,y)}\right)\\ &= \int_{\mathbb{R}} dx f\left(x - \frac{\xi\eta}{x+z_0}\right)\left(1 + \xi\eta\partial_x\frac{1}{z_0(x+z_0)}\right),\end{aligned}$$

where we also used (exercise)

$$\partial_x \frac{1}{x+z_0(x,y)} = -\frac{1}{z_0(x+z_0)}.$$

We obtain

$$\int_{\mathbb{R}^2} dxdy\partial_\xi\partial_\eta \; F(x,y,\xi,\eta) = \int_{\mathbb{R}^2} dxdy\partial_\xi\partial_\eta \; F\left(x - \frac{\xi\eta}{x+z_0}, y, \xi, \eta\right)\left(1 + \xi\eta\frac{1}{z_0(x+z_0)}\right).$$

We perform now the classical coordinate change $(x,y) \to (u,s)$ introduced in (6.13). We obtain

$$\begin{aligned}&\int_{\mathbb{R}^2} dxdy\partial_\xi\partial_\eta \; F\left(x - \frac{\xi\eta}{x+z_0}, y, \xi, \eta\right)\left(1 + \xi\eta\partial_x\frac{1}{z_0(x+z_0)}\right)\\ &= \int_{\mathbb{R}^2} duds\partial_\xi\partial_\eta \; (e^u z_0) \; F\left(x(u,s) - \frac{\xi\eta}{x(u,s)+z_0}, y(u,s), \xi, \eta\right)\left(1 + \xi\eta\frac{1}{z_0(x(u,s)+z_0)}\right),\end{aligned}$$

where $(e^u z_0)$ is the Jacobian (see (6.15)). Finally, for a fixed u we perform the linear coordinate change

$$\overline{\psi} = e^{-u}\xi, \qquad \psi = e^{-u}\eta.$$

In particular $\xi\eta = \overline{\psi}\psi e^{2u}$. The corresponding Jacobian is

$$\partial_\xi\partial_\eta \; = \partial_{\overline{\psi}}\partial_\psi \; e^{-2u}.$$

Inserting this in the integral we obtain

$$\begin{aligned}&\int_{\mathbb{R}^2} duds\partial_{\overline{\psi}}\partial_\psi \; (e^{-u}z_0) \; F\left(x(u,s) - \frac{\overline{\psi}\psi e^{2u}}{x(u,s)+z_0}, y(u,s), \; e^u\overline{\psi}, e^u\psi\right)\left(1 + \overline{\psi}\psi e^{2u}\frac{1}{z_0(x(u,s)+z_0)}\right)\\ &= \int_{\mathbb{R}^2} duds\partial_{\overline{\psi}}\partial_\psi \; (e^{-u}z) F\left(x(u,s,\overline{\psi},\psi), y(u,s), \; e^u\overline{\psi}, e^u\psi\right),\end{aligned}$$

where we used

$$\frac{e^{2u}}{x(u,s)+z_0} = \overline{\psi}\psi e^{2u}e^{-u} = \overline{\psi}\psi e^u \tag{6.16}$$

$$z_0\left(1 + \overline{\psi}\psi e^{2u}\frac{1}{z_0(x(u,s)+z_0)}\right) = \left(z_0 + \overline{\psi}\psi e^u\right) = z(u,s,\overline{\psi},\psi) \tag{6.17}$$

$$x(u,s) - \frac{\overline{\psi}\psi e^{2u}}{x(u,s)+z_0} = x(u,s) - \overline{\psi}\psi e^u = x(u,s,\overline{\psi},\psi). \tag{6.18}$$

This conclues the proof. □

Corollary 6.10. *Remember that $\forall v, v' \in H^{2|2}$ we have $(v,v') = \Phi\cdot\Phi' - z(\Phi)z(\Phi')$.*

(i) $\forall v, v' \in H^{2|2}$ we have, replacing Φ, Φ' with $\Phi(\tilde\Phi), \Phi'(\tilde\Phi')$,

$$-(v,v') = \cosh(u-u') + e^{u+u'}\left(\frac{(s-s')^2}{2} + (\overline{\psi} - \overline{\psi}')(\psi - \psi')\right).$$

(ii) It holds

$$\int \prod_{j\in\Lambda} dv_j \; e^{-H_\Lambda^{W,\varepsilon}(v)} = \int_{\mathbb{R}^{2\Lambda}} \left(\frac{duds}{2\pi}\partial_{\overline{\psi}}\partial_\psi \; e^{-u}\right)^\Lambda e^{-F_\Lambda^W(\nabla u)}e^{-M_\Lambda^\varepsilon(u)}e^{-\frac{1}{2}(s,D_\Lambda(u)s)}e^{-(\overline{\psi},D_\Lambda(u)\psi)},$$

where $F_\Lambda^W, M_\Lambda^\varepsilon, D_\Lambda(u)$ were defined in (6.1)(6.2)(6.3).

(iii) *For all functions $F(x, y, \xi, \eta)$ of the form $F(x, y, \xi, \eta) = f(\{x_j + z_j\}_{j\in\Lambda})$, with f regular enough, we have*

$$\langle f\rangle_{eucl} := \int d\nu_\Lambda^{W,\varepsilon}(v) f(v) = \int_{\mathbb{R}^\Lambda} d\rho_\Lambda^{W,\varepsilon}(u) f\left(\{e^{u_j}\}_{j\in\Lambda}\right)$$

where $d\rho_\Lambda^{W,\varepsilon}$ is the measure introduced in (6.4)

Proof.
(i) exercise

(ii) Remember the formula for $H(v)$ (6.9). Using (i), (6.12) and Theorem 6.9(i), we have

$$-\sum_{\{jk\}\in E_\Lambda^W} W_{jk}(1 + (v_j, v_k)) = \sum_{\{jk\}\in E_\Lambda^W} W_{jk}\left(\cosh(u_j - u_k) - 1\right) + \frac{1}{2}(s, -\Delta^{\omega(u)} s) + (\overline{\psi}, -\Delta^{\omega(u)}\psi)$$
$$\sum_{j\in\Lambda} \varepsilon_j (z_j - 1) = \sum_{j\in\Lambda} \varepsilon_j(\cosh u_j - 1) + \frac{1}{2}(s, \hat{\varepsilon} e^{\hat{u}} s) + (\overline{\psi}, \hat{\varepsilon} e^{\hat{u}}\psi)$$
$$dv = \frac{dxdy}{2\pi}\partial_\xi\partial_\eta\frac{1}{z} \to \frac{duds}{2\pi}\partial_{\overline{\psi}}\partial_\psi e^{-u}$$

The result follows.

(iii) Since $F = f(\{x_j + z_j\}_{j\in\Lambda}) = f(\{e^{u_j}\}_{j\in\Lambda})$, the function is independent of $s, \overline{\psi}, \psi$. These last three variables are Gaussian so we can integrate them out exactly.

$$\int_{\mathbb{R}^\Lambda} \left(\frac{ds}{\sqrt{2\pi}}\right)^\Lambda e^{\frac{1}{2}(s, D_\Lambda(u)s)} = \frac{1}{\sqrt{\det D_\Lambda(u)}}$$
$$\int (d\overline{\psi} d\psi)^\Lambda e^{-(\overline{\psi}, D_\Lambda(u)\psi)} = \left(\partial_{\overline{\psi}}\partial_\psi\right)^\Lambda e^{-(\overline{\psi}, D_\Lambda(u)\psi)} = \det D_\Lambda(u).$$

The result follows.

□

We are finally ready to complete the proof of Theorem 6.3.

Proof of Theorem 6.3. Setting $f = 1$ we argue, using also 6.7(ii),

$$1 = \int d\nu_\Lambda^{W,\varepsilon}(v) = \langle 1\rangle_{eucl} = \int_{\mathbb{R}^\Lambda} d\rho_\Lambda^{W,\varepsilon}(u),$$

which concludes the proof. □

[25: 21.01.2025]

[26: 24.01.2025]

6.4 Ward identities and some applications

We need some notation. Set

$$\langle f\rangle_{eucl} = \int (dv)^{\Lambda}\ e^{-H_{\Lambda}^{W,\varepsilon}(v)} f(v)$$

$$\langle f\rangle_{hor} = \int \left(\frac{duds}{2\pi} d\overline{\psi} d\psi\, e^{-u}\right)^{\Lambda} e^{-F_{\Lambda}^{W}(\nabla u)} e^{-M_{\Lambda}^{\varepsilon}(u)} e^{-\frac{1}{2}(s,D_{\Lambda}(u)s)} e^{-(\overline{\psi},D_{\Lambda}(u)\psi)}\ f(u,s,\overline{\psi},\psi)$$

$$\mathbb{E}^{us}[f] := \int \left(\frac{duds}{2\pi} e^{-u}\right)^{\Lambda} e^{-F_{\Lambda}^{W}(\nabla u)} e^{-M_{\Lambda}^{\varepsilon}(u)} e^{-\frac{1}{2}(s,D_{\Lambda}(u)s)} \det D_{\Lambda}(u)\ f(u,s)$$

$$\mathbb{E}^{u}[f] := \int_{\mathbb{R}^{\Lambda}} d\rho_{\Lambda}^{W,\varepsilon}(u) f(u) = \int_{\mathbb{R}^{\Lambda}} \left(\frac{duds}{\sqrt{2\pi}} e^{-u}\right)^{\Lambda} e^{-F_{\Lambda}^{W}(\nabla u)} e^{-M_{\Lambda}^{\varepsilon}(u)} (\det D_{\Lambda}(u))^{\frac{1}{2}}\ f(u).$$

Proposition 6.11. *For $i \neq j \in \Lambda$ we define*

$$B_{ij} = B_{ij}(u,s) = \cosh(u_i - u_j) + \frac{(s_i - s_j)^2}{2} e^{u_i + u_j}, \tag{6.19}$$
$$V_{ij}, F_{ij} \in \mathbb{R}^{\Lambda} \quad V_{ij}(k) := e^{\frac{u_i+u_j}{2}}(\delta_i(k) - \delta_j(k)), \quad F_{ij} := \frac{1}{\sqrt{B_{ij}}} V_{ij}.$$

With these notations, the following statements hold.

(i) For all $i \neq j$ it holds $-(v_i, v_j) = B_{ij} + (\overline{\psi}, V_{ij})(V_{ij}, \psi)$.

(ii) (Ward identity) For all $i \neq j$ and $m \in \mathbb{R}$, we have

$$\mathbb{E}^{us}\left[B_{ij}^m\left(1 - m(F_{ij}, D^{-1}F_{ij})\right)\right] = \mathbb{E}^{us}\left[B_{ij}^m\left(1 - \frac{m}{B_{ij}}(V_{ij}, D^{-1}V_{ij})\right)\right] = 1.$$

Remark By construction $B_{ij} \geq 1$ and the matrix D is invertible, hence the axpressions above are well defined.

Proof.
(i) follows from Cor. 6.10 and

$$e^{u_i+u_j}(\overline{\psi}_i - \overline{\psi}_j)(\psi_i - \psi_j) = (\overline{\psi}, V_{ij})(V_{ij}, \psi).$$

(ii) Remember that $\forall v, v' \in H^{2|2}$ we have $(v, v') = \Phi \cdot \Phi' - z(\Phi)z(\Phi')$. We argue

$$\langle(-(v_j, v_j))^m\rangle_{eucl} = \langle(-\Phi_i \cdot \Phi_j + z(\Phi_i)z(\Phi_j))\rangle_{eucl} = (-0+1)^m = 1,$$

where in the last step we applied the localization theorem 5.11. Passing to horospherical coordinates we obtain

$$1 = \langle(-(v_j, v_j))^m\rangle_{eucl} = \left\langle(B_{ij} + (\overline{\psi}, V_{ij})(V_{ij}, \psi))^m\right\rangle_{hor}$$

Since $(\overline{\psi}, V_{ij}) \in \mathcal{G}^{\text{odd}}$ we have $(\overline{\psi}, V_{ij})^2 = 0$ and hence

$$(B_{ij} + (\overline{\psi}, V_{ij})(V_{ij}, \psi))^m = B_{ij}^m + mB_{ij}^{m-1}(\overline{\psi}, V_{ij})(V_{ij}, \psi) = B_{ij}^m\left(1 + \frac{m}{B_{ij}}(\overline{\psi}, V_{ij})(V_{ij}, \psi)\right)$$
$$= B_{ij}^m e^{\frac{m}{B_{ij}}(\overline{\psi}, V_{ij})(V_{ij}, \psi)} = B_{ij}^m e^{m(\overline{\psi}, F_{ij})(F_{ij}, \psi)}.$$

It follows

$$1 = \left\langle B^m_{ij} e^{m(\overline{\psi},F_{ij})(F_{ij},\psi)} \right\rangle_{hor}.$$

Integrating over the Grassmann variables and using Lemma 6.12(ii) below, we obtain

$$\int \left(d\overline{\psi}d\psi\right)^{\Lambda} e^{-(\overline{\psi},D_\Lambda(u)\psi)}\ e^{m(\overline{\psi},F_{ij})(F_{ij},\psi)} = \det\left(D - mF_{ij}\otimes F_{ij}\right)$$
$$= \det D\ \det\left(1 - mD^{-1}F_{ij}\otimes F_{ij}\right) = \det D\ \left(1 - m(F_{ij},D^{-1}F_{ij})\right).$$

Therefore

$$1 = \left\langle B^m_{ij} e^{m(\overline{\psi},F_{ij})(F_{ij},\psi)} \right\rangle_{hor} = \mathbb{E}^{u,s}\left[B^m_{ij}\left(1 - m(F_{ij},D^{-1}F_{ij})\right)\right]$$

which concludes the proof. □

Remark Since $D > 0$ as a quadratic form we have $(F_{ij},D^{-1}F_{ij}) > 0$, hence, in the case $m > 0$, the expression $1 - m(F_{ij},D^{-1}F_{ij})$ may be negative. We will show that, when $\{ij\} \in E^W_\Lambda$ and $0 < m < W_{ij}$ this expression is strictly positive.

Lemma 6.12. *Let $M \in \mathbb{R}^{N\times N}_{sym}$ with $M > 0$ as a quadratic form.*
Given n vectors $v_1,\dots,v_n \in \mathbb{R}^N$ we define $K \in \mathbb{R}^{n\times n}_{sym}$ via

$$K_{ij} := (v_i, M^{-1}v_j).$$

Let P_{v_j} be the projection on v_j defined by $P_{v_j}(v) := m(v,v_j)v_j$. The following holds.

(i) $M - \sum_{i=1}^n P_{v_i} \geq 0 \Leftrightarrow 0 \leq K \leq \mathrm{Id}$, *where the inequalities are intended as quadratic forms,*

(ii) $\det(M - \sum_{i=1}^n P_{v_i}) = \det M\ \det(1-K)$.

Proof.
(i) Since M is real and symmetric, it is diagonalizable with $M = U\hat{\lambda}U^t$ where $\hat{\lambda} = \mathrm{diag}\,(\lambda_1,\dots,\lambda_n)$ and $U^tU = 1$. Since $M > 0$ all eigenvalues of M are strictly positive and hence

$$M^{\frac{1}{2}} := U\hat{\lambda}^{\frac{1}{2}}U^t$$

is well defined and strictly positive. Set $w_i := M^{-\frac{1}{2}}v_i$. We argue

$$M - \sum_{i=1}^n P_{v_i} \geq 0 \quad\Leftrightarrow\quad (v,Mv) - \sum_{i=1}^n (v,v_i)^2 \geq 0 \quad \forall v \in \mathbb{R}^n$$
$$\Leftrightarrow\quad |M^{\frac{1}{2}}v|^2 - \sum_{i=1}^n (M^{\frac{1}{2}}v,w_i)^2 \geq 0 \quad \forall v \in \mathbb{R}^n \quad\Leftrightarrow\quad |v|^2 - \sum_{i=1}^n (v,w_i)^2 \geq 0 \quad \forall v \in \mathbb{R}^n$$
$$\Leftrightarrow\quad \mathrm{Id} - \sum_{i=1}^n P_{w_i} \geq 0.$$

Each vector $v \in \mathbb{R}^n$ can be decomposed as $v = \sum_{i=1}^n \alpha_i w_i + v^\perp$ where $\alpha_1,\dots\alpha_n \in \mathbb{R}$ and $(v^\perp,w_i) = 0\ \forall i = 1,\dots,n$. Therefore

$$|v^2| - \sum_{i=1}^n (v,w_i)^2 = |v^\perp|^2 + |\sum_{i=1}^n \alpha_i w_i|^2 - \sum_{i=1}^n (\sum_{j=1}^n \alpha_j w_j, w_i)^2 \geq |\sum_{i=1}^n \alpha_i w_i|^2 - \sum_{i=1}^n (\sum_{j=1}^n \alpha_j w_j, w_i)^2.$$

Therefore it is sufficient to consider $v = \sum_{i=1}^{n} \alpha_i w_i$. We argue

$$|\sum_{i=1}^{n} \alpha_i w_i|^2 = \sum_{ij} \alpha_i \alpha_j (w_i, w_j) = \sum_{ij} \alpha_i \alpha_j (v_i, M^{-1} v_j) = (\alpha, K\alpha),$$
$$\sum_{i=1}^{n} (\sum_{j=1}^{n} \alpha_j w_j, w_i)^2 = \sum_{jk} \alpha_j \alpha_k \sum_i (w_j, w_i)(w_i, w_k) = (\alpha, K^2 \alpha).$$

Hence

$$\mathrm{Id} - \sum_{i=1}^{n} P_{w_i} \geq 0 \qquad \Leftrightarrow \qquad K - K^2 \geq 0.$$

Sice K is real symmetric, it is diagonalizable and admits an orthonormal basis of eigenvectors. Therefore

$$K - K^2 \geq 0 \qquad \Leftrightarrow \qquad \mu_j - \mu_j^2 \geq 0$$

for each eigenvalue μ_j. This is possible only if $0 \leq \mu_j \leq 1$ $\forall j = 1, \ldots, n$. The result follows.

(ii) We argue,

$$\det(M - \sum_{i=1}^{n} P_{v_i}) = \det M \ \det\left(1 - \sum_{i=1}^{n} M^{-\frac{1}{2}} P_{v_i} M^{-\frac{1}{2}}\right)$$

We consider now the two functions $F_1, F_2 \colon \mathbb{R} \to \mathbb{R}$ defined by

$$F_1(t) := \det\left(1 - t\sum_{i=1}^{n} M^{-\frac{1}{2}} P_{v_i} M^{-\frac{1}{2}}\right), \qquad F_2(t) := \det(1 - tK),$$

and prove that there is a $\gamma > 0$ such that

$$F_1(t) = F_2(t) \qquad \forall |t| < \gamma.$$

The identity for general t follows noting that both functions are polynomes in t.

There is a $\delta > 0$ such that

$$F_j(t) > 0 \quad \forall t \in [-\delta, \delta] \ \forall j = 1, 2.$$

Set $X := \sum_{i=1}^{n} M^{-\frac{1}{2}} P_{v_i} M^{-\frac{1}{2}}$. For $|t| < \min\{\delta, \|X\|^{-1}\}$, we have

$$\ln F_1(t) = \ln\det(1 - tX) = \mathrm{tr}\ \ln(1 - tX) = -\sum_{q \geq 0} \frac{t^q}{q} \mathrm{tr}\, X^q.$$

The series above is absolutely convergent since $|t| < \|X\|^{-1}$, with $\|X\| := \sup_j \sum_{j'} |X_{jj'}|$. By direct computation we obtain

$$\mathrm{tr}\, X^q = \mathrm{tr}\, K^q \qquad \forall q,$$

hence

$$\ln F_1(t) = -\sum_{q \geq 0} \frac{t^q}{q} \mathrm{tr}\, X^q = -\sum_{q \geq 0} \frac{t^q}{q} \mathrm{tr}\, K^q = \ln F_2(t).$$

Therefore $F_1(t) = F_2(t)$ $\forall |t| < \min\{\delta, \|X\|^{-1}\}$. This concludes the proof. □

6.4.1 One pair with positive W

Proposition 6.13. *Fix an ordering on the edges in $E >_\Lambda$.*
For each $e = \{ij\} \in E_\Lambda$ with $i < j$ and each $u \in \mathbb{R}^\Lambda$ let $V_e = V_e(u) := e^{\frac{u_i+u_j}{2}}(\delta_i - \delta_j) \in \mathbb{R}^\Lambda$.
If $e \in E_\Lambda^W$ (i.e. $W_e > 0$) it holds

$$0 < (V_e, D(u)^{-1}V_e) \leq \frac{1}{W_e} \qquad \forall u \in \mathbb{R}^\Lambda.$$

Proof. Set $\tilde{V}_e := \sqrt{W_e}V_e$. Then

$$(V_e, D(u)^{-1}V_e) \leq \frac{1}{W_e} \quad \Leftrightarrow \quad (\tilde{V}_e, D(u)^{-1}\tilde{V}_e) \quad \Leftrightarrow \quad D - P_{\tilde{V}_e} \geq 0,$$

where in the last step we used Lemma 6.12. We have

$$(v, P_{\tilde{V}_e}v) = (v, \tilde{V}_e)^2 = W_{ij}e^{u_i+u_j}(v_i - v_j)^2.$$

Therefore

$$\begin{aligned}(v, (D - P_{\tilde{V}_e})v) = (v, Dv) - (v, \tilde{V}_e)^2 &= \sum_{kl \in E_\Lambda^W} W_{kl}e^{u_k+u_l}(v_k - v_l)^2 + \sum_{k\in\Lambda} \varepsilon_k e^{u_k} v_k^2 - W_{ij}e^{u_i+u_j}(v_i - v_j)^2 \\ &\geq \sum_{kl\in E_\Lambda^W \setminus\{ij\}} W_{kl}e^{u_k+u_l}(v_k - v_l)^2 \geq 0, \end{aligned} \tag{6.20}$$

where in the last step we used $e \in E_\Lambda^W$. This concludes the proof. □

Corollary 6.14. *For each $e \in E_\Lambda^W$ (i.e. $W_e > 0$) it holds*

$$\mathbb{E}^{u,s}[B_e^m] \leq \frac{1}{1 - \frac{m}{W_e}} \qquad \forall 0 < m < W_e.$$

Remark. The bound holds *uniformly* in the volume $|\Lambda|$.

Proof. By Proposition 6.11 we have

$$1 = \mathbb{E}^{u,s}\left[B_e^m(1 - m(F_e, D^{-1}F_e))\right].$$

Proposition 6.13 and $B_e \geq 1$, yield

$$(F_e, D^{-1}F_e) = \frac{1}{B_e}(V_e, D^{-1}V_e) \leq \frac{1}{B_e W_e} \leq \frac{1}{W_e}.$$

Putting all this together we obtain

$$1 = \mathbb{E}^{u,s}\left[B_e^m(1 - m(F_e, D^{-1}F_e))\right] \geq \left(1 - \frac{m}{W_e}\right)\mathbb{E}^{u,s}[B_e^m],$$

from which the result follows.

□

Using this estimate we can prove probability bounds.
For any $\mathcal{E}$ measurable set in $\mathbb{R}^\Lambda$, its probability is defined as

$$\mathbb{P}_\Lambda^{W,\varepsilon}(\mathcal{E}) := \mathbb{E}^u[1_\mathcal{E}(u)] = \rho_\Lambda^{W,\varepsilon}(\mathcal{E}).$$

Theorem 6.15. *Let $e = \{ij\} \in E_\Lambda^W$. Then, for all $\delta > 0$ we have*

$$\mathbb{P}_\Lambda^{W,\varepsilon}\Big(\cosh(u_i - u_j) \ge 1 + \delta\Big) \le \frac{2}{(1+\delta)^{\frac{W_e}{2}}}. \tag{6.21}$$

In particular, for any $0 < \eta < 1$ and $W_e > 1$ we have

$$\mathbb{P}_\Lambda^{W,\varepsilon}\Big(\cosh(u_i - u_j) \ge 1 + \frac{1}{W_e^\eta}\Big) \le 2\ e^{-\frac{W^{1-\eta}}{4}}. \tag{6.22}$$

Proof. It holds

$$1_{[1+\delta,\infty)}(x) \le \left(\frac{x}{1+\delta}\right)^m \qquad \forall m > 0$$

pointwise. Assume now $0 < m < W_e$. Inserting this in the probability we obtain

$$\begin{aligned}\mathbb{P}_\Lambda^{W,\varepsilon}\Big(\cosh(u_i - u_j) \ge 1 + \delta\Big) &\le \mathbb{E}^u\left[\left(\frac{\cosh(u_i - u_j)}{1+\delta}\right)^m\right] \\ &= \frac{1}{(1+\delta)^m}\mathbb{E}^u\left[\cosh(u_i - u_j)^m\right] \le \frac{1}{(1+\delta)^m}\mathbb{E}^{u,s}\left[B_{ij}^m\right] \le \frac{1}{1 - \frac{m}{W_{ij}}}\frac{1}{(1+\delta)^m},\end{aligned}$$

where in the last two steps we used $\cosh(u_i - u_j) \le B_{ij}$ pointwise, $\mathbb{E}^u\left[\cosh(u_i - u_j)^m\right] = \mathbb{E}^{u,s}\left[\cosh(u_i - u_j)^m\right]$ and Proposition 6.14, which is applicable since $e = \{ij\} \in E_\Lambda^W$ and we assumed $0 < m < W_e$. Taking $m := \frac{W_e}{2}$ yields (6.21). To prove (6.22) set $\delta = W^{-\eta}$ and argue

$$\frac{1}{(1+\delta)^{\frac{W_e}{2}}} = e^{-\frac{W_e}{2}\ln(1+\delta)} \le e^{-\frac{W_e}{2}\frac{1}{2W^\eta}}$$

where we used $\ln(1+\delta) \ge \frac{1}{2}\delta$ for $0 < \delta < 1$.

□

[26: 24.01.2025]

[27: 28.01.2025]

6.4.2 Many pairs with positive W

Theorem 6.16. *For each $e \in E_\Lambda^W$ we introduce a power $m_e \in [0, W_e)$. Then it holds*

$$\mathbb{E}^{u,s}\left[\prod_{e \in E_\Lambda^W} B_e^{m_e}\right] \le \prod_{e \in E_\Lambda^W} \frac{1}{1 - \frac{m_e}{W_e}}$$

Proof. Fix an ordering on Λ and for $e \in E_\Lambda$ set $e = \{i_e j_e\}$ with $i_e < j_e$. Define $V_e := V_{i_e j_e}$, $F_e := F_{i_e j_e}$. We introduce the two matrices $M, K \in \mathbb{R}_{sym}^{E_\Lambda^W \times E_\Lambda^W}$ defined by

$$K_{ee'} = K_{ee'}(u,s) := (F_e, D^{-1}F_{e'}), \qquad M_{ee'} =: \delta_{ee'}m_e.$$

Using the same arguments as in Proposition 6.11, we argue

$$\begin{aligned}1 = \left\langle \prod_{e \in E_\Lambda^W} (-(v_{i_e}, v_{j_e}))^{m_e} \right\rangle_{eucl} &= \left\langle \prod_{e \in E_\Lambda^W} (B_{i_e j_e} + (\overline{\psi}, V_e)(V_e, \psi))^{m_e} \right\rangle_{hor} \\ = \mathbb{E}^{u,s}\left[\prod_{e \in E_\Lambda^W} B_e^{m_e}\ \det\Big(\mathrm{Id} - \sqrt{M}K\sqrt{M}\Big)\right],\end{aligned}$$

where in the first step we used the localization theorem 5.11, in the second step we passed to horospherical coordinates and in the last step we integrated out the Grassmann variables and used

$$\det\left(D-\sum_{e\in E_\Lambda^W} m_e F_e\otimes F_e\right)=\det\left(D-\sum_{e\in E_\Lambda^W}\sqrt{m_e}F_e\otimes F_e\sqrt{m_e}\right)=\det D\ \det(\mathrm{Id}-\sqrt{M}K\sqrt{M})$$

The result now follows from the following two claims.

Claim 1. Set $\hat{W}\in\mathbb{R}^{E_\Lambda^W\times E_\Lambda^W}$ with $\hat{W}_{ee'}:=\delta_{ee'}W_e$. Then

$$K\leq\frac{1}{\hat{W}} \tag{6.23}$$

as a quadratic form, for almost every configuration $(u,s)\in\mathbb{R}^{2\Lambda}$

Claim 2. Let $A,B\in\mathbb{R}^{N\times N}_{sym}$ two given matrices with $A>B>0$ as a quadratic form. Then

$$\det A\geq\det B. \tag{6.24}$$

We show how the two Claims imply the result.
By Claim 1 we have

$$\sqrt{M}K\sqrt{M}\leq\sqrt{M}\frac{1}{\hat{W}}\sqrt{M}=\frac{M}{\hat{W}}=\mathrm{diag}\left(\frac{m_e}{W_e}\right).$$

Since $m_e<W_e$ we also have $\frac{M}{\hat{W}}<\mathrm{Id}$ and hence

$$\mathrm{Id}\,-\sqrt{M}K\sqrt{M}\geq\mathrm{Id}\,-\frac{M}{\hat{W}}>0.$$

Then, by Claim 2, we have

$$\det\left(\mathrm{Id}\,-\sqrt{M}K\sqrt{M}\right)\geq\det\left(\mathrm{Id}\,-\frac{M}{\hat{W}}\right)=\prod_{e\in E_\Lambda^W}\left(1-\frac{m_e}{W_e}\right)$$

pointwise a.s. This concludes the proof of the Theorem.

□

Proof of Claim 1. We argue

$$K\leq\frac{1}{\hat{W}}\quad\Leftrightarrow\quad\sqrt{\hat{W}}K\sqrt{\hat{W}}\leq\mathrm{Id}\quad\Leftrightarrow\quad D-\sum_{e\in E_\Lambda^W}P_{\sqrt{W_eF_e}}\geq 0,$$

where the last ⇔ holds by Lemma 6.12. We compute

$$(v,(D-\sum_{e\in E_\Lambda^W}P_{\sqrt{W_eF_e}})v)=\sum_{j\in\Lambda}\varepsilon_j e^{u_j}v_j^2\geq 0\qquad\forall v\in\mathbb{R}^\Lambda.$$

This completes the proof of the claim. □

Proof of Claim 2. Since A, B are real, symmetric and positive, they are both diagonalizable with real eigenvalues $a_1 \geq a_2 \geq \cdots \geq a_N > 0$, $b_1 \geq b_2 \geq \cdots \geq b_N > 0$. By minmax theorem for each $k = 1, \dots, N$,

$$a_k = \max_{\mathcal{M} \in \mathbb{R}^N,\ \dim M = k} \min_{v \in \mathcal{M}} (v, A, v), \qquad b_k = \max_{\mathcal{M} \in \mathbb{R}^N,\ \dim M = k} \min_{v \in \mathcal{M}} (v, B, v).$$

Since $A \geq B$ we argue $(v, Av) \geq (v, Bv)$ $\forall v$, hence $a_k \geq b_k$ $\forall k = 1, \dots, N$. Since in addition $B > 0$ we have $a_k \geq b_k > 0$ for all k and therefore

$$\det A = \prod_{k=1}^{N} a_k \geq \prod_{k=1}^{N} b_k = \det B.$$

□

Corollary 6.17. *For each $e \in E_\Lambda^W$ we introduce a parameter $\gamma_e \geq 0$. It holds*

$$\mathbb{P}_\Lambda^{W,\varepsilon}\Big(|u_{i_e} - u_{j_e}| \geq \sqrt{\gamma_e},\ \forall e \in E_\Lambda^W \Big) \leq \prod_{e \in E_\Lambda^W, s.t. \gamma_e > 0} \frac{2}{\left(1 + \frac{\gamma_e}{2}\right)^{\frac{W_e}{2}}}.$$

Proof. For each $x \in \mathbb{R}$ we have

$$\cosh x - 1 = \frac{x^2}{2} + \sum_{n \geq 2} \frac{x^{2n}}{(2n)!} \geq \frac{x^2}{2},$$

hence

$$1 + \frac{(u_i - u_j)^2}{2} \leq \cosh(u_i - u_j) \leq B_{ij} = B_{ij}(u, s) \qquad \forall i, j \in \Lambda, \forall u, s \in \mathbb{R}^{2\Lambda}.$$

Inserting this bound in the probability above we obtain

$$\mathbb{P}_\Lambda^{W,\varepsilon}\Big(|u_{i_e} - u_{j_e}| \geq \sqrt{\gamma_e},\ \forall e \in E_\Lambda^W \Big) \leq \mathbb{E}^{u,s}\left[\prod_{e \in E_\Lambda^W} \left(\frac{1 + \frac{(u_{i_e} - u_{j_e})^2}{2}}{1 + \frac{\gamma_e}{2}} \right)^{m_e} \right] \leq \frac{\mathbb{E}^{u,s}\left[\prod_{e \in E_\Lambda^W} B_{i_e j_e}^{m_e} \right]}{\prod_{e \in E_\Lambda^W} \left(1 + \frac{\gamma_e}{2}\right)^{m_e}},$$

where we take $m_e > 0$ $\forall e$ such that $\gamma_e > 0$ and $m_e = 0$ otherwise.
The result follows from Theorem 6.16 by setting $m_e := \frac{W_e}{2}$ $\forall e$ such that $\gamma_e > 0$. □

6.4.3 Pairs with zero W

We consider now the case of a pair $\{ij\} \notin E_\Lambda^W$. To simplify the notation, in this section we consider only the case of *uniform nearest-neighbor* interaction W, i.e.

$$W_{ij} = \begin{cases} W & \forall |i-j| = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{6.25}$$

Remember that, by Proposition 6.11,

$$\mathbb{E}^{u,s}\left[B_{ij}^m \big(1 - m(F_{ij}, D^{-1} F_{ij})\big) \right] = 1$$

for any $m \in \mathbb{R}$ and for any $i, j \in \Lambda$. Setting $C > 0$ a constant we argue, using Lemma 6.12,

$$(F_{ij}, D^{-1} F_{ij}) \leq \frac{1}{C} \quad \Leftrightarrow \quad D - P_{\sqrt{C} F_{ij}} \geq 0.$$

We compute, as in (6.20),

$$(v,(D-P_{\sqrt{C}F_{ij}})v) = (v,Dv) - C\,(v,F_{ij})^2 = \sum_{kl\in E_\Lambda^W} W_{kl} e^{u_k+u_l}(v_k-v_l)^2$$
$$+\sum_{k\in\Lambda} \varepsilon_k e^{u_k} v_k^2 - \frac{C}{B_{ij}} e^{u_i+u_j}(v_i-v_j)^2$$
$$= W\left(\sum_{kl\in E_\Lambda^W} W_{kl} e^{u_k+u_l}(v_k-v_l)^2\right) - \frac{C}{B_{ij}} e^{u_i+u_j}(v_i-v_j)^2 + \sum_{k\in\Lambda} \varepsilon_k e^{u_k} v_k^2,$$

where in the last step we used $W_{kl} = W\mathbf{1}_{|k-l|=1}$.

- If $|i-j|=1$, then we have $D - P_{\sqrt{C}F_{ij}} \geq 0$ for any $0 < C \leq W$.
- If $|i-j| > 1$, then for any given $C > 0$ there may be u, s configurations such that $D - P_{\sqrt{C}F_{ij}} < 0$.

In the case $|i-j|>1$ we argue in three steps.

Step 1 We identify a constant $C = C_{ij} > 0$ and a subset $\mathcal{A}_{ij} \subset \mathbb{R}^{2\Lambda}$ of configurations such that $(F_{ij}, D^{-1}F_{ij}) \leq \frac{1}{C}$ $\forall (u,s) \in \mathcal{A}_{ij}$. As a result

$$\mathbb{E}^{u,s}\left[\mathbf{1}_{\mathcal{A}_{ij}}(u,s)\, B_{ij}^m\big(1 - m(F_{ij}, D^{-1}F_{ij})\big)\right] \geq \left(1 - \frac{m}{C}\right)\mathbb{E}^{u,s}\left[\mathbf{1}_{\mathcal{A}_{ij}}(u,s)\, B_{ij}^m\right] > 0,$$

for $m > C$.

Step 2 We reformulate the characteristic function $\mathbf{1}_{\mathcal{A}_{ij}}$ in such a way to be able to apply the localization theorem 5.11. As a result we will show (see below)

$$1 \geq \mathbb{E}^{u,s}\left[\mathbf{1}_{\mathcal{A}_{ij}}(u,s)\, B_{ij}^m\big(1 - m(F_{ij}, D^{-1}F_{ij})\big)\right].$$

Step 3 We use other arguments to show

$$\mathbb{E}^{u,s}\left[\mathbf{1}_{\mathcal{A}_{ij}^c}(u,s)\, B_{ij}^m\right] \ll 1.$$

Putting these three steps together we obtain $\mathbb{E}^{u,s}\left[\mathbf{1}_{\mathcal{A}_{ij}^c}(u,s)\, B_{ij}^m\right] \leq \mathit{const}$, which allows to derive probability bounds similar to the ones proved in Corollary 6.17. We will prove these three steps in the case when $d=1$ and W is large enough.

Heuristics If $W \gg 1$ we expect the dominant configuration is $u_j = u$ and $s_j = s$ $\forall j \in \Lambda$. On this configuration we have $B_{ij} = 1$ and

$$D = We^{2u}(-\Delta) + e^u\hat{\varepsilon} = We^{2u}\left(-\Delta + \frac{e^{-u}}{W}\hat{\varepsilon}\right),$$

where $\hat{\varepsilon} = \mathrm{diag}\,(\{\varepsilon_j\}_{j\in\Lambda})$. Therefore

$$(F_{ij}, D^{-1}F_{ij}) = \frac{1}{W}\left((\delta_i - \delta_j), \left(-\Delta + \frac{e^{-u}}{W}\hat{\varepsilon}\right)^{-1}(\delta_i - \delta_j)\right) \leq \frac{1}{W}\left((\delta_i-\delta_j), (-\Delta)^{-1}(\delta_i - \delta_j)\right).$$

Note that although $-\Delta$ has a zero eigenvalue and hence is not invertible, it is invertible when restricted to the subspace

$$1^{\perp} = \{\varphi \in \mathbb{R}^{\Lambda} | \, (\varphi, 1) = \sum_j \varphi_j = 0\}. \tag{6.26}$$

Since $\delta_i - \delta_j \in 1^{\perp}$, $\Big((\delta_i - \delta_j), (-\Delta)^{-1}(\delta_i - \delta_j)\Big)$ is well defined and finite. By general properties of the infinite volume discrete Laplacian it holds

$$\lim_{\Lambda \to \mathbb{Z}^d} \Big((\delta_i - \delta_j), (-\Delta)^{-1}(\delta_i - \delta_j)\Big) \overset{|i-j| \gg 1}{\simeq} \begin{cases} |i-j| & d = 1 \\ \ln|i-j| & d = 2 \\ C_d & d \geq 3 \end{cases}$$

where $C_d > 0$ is a constant depending only on the dimension. Hence, for $d \geq 3$ we argue, on the constant configuration and for large enough volume,

$$(F_{ij}, D^{-1}F_{ij}) \geq \frac{C_d}{W} \quad \Rightarrow \quad 1 - m(F_{ij}, D^{-1}F_{ij}) \geq 1 - \frac{mC_d}{W} > 0 \quad \forall 0 < m < \frac{W}{C_d}, \forall i, j \in \Lambda.$$

If $W \gg 1$ the bound above holds for large power m. On the contrary, for $d = 1$ we get

$$(F_{ij}, D^{-1}F_{ij}) \geq \frac{C_d}{W} \quad \Rightarrow \quad 1 - m(F_{ij}, D^{-1}F_{ij}) \geq 1 - \frac{m|i-j|}{W} > 0 \quad \forall 0 < m < \frac{W}{C_d|i-j|}.$$

Therefore we can take the power m large only if $W/|i-j| \gg 1$ i.e. for $|i-j| \ll W$.

[27: 28.01.2025]

[28: 31.01.2025]

Fluctuation bounds in the case of dimension 1 We assume now $\Lambda = \Lambda_L = [-L, L] \cap \mathbb{Z}$.

Proposition 6.18. *Let $\omega = \{\omega_{ij}\}_{\{ij\} \in E_\Lambda} \in [0, \infty)^{E_\Lambda}$ a family of edge weights such that $\omega_{ij} = \omega_{ji} > 0$ iff $|i-j| = 1$.*
Remember that the corresponding weighted discrete Laplacian $-\Delta^{\omega}$ is defined by

$$(\varphi, (-\Delta^{\omega})\varphi) = \sum_{q=-L}^{L-1} \omega_{qq+1}(\varphi_q - \varphi_{q+1})^2.$$

(i) On the set $1^{\perp}$ defined in (6.26), the matrix $-\Delta^{\omega}$ is invertible.

(ii) $\forall i < j \in \Lambda$ it holds

$$\Big((\delta_i - \delta_j), (-\Delta^{\omega})^{-1}(\delta_i - \delta_j)\Big) \leq \sum_{q=i}^{j-1} \frac{1}{\omega_{qq+1}}.$$

Proof.
(i) exercise

(ii) Set $C := \sum_{q=i}^{j-1} \frac{1}{\omega_{qq+1}}$. We argue, using Lemma 6.12,

$$\Big((\delta_i - \delta_j), (-\Delta^\omega)^{-1} (\delta_i - \delta_j)\Big) \leq C \quad \Leftrightarrow \quad -\Delta^\omega - \frac{1}{C} P_{\delta_i - \delta_j} \geq 0$$
$$\Leftrightarrow \quad (\varphi, (-\Delta^\omega)\varphi) - \frac{1}{C}(\varphi_i - \varphi_j)^2 \geq 0 \quad \forall \varphi \in \mathbb{R}^\Lambda.$$

We compute

$$\varphi_i - \varphi_j = \sum_{q=i}^{j-1} \varphi_q - \varphi_{q+1} = \sum_{q=i}^{j-1} (\varphi_q - \varphi_{q+1}) \sqrt{\omega_{qq+1}} \frac{1}{\sqrt{\omega_{qq+1}}} = V \cdot \tilde{V},$$

where $V, \tilde{V} \in \mathbb{R}^{\{i,\ldots,j-1\}}$ are defined as

$$V_q := (\varphi_q - \varphi_{q+1}) \sqrt{\omega_{qq+1}}, \qquad \tilde{V}_q := \frac{1}{\sqrt{\omega_{qq+1}}}.$$

It follows, by Cauchy-Schwartz inequality,

$$(\varphi_i - \varphi_j)^2 = (V \cdot \tilde{V})^2 \leq |\tilde{V}|^2 |V|^2 = C \left[\sum_{q=i}^{j-1} \omega_{qq+1} (\varphi_q - \varphi_{q+1})^2 \right].$$

We also have

$$\varphi, (-\Delta^\omega)\varphi) = \sum_{q=-L}^{L-1} \omega_{qq+1} (\varphi_q - \varphi_{q+1})^2 = \sum_{q=i}^{j-1} \omega_{qq+1} (\varphi_q - \varphi_{q+1})^2 + \sum_{q<i, q\geq j} \omega_{qq+1} (\varphi_q - \varphi_{q+1})^2.$$

Putting all this together we obtain

$$(\varphi, (-\Delta^\omega)\varphi) - \frac{1}{C}(\varphi_i - \varphi_j)^2 \geq \sum_{q<i, q\geq j} \omega_{qq+1} (\varphi_q - \varphi_{q+1})^2 \geq 0.$$

This completes the proof of the proposition. □

Remark Remember the definition of D in (6.3). For $i < j$ we argue, using Proposition 6.18 above,

$$\begin{aligned}
(F_{ij}, D^{-1} F_{ij}) &= \frac{e^{u_i + u_j}}{B_{ij}} \left((\delta_i - \delta_j), \left(-\Delta^{\omega(u)} + \hat{\varepsilon} e^{\hat{u}} \right)^{-1} (\delta_i - \delta_j) \right) \\
&\leq e^{u_i + u_j} \left((\delta_i - \delta_j), \left(-\Delta^{\omega(u)} \right)^{-1} (\delta_i - \delta_j) \right) \\
&\leq \frac{e^{u_i + u_j}}{W} \sum_{q=i}^{j-1} \frac{1}{e^{u_q + u_{q+1}}} = \frac{1}{W} \sum_{q=i}^{j-1} e^{u_i - u_q} e^{u_j - u_{q+1}} \leq \frac{1}{W} \sum_{q=i}^{j-1} e^{|u_i - u_q|} e^{|u_j - u_{q+1}|}.
\end{aligned}$$

It follows, using $|u_i - u_q| + |u_j - u_{q+1}| \leq \sum_{k=i}^{j-1} |u_k - u_{k+1}| \ \forall q = i, \ldots, j-1$,

$$(F_{ij}, D^{-1} F_{ij}) \leq \frac{|j-i|}{W} \ e^{\sum_{k=i}^{j-1} |u_k - u_{k+1}|}. \tag{6.27}$$

In particular, if $|u_k - u_{k+1}| < \sqrt{\gamma}\ \forall k = i, \dots, j-1$ we obtain the bound

$$(F_{ij}, D^{-1}F_{ij}) \leq \frac{|j-i|}{W}\ e^{|j-i|\sqrt{\gamma}}.$$

We formulate now the constraint $|u_k - u_{k+1}| < \sqrt{\gamma}$ in terms of B_{kk+1} as follows. Since

$$1 + \frac{(u_k - u_{k+1})^2}{2} \leq B_{kk+1},$$

it holds

$$B_{kk+1} \leq 1 + \frac{\gamma}{2} \quad \Rightarrow \quad |u_k - u_{k+1}| \leq \sqrt{\gamma}.$$

Let $0 < \gamma < 1$ be a parameter. For $i < j$ we define

$$\mathcal{A}_{ij}^{\gamma} := \left\{ (u,s) \in \mathbb{R}^{2\Lambda} |\ B_{kk+1} \leq 1 + \frac{\gamma}{2}\ \forall k = i, \dots, j-1 \right\} \tag{6.28}$$

$$\chi_{ij}^{\gamma} := \mathbf{1}_{\mathcal{A}_{ij}^{\gamma}} = \prod_{k=i}^{j-1} \mathbf{1}_{B_{kk+1} \leq 1 + \frac{\gamma}{2}}. \tag{6.29}$$

Theorem 6.19 (constrained estimate)**.** *Remember that, for each $i < j \in \Lambda_L = [-L, L] \cap \mathbb{Z}$, the notation $\{i,j\}$ denotes an unordered edge in E^{Λ} while (i,j) denotes an open interval in $\mathbb{R}$. Let $e_1 = \{i_1, j_1\}, \dots e_n = \{i_n, j_n\} \in E^{\Lambda}$ be n not-nearest-neighbor pairs such that*

- $i_k < j_k$ *and* $j_k - i_k \geq 2\ \forall k = 1, \dots, n$,
- $(i_k, j_k) \cap (i_{k'}, j_{k'}) = \varnothing\ \forall k \neq k',\ k, k' = 1, \dots, n$.

Let $e'_1 = \{i'_1, i'_1 + 1\}, \dots e'_{n'} = \{i'_{n'}, i'_{n'} + 1\}$ be n' nearest-neighbor pairs such that

- $(i'_k, i'_k + 1) \cap (i'_{k'}, i'_{k'} + 1) = \varnothing\ \forall k \neq k'\ k, k' = 1, \dots, n'$,
- $(i'_k, i'_k + 1) \cap (i_{k'}, j_{k'}) = \varnothing\ \forall k = 1, \dots, n', k' = 1, \dots, n'$.

Setting $\tilde{E} := (\cup_{k=1}^{n} (i_k, j_k)) \cup \left(\cup_{k'=1}^{n'} (i_{k'}, i_{k'} + 1) \right)$, we consider the matrices $K, \hat{C} \in \mathbb{R}^{\tilde{E} \times \tilde{E}}$ defined by

$$K_{ee'} := (F_e, D^{-1}F_{e'}), \qquad \hat{C}_{ee'} := \delta_{ee'} C_e,$$

$$C_e := \begin{cases} \frac{1}{W} \sum_{q=i_k}^{j_k - 1} e^{u_{i_k} - u_q} e^{u_{j_k} - u_{q+1}} & \text{if } e = e_k,\ k = 1, \dots, n \\ \frac{1}{W} & \text{if } e = e'_{k'},\ k' = 1, \dots, n'. \end{cases}$$

Note that K is a function of (u,s) and C is a function of u. The following statements hold.

(i) $K(u,s) \leq \hat{C}(u)$ *as a quadratic form for all* $(u,s) \in \mathbb{R}^{2\Lambda}$.

(ii) It holds

$$C_{e_k}(u) \leq C_k \qquad \forall (u,s) \in \cap_{k=1}^{n} \mathcal{A}_{i_k j_k}^{\gamma}$$

where, setting $l_k := j_k - i_k$, the constant C_k is defined as

$$C_k := \frac{l_k}{W} e^{l_k \sqrt{\gamma}}. \tag{6.30}$$

(iii) Fix $m_1, \ldots, m_n$ and $p_1, \ldots, p_{n'}$ such that

$$0 < m_k < \frac{1}{C_k} \quad \forall k = 1, \ldots, n$$
$$0 < p_{k'} < W \quad \forall k' = 1, \ldots, n'.$$

We consider the matrix $M \in \mathbb{R}^{\tilde{E} \times \tilde{E}}$ defined by $M_{ee'} = \delta_{ee'} M_e$ with

$$M_e := \begin{cases} m_k & \text{if } e = e_k,\ k = 1, \ldots, n \\ p_{k'} & \text{if } e = e'_{k'},\ k' = 1, \ldots, n'. \end{cases}$$

Remember the definition of χ^γ_{ij} in (6.28). It holds

(a) $1 \geq \mathbb{E}^{u,s}\left[\prod_{k=1}^n \chi^\gamma_{i_k j_k} B^{m_k}_{i_k j_k} \ \prod_{k'=1}^{n'} B^{p_{k'}}_{i_{k'} i_{k'}+1} \ \det\left(1 - \sqrt{M} K \sqrt{M}\right)\right]$

(b) $\mathbb{E}^{u,s}\left[\prod_{k=1}^n \chi^\gamma_{i_k j_k} B^{m_k}_{i_k j_k} \ \prod_{k'=1}^{n'} B^{p_{k'}}_{i_{k'} i_{k'}+1}\right] \leq \prod_{k=1}^n \frac{1}{1 - m_k C_k} \prod_{k'=1}^{n'} \frac{1}{1 - \frac{p_{k'}}{W}}$

Proof.
$(i), (ii)$ exercise

(iii) To simplify the formulas we assume $n = 1$ and $n' = 0$. In this case we study

$$\mathbb{E}^{u,s}\left[\chi^\gamma_{ij} B^m_{ij} \left(1 - m(F_{ij}, D^{-1} F_{ij})\right)\right].$$

We have

$$\mathbf{1}_{B_{kk+1} \leq 1 + \frac{\gamma}{2}} = \mathbf{1}_{(-\infty, 1]}\left(\frac{B_{kk+1}}{1 + \frac{\gamma}{2}}\right).$$

The function $\mathbf{1}_{(-\infty,1]}$ is not C^1, hence we cannot upgrade it to a map on even elements of the Grassmann algebra. To solve the problem we introduce a smoothing as follows.
Let $\chi \in C^\infty(\mathbb{R}; [0,1])$ be a smooth function such that

$$\chi(x) = 1,\ \forall x \leq 1, \quad \chi(x) = 0, \forall x > 2, \qquad \chi'(x) \leq 0\ \forall x.$$

We consider the family $\{\chi_\varepsilon\}_{\varepsilon > 0}$ the family of smooth functions defined as

$$\chi_\varepsilon(x) := \chi\left(1 + \frac{x - 1}{\varepsilon}\right).$$

This function satisfies

- $\chi_\varepsilon(x) = 1\ \forall x \leq 1$, $\chi_\varepsilon(x) = 0\ \forall x > 1 + \varepsilon$ and $\lim_{\varepsilon \downarrow 0} \chi_\varepsilon = \mathbf{1}_{(-\infty,1]}$ pointwise,
- $\chi'_\varepsilon(x) \leq 0\ \forall x \in \mathbb{R}$.

With the above notation, it holds

$$\mathbf{1}_{(-\infty,1]}\left(\tfrac{B_{kk+1}(u,s)}{1 + \frac{\gamma}{2}}\right) = \lim_{\varepsilon \downarrow 0} \chi_\varepsilon\left(\tfrac{B_{kk+1}(u,s)}{1 + \frac{\gamma}{2}}\right)$$

pointwise for all $(u,s) \in \mathbb{R}^2$. Since $\chi_\varepsilon(x) \leq \mathbf{1}_{(-\infty,2]}(x)\ \forall x \in \mathbb{R}$ and $\forall \varepsilon > 0$, by dominated convergence we have

$$\mathbb{E}^{u,s}\left[B^m_{ij}\left(1 - m(F_{ij}, D^{-1} F_{ij})\right) \chi^\gamma_{ij}\right] = \lim_{\varepsilon \downarrow 0} \mathbb{E}^{u,s}\left[B^m_{ij}(1 - m(F_{ij}, D^{-1} F_{ij}) \prod_{k=i}^{j-1} \chi_\varepsilon\left(\tfrac{B_{kk+1}}{1 + \frac{\gamma}{2}}\right)\right].$$

Remember the definition of V_{ij} from (6.19) and v with its product (6.5)(6.6). Since χ_ε is differentiable, the function

$$\chi_\varepsilon\left(\tfrac{B_{kk+1}+(\overline{\psi},V_{ij})(V_{ij},\psi)}{1+\frac{\gamma}{2}}\right)=\chi_\varepsilon\left(\tfrac{B_{kk+1}}{1+\frac{\gamma}{2}}\right)+\chi_\varepsilon'\left(\tfrac{B_{kk+1}}{1+\frac{\gamma}{2}}\right)\tfrac{(\overline{\psi},V_{ij})(V_{ij},\psi)}{1+\frac{\gamma}{2}}$$

is well-defined. Moreover, since $\chi_\varepsilon(x)=1\ \forall x\leq 1$, we have

$$\chi_\varepsilon\left(\frac{1}{1+\frac{\gamma}{2}}\right)=1\qquad\forall\gamma\geq 0.$$

We argue, using Theorem 5.11,

$$\left\langle\Big(B_{ij}+(\overline{\psi},V_{ij})(V_{ij},\psi)\Big)^m\ \prod_{k=i}^{j-1}\chi_\varepsilon\left(\tfrac{B_{kk+1}+(\overline{\psi},V_{qq+1})(V_{qq+1},\psi)}{1+\frac{\gamma}{2}}\right)\right\rangle_{hor}$$
$$=\left\langle(-(v_i,v_j))^m\prod_{k=i}^{j-1}\chi_\varepsilon\left(\tfrac{-(v_k,v_{k+1})}{1+\frac{\gamma}{2}}\right)\right\rangle_{eucl}=\chi_\varepsilon\left(\frac{1}{1+\frac{\gamma}{2}}\right)^{j-i}=1.$$

Integrating out the Grassmann variables $\overline{\psi},\psi$ we obtain (exercise)

$$1=\mathbb{E}^{u,s}\left[B_{ij}^m\ \prod_{k=i}^{j-1}\chi_\varepsilon\left(\tfrac{B_{kk+1}}{1+\frac{\gamma}{2}}\right)\ \frac{\det\Big(D-\sum_{q=i}^{j-1}P_q-mP_{F_{ij}}\Big)}{\det D}\right].$$

where we defined

$$P_q:=\frac{\chi_\varepsilon'\left(\frac{B_{kk+1}}{1+\frac{\gamma}{2}}\right)}{\chi_\varepsilon\left(\frac{B_{kk+1}}{1+\frac{\gamma}{2}}\right)}P_{F_{qq+1}}.$$

Since $\chi'\leq 0$ and $P_{F_{qq+1}}\geq 0$ as a quadratic form we have

$$D-\sum_{q=i}^{j-1}P_q-mP_{F_{ij}}\geq D-mP_{F_{ij}}.$$

Setting

$$C:=\frac{l}{W}e^{l\sqrt{\gamma}},\qquad l:=j-i,$$

and using (i),(ii) and the assumption on $0<m<C^{-1}$ we have

$$m(F_{ij},D^{-1}F_{ij})\leq mC<1\qquad\forall(u,s)\in\mathcal{A}_{ij}^\gamma.$$

Hence, by Lemma 6.12, $D-mP_{F_{ij}}>0$ as a quadratic form. By Claim 2 in the proof of Theorem 6.16, we have

$$\det\left(D-\sum_{q=i}^{j-1}P_q-mP_{F_{ij}}\right)\ \geq\ \det\left(D-mP_{F_{ij}}\right)\ =\ \det D\ \Big(1-m(F_{ij},D^{-1}F_{ij})\Big),$$

on the (u,s) configurations in $\mathcal{A}_{ij}^\gamma$. It follows

$$1\geq\mathbb{E}^{u,s}\left[B_{ij}^m\ \prod_{k=i}^{j-1}\chi_\varepsilon\left(\tfrac{B_{kk+1}}{1+\frac{\gamma}{2}}\right)\ \Big(1-m(F_{ij},D^{-1}F_{ij})\Big)\right].$$

Taking the limit $\varepsilon \downarrow 0$ we obtain

$$1 \geq \mathbb{E}^{u,s}\left[B_{ij}^m \chi_{ij}^\gamma \left(1 - m(F_{ij}, D^{-1}F_{ij})\right)\right].$$

Finally, inserting $1 - m(F_{ij}, D^{-1}F_{ij}) \geq 1 - mC > 0$ for all $(u,s) \in \mathcal{A}_{ij}^\gamma$, we argue

$$1 \geq \mathbb{E}^{u,s}\left[B_{ij}^m \chi_{ij}^\gamma \left(1 - m(F_{ij}, D^{-1}F_{ij})\right)\right] \geq (1 - mC)\, \mathbb{E}^{u,s}\left[B_{ij}^m \chi_{ij}^\gamma\right],$$

and hence

$$\mathbb{E}^{u,s}\left[B_{ij}^m \chi_{ij}^\gamma\right] \leq \frac{1}{1 - mC}.$$

This proves (iii) in the special case $n = 1, n' = 0$. The general case is proved in the same way. □

Theorem 6.20 (unconstrained estimate)**.** *There exists a constant $W_0 > 0$ such that $\forall W \geq W_0$ it holds*

$$\mathbb{E}^{u,s}\left[B_{ij}^m\right] \leq 2$$

for all $m \leq W^{\frac{1}{4}}$ and all $i < j$ with $j - i = l \leq W^{\frac{1}{4}}$.

Proof. We argue, for any $\gamma > 0$,

$$1 = \chi_{ij}^\gamma + (1 - \chi_{ij}^\gamma) \leq \chi_{ij}^\gamma + \sum_{k=i}^{j-1} \mathbf{1}_{B_{kk+1} > 1 + \frac{\gamma}{2}} \leq \chi_{ij}^\gamma + \sum_{k=i}^{j-1} \frac{B_{kk+1}^p}{(1 + \frac{\gamma}{2})^p},$$

where the power $p \geq 1$ will be fixed later. Inserting this in the average above we obtain

$$\mathbb{E}^{u,s}\left[B_{ij}^m\right] \leq \mathbb{E}^{u,s}\left[B_{ij}^m \chi_{ij}^\gamma\right] + \sum_{k=i}^{j-1} \frac{1}{(1 + \frac{\gamma}{2})^p} \mathbb{E}^{u,s}\left[B_{ij}^m B_{kk+1}^p\right] \tag{6.31}$$

Remember $l = j - i \leq W^{\frac{1}{4}}$ and $m \leq W^{\frac{1}{4}}$. Using Theorem 6.19 we obtain

$$\mathbb{E}^{u,s}\left[B_{ij}^m \chi_{ij}^\gamma\right] \leq \frac{1}{1 - mC}.$$

where $C := \frac{l}{W} e^{l\sqrt{\gamma}}$ and the inequality above holds $\forall 0 < m < C^{-1}$. Set now

$$\gamma := \frac{c}{\sqrt{W}},$$

where $c > 0$ is a parameter (independent of W) to be fixed later. With this choice we get

$$C^{-1} = \frac{W}{l} e^{-l\sqrt{\gamma}} = \frac{W}{l} e^{-c} \geq W^{\frac{3}{4}} e^{-c}.$$

Hence $m < C^{-1}$ for all $m \leq W^{\frac{1}{4}}$, with W large enough. The bound above becomes

$$\mathbb{E}^{u,s}\left[B_{ij}^m \chi_{ij}^\gamma\right] \leq \frac{1}{1 - mC} \leq \frac{1}{1 - \frac{e^c}{\sqrt{W}}}.$$

To bound the sum in (6.31) we argue, using Lemma 6.21 below,

$$B_{ij} \leq 2^{j-i} \prod_{q=i}^{j-1} B_{qq+1}.$$

Therefore, for $q = i, \dots j-1$, we have

$$\mathbb{E}^{u,s}\left[B_{ij}^m B_{kk+1}^p\right] \;\le\; 2^{ml}\;\mathbb{E}^{u,s}\left[\prod_{q=i}^{k-1} B_{qq+1}^m \; B_{kk+1}^{p+m} \prod_{q=k+1}^{j-1} B_{qq+1}^m\right].$$

Setting $p = \frac{W}{4}$ we have $m < m+p < \frac{W}{2}$ and hence, by Theorem 6.16,

$$\mathbb{E}^{u,s}\left[\prod_{q=i}^{k-1} B_{qq+1}^m \; B_{kk+1}^{p+m} \prod_{q=k+1}^{j-1} B_{qq+1}^m\right] \le 2^l.$$

Putting all this together we obtain

$$\mathbb{E}^{u,s}\left[B_{ij}^m\right] \le \frac{1}{1-\frac{e^c}{\sqrt{W}}} + \frac{2^{ml}2^l l}{(1+\frac{\gamma}{2})^p}.$$

Note that

$$\left(1+\frac{\gamma}{2}\right)^{-p} = \left(1+\frac{c}{2\sqrt{W}}\right)^{-\frac{W}{4}} = e^{-\frac{W}{4}\ln(1+\frac{c}{2\sqrt{W}})} \le e^{-\sqrt{W}\frac{c}{16}}$$

and

$$2^l \le 2^{W^{\frac{1}{4}}} = e^{W^{\frac{1}{4}}\; ln2}, \qquad 2^{ml} \le 2^{\sqrt{W}} = e^{\sqrt{W}\; ln2}.$$

Setting $c = 32\; ln2$ we get

$$\frac{2^{ml}2^l l}{(1+\frac{\gamma}{2})^p} \le W^{\frac{1}{4}} e^{W^{\frac{1}{4}}\; ln2} e^{-\sqrt{W}(\frac{c}{16}-\; ln2)} = W^{\frac{1}{4}} e^{W^{\frac{1}{4}}\; ln2} e^{-\sqrt{W} ln2} \le \frac{1}{2}$$

for W large enouhg. So we choose W_0 such that the above bound holds and in addition

$$\frac{1}{1-\frac{e^c}{\sqrt{W}}} \le 1 + \frac{1}{2}$$

holds for all $W \ge W_0$. This concludes the proof.

□

Remark. Setting $W \gg 1$ the bound above imples, for all $|j-i| < W^{\frac{1}{4}}$ and $0 < \varepsilon < \frac{1}{4}$,

$$\mathbb{P}\left(|u_i - u_j| \ge W^{-\frac{1}{8}+\frac{\varepsilon}{2}}\right) \le \frac{\mathbb{E}^{u,s}\left[B_{ij}^{W^{\frac{1}{4}}}\right]}{\left(1+\frac{1}{2W^{\frac{1}{4}-\varepsilon}}\right)^{W^{\frac{1}{4}}}} \le\; 2e^{-W^{\frac{1}{4}}\frac{1}{4W^{\frac{1}{4}-\varepsilon}}} = 2\; e^{-\frac{W^\varepsilon}{4}} \ll 1,$$

where we used $\ln(1+\delta) \ge 1 + \frac{\delta}{2}$ for $0 < \delta < 1$.

Lemma 6.21. *For any three points $i, j, k \in \mathbb{Z}^d$ we have*

$$B_{ij} \;\le\; 2\; B_{ik} B_{kj} \qquad \forall (u,s) \in \mathbb{R}^{2\mathbb{Z}^d}.$$

Proof. Remember that $H^2 \cap \{z > 0\}$ can be parametrized via a vector $v = (x, y, z)$ where the independent variables are $x, y \in \mathbb{R}$, $z = \sqrt{1 + x^2 + y^2}$ and the bilinear form is

$$(v_i, v_j) = x_i x_j + y_i y_j - z_i z_j.$$

Passing to horospherical coordinates, the independent variables are $u, s \in \mathbb{R}$, and the bilinear form becomes

$$(v_i, v_j) = -B_{ij}.$$

The expression $B_{ij} = -(v_i, v_j) \geq 1$ has an interpretation as the hyperbolic cosine of the geodesic distance on H^2

$$\mathrm{dist}\,(v_i, v_j) = \cosh^{-1}(-(v_i, v_j)) = \cosh^{-1}(B_{ij}),$$

where $\mathrm{dist}\,(v_i, v_j)$ is the minimal length of any curve on H^2 connecting v_i and v_j. This distance satisfies the triangle inequality hence

$$\mathrm{dist}\,(v_i, v_j) \leq \mathrm{dist}\,(v_i, v_k) + \mathrm{dist}\,(v_k, v_j)$$

for all i, j, k. It follows, using $\cosh(a + b) \leq 2 \cosh a \cosh b$ for all $a, b \geq 0$,

$$B_{ij} = \cosh\Big(\mathrm{dist}\,(v_i, v_j)\Big) \leq \cosh\Big(\mathrm{dist}\,(v_i, v_k) + \mathrm{dist}\,(v_k, v_j)\Big) \leq 2 B_{ik} B_{kj},$$

which concludes the proof. $\square$