Chapter 3

Higher dimensional problems

In one dimension, the transfer matrix approach garantees the existence of the infinite volume limit, as long as the transfer operator is regular enough. When in addition we can show that this operator is "near" to the harmonic cristal, then we can obtain precise estimates of the limit.

In dimension larger than one, the transfer matrix approach does not apply but as in the 1d case, many techniques use some kind of comparison with the harmonic cristal. We will then start the chapter reviewing the results we obtained for the harmonic cristal in $d = 1$ with a different approach that, contrary to the transfer matrix, can be directly generalized to any dimension.

3.1 Gaussian integrals in 1d

3.1.1 The harmonic cristal as a gaussian integral

The Hamiltonian for the harmonic cristal we introduced in the previous chapter can be written as a quadratic form

$$\beta H_\Lambda^{(har)}(\phi) = (\phi, A_\Lambda^{(har)}\phi)_\Lambda = \sum_{j,k=-L}^{L} \phi_j A_{\Lambda\ jk}^{(har)} \phi_k = \sum_{j=-L}^{L-1} \beta(\phi_j - \phi_{j+1})^2 + \sum_{j=-L}^{L} m^2 \phi_j^2$$
$$= (\phi, -\beta\Delta_\Lambda\phi) + (\phi, m^2 \mathrm{I}_\Lambda \phi),$$

where $(\phi, \psi)_\Lambda = \sum_{j=-L}^{L} \phi_j \psi_j$ is the real euclidean scalar product on Λ and $-\Delta_\Lambda$ is the discrete Laplacian defined by

$$(-\Delta_\Lambda)_{ij} = \begin{cases} -1 & |i-j| = 1 \\ \sum_{k\in\Lambda,\ |k-j|=1} 1 & i = j \end{cases}$$

Inserting the boundary conditions the Hamiltonian becomes

$$\begin{array}{ll} \text{Dirichlet:} & H^D_\Lambda(\phi) = H^{har}_\Lambda(\phi) + \phi^2_{-L} + \phi^2_L = (\phi, [-\beta\Delta^D_\Lambda + m^2 \mathrm{I}_\Lambda]\phi) \\ \text{periodic:} & H^{per}_\Lambda(\phi) = H^{har}_\Lambda(\phi) + (\phi_L - \phi_{-L})^2 = (\phi, [-\beta\Delta^{per}_\Lambda + m^2 \mathrm{I}_\Lambda]\phi) \\ \text{Neuman:} & H^N_\Lambda(\phi) = H^{har}(\phi) = (\phi, [-\beta\Delta^N_\Lambda + m^2 \mathrm{I}_\Lambda]\phi) \end{array}$$

where

$$-\Delta^N = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \quad -\Delta^D = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$-\Delta^{per} = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

Note that

$$0 \leqslant (f, -\Delta^N f) \leqslant (f, -\Delta^{per} f) \leqslant 2(f, -\Delta^D f)$$

where in the last inequality we used $2(f^2_{-L} + f^2_L) \geqslant (f_{-L} - f_{-L})^2$. Moreover the constant vector is in the kernel of both Δ^N and Δ^{per}

$$-\Delta^N f = -\Delta^{per} f = 0 \quad \text{if} \quad f_j = f \ \forall j,$$

while $(f, -\Delta^D f) > 0 \ \forall f \in \mathbb{R}^\Lambda$. Therefore only the meaure $d\mu^D_\Lambda(\phi)$ with Dirichlet boundary conditions is well defined also for $m = 0$.

3.1.2 Gaussian integrals and correlations

In the following we will need some basic facts about gaussian meaures.

Lemma 1 *Let A be a $N \times N$ real symmetric matrix such that $A > 0$ as a quadratic form. Let $d\phi = \prod_{j=1}^N d\phi_j$ the Lebesgue measure on $\mathbb{R}^N$. Then*

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi, A\phi)} d\phi = \frac{(2\pi)^{N/2}}{\sqrt{\det A}}, \qquad \frac{\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi, A\phi)} \phi_{j_1}\phi_{j_2} d\phi}{\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi, A\phi)} d\phi} = A^{-1}_{ij}.$$

More generally let $j_1, \ldots, j_n \in \{1, \ldots, N\}$ n (not necessarily different) points. Then

$$\frac{\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi, A\phi)} \phi_{j_1}\phi_{j_2}\cdots\phi_{j_n} d\phi}{\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi, A\phi)} d\phi} = \begin{cases} 0 & n \ odd \\ \sum_P \prod_{(\alpha,\beta)\in P} A^{-1}_{j_\alpha j_\beta} & n = 2m \end{cases}$$

where P is a pairing of the set the set $\{1, \ldots, 2m\}$, i.e. a partition into m subsets of size 2.

Example

Proof Since A is real and symmetric, there exist an real orthogonal matrix U ($U^t = U^{-1}$) and a real diagonal matrix $\hat\lambda$ such that $A = U^t\hat\lambda U$ and

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi,A\phi)}d\phi = \int_{\mathbb{R}^N} e^{-\frac{1}{2}(U\phi,\hat\lambda U\phi)}d\phi = \int_{\mathbb{R}^N} e^{-\frac{1}{2}(\tilde\phi,\hat\lambda\tilde\phi)}|\det U^{-1}|d\tilde\phi$$

$$= \prod_{i=1}^N \int_{\mathbb{R}} e^{-\frac{1}{2}\lambda_i\tilde\phi_j^2}d\tilde\phi_i = \frac{(2\pi)^{N/2}}{\prod_{i=1}^N\sqrt{\lambda_i}} = \frac{(2\pi)^{N/2}}{\sqrt{\det A}}.$$

where we performed the change of variable $\tilde\phi = U\phi$ and we used $|\det U| = 1$. To prove the other relation we may use integration by parts. We have

$$\phi_{j_1} e^{-\frac{1}{2}(\phi,A\phi)} = -\sum_{i=1}^N A^{-1}_{j_1 i}\ \frac{\partial}{\partial\phi_i} e^{-\frac{1}{2}(\phi,A\phi)}.$$

Inserting this relation in the integral we obtain

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi,A\phi)}\phi_{j_1}\phi_{j_2}d\phi = -\sum_{i=1}^N A^{-1}_{j_1 i}\int_{\mathbb{R}^N}\phi_{j_2}\frac{\partial}{\partial\phi_i}e^{-\frac{1}{2}(\phi,A\phi)}d\phi$$

$$= +\sum_{i=1}^N A^{-1}_{j_1 i}\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi,A\phi)}\frac{\partial}{\partial\phi_i}\phi_{j_2}d\phi = A^{-1}_{j_1 j_2}\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi,A\phi)}d\phi$$

The proof for the general case is similar. Alternatively one may use the *generating function* $S : \{f_j\}_{j=1}^N \to \mathbb{R}$

$$S(f) = \frac{\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi,A\phi)}e^{(\phi,f)}d\phi}{\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi,A\phi)}d\phi}$$

$$= e^{\frac{1}{2}(f,A^{-1}f)}\frac{\int_{\mathbb{R}^N} e^{-\frac{1}{2}([\phi-A^{-1}f],A[\phi-A^{-1}f])}d\phi}{\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi,A\phi)}d\phi} = e^{\frac{1}{2}(f,A^{-1}f)}$$

Since S is smooth in f_j $\forall j$ we have

$$\frac{\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi,A\phi)}\phi_{j_1}\phi_{j_2}\cdots\phi_{j_n}d\phi}{\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi,A\phi)}d\phi} = \frac{\partial^n}{\partial f_{j_1}\cdots\partial f_{j_n}}S(f)_{|f=0}.$$

□

3.1.3 Partition function and correlations

With these formulas we can now compute the partition function and correlation functions for the harmonic cristal in $d = 1$

$$Z_\Lambda^{(b.c.)} = \int e^{-\beta H_\Lambda^{(b.c.)}(\phi)}d\phi = \int e^{-(\phi,A_\Lambda\phi)_\Lambda}d\phi = \frac{(\pi)^{\frac{2L+1}{2}}}{\sqrt{\det A_\Lambda}}$$

$$\mathbb{E}_\Lambda^{(b.c.)}[\phi_x^2] = \frac{\int e^{-(\phi,A_\Lambda\phi)_\Lambda}\ \phi_x^2\ d\phi}{\int e^{-(\phi,A_\Lambda\phi)_\Lambda}\ d\phi} = \frac{1}{2}(A_\Lambda^{-1})_{xx}$$

$$\mathbb{E}_\Lambda^{(b.c.)}[\phi_x\phi_y] = \frac{\int e^{-(\phi,A_\Lambda\phi)_\Lambda}\ \phi_x\phi_y\ d\phi}{\int e^{-(\phi,A_\Lambda\phi)_\Lambda}\ d\phi} = \frac{1}{2}(A_\Lambda^{-1})_{xy}.$$

where the matrix A_Λ depends on the boundary conditions. The problem is then converted in the study of the determinant and inverse of A_Λ as $\Lambda \to \mathbb{Z}$.

3.1.4 Finite volume computation: periodic boundary conditions

In the case of periodic boundary conditions we can compute the eigenvalues and eigenvectors of the discrete Laplacian by taking the Fourier transform.

Discrete Fourier transform

Any function $f \in \mathbb{R}^\Lambda$ can be seen as a periodic function of period $T = 2L + 1$, i.e. $f \in \mathbb{R}^\mathbb{Z}$ with $f(x + nT) = f(x)$ $\forall n \in \mathbb{Z}$. Let $\mathcal{P}_T(\mathbb{Z})$ the corresponding set of functions.

Definition 1 (Discrete Fourier transform) *The discrete Fourier transform is a linear functional*

$$\begin{aligned} \mathcal{F}: \quad & \mathcal{P}_T(\mathbb{Z}) && \to \mathcal{P}_T(\mathbb{Z}) \\ & f && \to \mathcal{F}[f](n) = \hat{f}(n) = c_1 \sum_{x=-L}^{L} f(x) e^{-ik_n x} \end{aligned}$$

where $n \in \Lambda = \{-L, \ldots, L\}$, $k_n = \frac{2\pi n}{2L+1}$ *and* $c_1 > 0$ *is a normalization constant. This functional is invertible and*

$$\begin{aligned} \mathcal{F}^{-1}: \quad & \mathcal{P}_T(\mathbb{Z}) && \to \mathcal{P}_T(\mathbb{Z}) \\ & g && \to \mathcal{F}^{-1}[g](x) = \check{g}(x) = c_2 \sum_{n=-L}^{L} g(k_n) e^{+ik_n x} \end{aligned}$$

where the constants $c_1, c_2 > 0$ *must satisfy* $c_1 c_2 = \frac{1}{2L+1}$.

There are several possible conventions. One may take $c_1 = c_2 = (2L + 1)^{-1/2}$, or $c_1 = 1$ and $c_2 = (2L + 1)^{-1}$.

With these definitions we have the following properties

Convolution. Let $f, g \in \mathcal{P}_T(\mathbb{Z})$. The (discrete) convolution is defined by

$$f * g(x) = \sum_{y=-L}^{L} f(x - y) g(y)$$

The corresponding Fourier transform is

$$[\mathcal{F}(f * g)](k_n) = c_1 c_2^2 (2L + 1)^2 \hat{f}(k_n) \hat{g}(k_n) = \frac{1}{c_1} \hat{f}(k_n) \hat{g}(k_n).$$

Then

$$\mathcal{F}^{-1}[f \cdot g](x) = c_1 (\check{f} * \check{g})(x).$$

Scalar product. Let $f, g \in \mathcal{P}_T(\mathbb{Z})$. We consider the real scalar product on Λ $(f, g) = \sum_{x=-L}^{L} f(x)g(x)$. Then we have

$$(f, g) = c_2^2(2L+1) \sum_{n=-L}^{L} \overline{\hat{f}(k_n)} \hat{g}(k_n) = \frac{1}{c_1^2} \frac{1}{2L+1} \sum_{n=-L}^{L} \overline{\hat{f}(k_n)} \hat{g}(k_n)$$

Fourier transform of the Laplacian. Note that the matrix $-\Delta_\Lambda^{per}$ is translation invariant i.e.

$$(-\Delta_\Lambda^{per})_{x,y} = (-\Delta_\Lambda^{per})_{x-y,0} F(|x-y|)$$

since the value of this matrix element depends only on the distance $|x-y|$, then it acts as a convolution

$$[(-\Delta_\Lambda^{per})f](x) = \sum_y (-\Delta_\Lambda^{per})_{x,y} f(y) = [F * f](y).$$

The Fourier transform is then

$$\mathcal{F}[(-\Delta_\Lambda^{per})f](k_n) = [\mathcal{F}(F * f)](k_n) = \frac{1}{c_1} \hat{F}(k_n) \hat{f}(k_n).$$

Therefore, by translation invariance the Laplacian is a diagonal matrix in Fourier space and the eigenvalues are given by

$$\lambda_n = \frac{1}{c_1} \hat{F}(k_n).$$

To compute the eigenvalues

$$\frac{1}{c_1} \hat{F}(k_n) = \sum_{x=-L}^{L} e^{-ik_n x} (-\Delta_\Lambda^{per})_{x,0} = \left[2 - e^{-ik_n} - e^{ik_n}\right] = 2[1 - \cos(k_n)]$$

Note that by symmetry there are $L+1$ distinct eigenvalues: $\lambda_n = 2[1 - \cos(k_n)]$ with $n = 1, .., L$ each of multiplicity 2 and $\lambda_0 = 0$ of multiplicity 1.

Let $M = -\beta \Delta_\Lambda^{per} + m^2 \mathrm{I}_\Lambda$. From above we have

$$\widehat{[Mf]}(k_n) = \sum_m \hat{M}_{k_n k_m} \hat{f}(k_m) = \mu(k_n) \hat{f}(k_n) = [\mu \cdot \hat{f}](k_n) \qquad \text{where } \mu(k_n) = 2\beta(1 - \cos k_n) + m^2.$$

Hence $\hat{M}_{k_n k_m} = \delta_{nm} \mu(k_n)/c_1$ is a diagonal matrix and

$$[\hat{M}^{-1} \hat{f}](k_n) = [\mu^{-1} \cdot \hat{f}](k_n) = \frac{1}{\mu(k_n)} \hat{f}(k_n).$$

Therefore

$$\begin{aligned}[M^{-1} f](x) &= \sum_y M_{xy}^{-1} f(y) = \mathcal{F}^{-1}[\hat{M}^{-1} \hat{f}](x) = \mathcal{F}^{-1}[\mu^{-1} \cdot \hat{f}](x) \\ &= c_1 \left[\mathcal{F}^{-1}(\mu^{-1}) * f\right](x) = c_1 \sum_y \mathcal{F}^{-1}(\mu^{-1})(x-y) f(y)\end{aligned}$$

As a conclusion we obtain

$$
\begin{aligned}
M_{xy}^{-1} &= c_1\mathcal{F}^{-1}(\mu^{-1})(x-y) = c_1c_2\sum_{n=-L}^{L}\tfrac{1}{\mu(k_n)}e^{ik_n(x-y)} \\
&= \tfrac{1}{2L+1}\sum_{n=-L}^{L}\tfrac{1}{\mu(k_n)}e^{ik_n(x-y)} = \tfrac{1}{2L+1}\sum_{n=-L}^{L}\tfrac{e^{ik_n(x-y)}}{2\beta(1-\cos k_n)+m^2}.
\end{aligned}
$$

This result is independent from the choice of c_1, c_2.

Remark. The arguments above are false if we take Dirichlet or Neuman boundary conditions.

Finite volume partition function and correlations

With the definitions above we can now explicitely compute some quantities. Since we are considering periodic boundary conditions we have $A_\Lambda = M$. Moreover each eigenvalue except $\mu(0)$ has multiplicity 2, then

$$
\det M = \mu(0)\prod_{n=1}^{L}\mu(k_n)^2 = m^2\prod_{n=1}^{L}\mu(k_n)^2.
$$

Then

$$
\begin{aligned}
\tfrac{1}{2L+1}\ln Z_\Lambda^{(per)} &= \ln\sqrt{\pi} - \tfrac{\ln m}{2L+1} - \tfrac{1}{2L+1}\sum_{n=1}^{L}\ln\mu(k_n) \\
\mathbb{E}_\Lambda^{per}[\phi_x^2] &= \tfrac{1}{2}(M^{-1})_{xx} = \tfrac{1}{2m^2|\Lambda|} + \tfrac{1}{|\Lambda|}\sum_{n=1}^{L}\frac{1}{\mu(k_n)} \\
\mathbb{E}_\Lambda^{per}[\phi_x\phi_y] &= \tfrac{1}{2}(M^{-1})_{xy} = \tfrac{1}{2m^2|\Lambda|} + \tfrac{1}{|\Lambda|}\sum_{n=1}^{L}\frac{\cos(k_n(x-y))}{\mu(k_n)}.
\end{aligned}
$$

where $|\Lambda| = 2L+1$ and we used $k_{-n} = -k_n$ and $\mu(k_n) = \mu(-k_n)$.

Some elementary estimates on the two point function: spectral gap
Contrary to the continuous Laplacian, the discrete Laplacian has a spectral gap,

$$
\mu(k_n) - \mu(k_0) \geqslant 2\beta(1-\cos(\tfrac{2\pi}{2L+1})) = O(L^{-2}) > 0 \qquad \forall n \neq 0.
$$

Using this fact we can prove the following estimates.

Lemma 2 *There exist constants C_1, C_2 such that*

$$
|\mathbb{E}_\Lambda^{per}[\phi_x\phi_y] - \mathbb{E}_\Lambda^{per}[\phi_x\phi_x]| \leqslant \frac{C_1}{m} \tag{3.1.1}
$$

$$
|\mathbb{E}_\Lambda^{per}[\phi_x\phi_y] - \mathbb{E}_\Lambda^{per}[\phi_x\phi_{y+1}]| \leqslant C_2 \tag{3.1.2}
$$

for any choice of m*,* $|\Lambda|$*,* x *and* y*. The factor* m^{-1} *is due to the properties of the one dimensional Laplacian and cannot be avoided. Precisely, there exist two constants* K_1, K_2 *such that*

$$\frac{K_1}{m} \leqslant \left| \mathbb{E}_{\Lambda}^{per}[\phi_x \phi_x] - \tfrac{1}{2m^2|\Lambda|} \right| \leqslant \frac{K_2}{m} \tag{3.1.3}$$

for any choice of m *and* $|\Lambda|$*. The points on the boundary play a special role and the corresponding two point function has nicer a priori bounds. Precisely there exists a constant* C_3 *such that*

$$m \left| \mathbb{E}_{\Lambda}^{per}[\phi_x \phi_{\pm L}] - \tfrac{1}{2m^2|\Lambda|} \right| \leqslant \frac{C_3}{mL} \tag{3.1.4}$$

for any choice of m Λ *and* x*.*

Proof.

$$\begin{aligned}
|\mathbb{E}_{\Lambda}^{per}[\phi_x \phi_y] - \mathbb{E}_{\Lambda}^{per}[\phi_x \phi_x]| &= \left| \tfrac{1}{|\Lambda|} \sum_{n=1}^{L} \tfrac{[\cos(k_n(x-y))-1]}{\mu(k_n)} \right| \\
&\leqslant \tfrac{2}{|\Lambda|} \sum_{n=1}^{L} \tfrac{1}{2\beta(1-\cos k_n)+m^2} \\
&\leqslant \tfrac{2}{|\Lambda|} \sum_{1\leqslant n \leqslant L/10} \tfrac{1}{2\beta(1-\cos k_n)+m^2} + \tfrac{2}{|\Lambda|} \sum_{L/10 < n \leqslant L} \tfrac{1}{2\beta(1-\cos k_n)+m^2}
\end{aligned}$$

To estimate the second sum notice that

$$1 - \cos(k_n) \geqslant 1 - \cos \pi/10 + O(L^{-1}) \geqslant Const \qquad \forall \quad L/10 < n \leqslant L.$$

Then

$$\tfrac{2}{|\Lambda|} \sum_{L/10 < n \leqslant L} \tfrac{1}{2\beta(1-\cos k_n)+m^2} \leqslant \tfrac{2L}{2L+1} \sup_{L/10 < n \leqslant L} \tfrac{1}{2\beta(1-\cos k_n)+m^2} \leqslant Const.$$

To estimate the first sum notice that we can find a small number $\rho > 0$ such that

$$1 - \cos(k_n) \geqslant \rho k_n^2 \qquad \forall \quad n \leqslant L/10.$$

Then

$$\begin{aligned}
&\tfrac{2}{|\Lambda|} \sum_{1\leqslant n \leqslant L/10} \tfrac{1}{2\beta(1-\cos k_n)+m^2} \leqslant \tfrac{1}{\pi} \tfrac{2\pi}{|\Lambda|} \sum_{1\leqslant n \leqslant L/10} \tfrac{1}{2\beta\rho k_n^2+m^2} \\
&\leqslant \tfrac{1}{\pi} \int_{a_L}^{b_L} \tfrac{1}{2\beta\rho k^2+m^2} dk = \tfrac{1}{m\pi\sqrt{2\beta\rho}} \int_{\frac{a_L\sqrt{2\beta\rho}}{m}}^{\frac{b_L\sqrt{2\beta\rho}}{m}} \tfrac{1}{k^2+1} dk \\
&= \tfrac{1}{m\pi\sqrt{2\beta\rho}} [\arctan(k)]_{\frac{a_L\sqrt{2\beta\rho}}{m}}^{\frac{b_L\sqrt{2\beta\rho}}{m}} \leqslant \tfrac{Const}{m}
\end{aligned}$$

where we set $a_L = \frac{2\pi}{2L+1}$, $b_L = \frac{\pi+O(L^{-1})}{10}$. The estimate (3.1.3) is proved in the same way. To obtain the lower bound notice that $1-\cos(k_n) \leqslant \rho k_n^2$ for some constant ρ for all $1 \leqslant n \leqslant L/10$ and $1-\cos(k_n) \leqslant 2$ for all $0 \leqslant n \leqslant L$. To prove (3.1.2)

$$\mathbb{E}_\Lambda^{per}[\phi_x\phi_y] - \mathbb{E}_\Lambda^{per}[\phi_x\phi_{y+1}] = \frac{1}{|\Lambda|}\sum_{n=1}^{L} \frac{[\cos(k_n(x-y)) - \cos(k_n(x-y-1))]}{\mu(k_n)}$$

$$= \frac{1}{|\Lambda|}\sum_{n=1}^{L} \frac{\cos(k_n(x-y))[1-\cos(k_n)] - \sin(k_n(x-y))\sin(k_n)}{\mu(k_n)}$$

Then

$$|\mathbb{E}_\Lambda^{per}[\phi_x\phi_y] - \mathbb{E}_\Lambda^{per}[\phi_x\phi_{y+1}]| \leqslant \frac{1}{|\Lambda|}\sum_{n=1}^{L} \frac{[1-\cos(k_n)]}{2\beta(1-\cos(k_n))+m^2} + \frac{1}{|\Lambda|}\sum_{n=1}^{L} \frac{|\sin(k_n(x-y))\sin(k_n)|}{2\beta(1-\cos(k_n))+m^2}$$

$$\leqslant \frac{L}{2\beta(2L+1)} + \frac{1}{|\Lambda|}\sum_{n=1}^{L} \frac{|\sin(k_n(x-y))\sin(k_n)|}{2\beta(1-\cos(k_n))+m^2}$$

To estimate the last term we break it as before in two sums $\sum_{n\leqslant\bar{n}}$ and $\sum_{\bar{n}\leqslant n\leqslant L}$, where

$$\bar{n} = \frac{L}{10|x-y|} \quad ,1 \leqslant |x-y| \ll L \Rightarrow \quad |k_n| \leqslant \frac{L}{10}, \text{ and } |k_n(x-y)| \leqslant \frac{L}{10}\ \forall\ n \leqslant \bar{n}.$$

The last sum is bounded by a constant. The first sum is bounded by

$$\frac{K}{|\Lambda|}\sum_{1\leqslant n\leqslant\bar{n}} \frac{k_n^2|x-y|}{2\beta\rho k_n^2+m^2} \leqslant K\frac{|x-y|\bar{n}}{2\beta\rho|\Lambda|} \leqslant K$$

where K and K are some constants. Finally to prove (3.1.4) note that

$$\left|\mathbb{E}_\Lambda^{per}[\phi_x\phi_L] - \frac{1}{2m^2|\Lambda|}\right| = \frac{1}{|\Lambda|}\left|\sum_{n=1}^{L} \frac{\cos k_n(x-L)}{2\beta(1-\cos(k_n))+m^2}\right| = \frac{1}{|\Lambda|}\left|\sum_{n=1}^{L} \frac{\cos[k_n(x-1/2)+n\pi]}{2\beta(1-\cos(k_n))+m^2}\right|$$

$$= \frac{1}{|\Lambda|}\left|\sum_{n=1}^{L}(-1)^n f(k_n)\right| = \frac{1}{|\Lambda|}\left|\sum_{1\leqslant n\leqslant L/2} f(k_{2n}) - f(k_{2n-1})\right| \leqslant \frac{1}{|\Lambda|}\sum_{n=1}^{L}|f(k_{n+1}) - f(k_n)|\,.$$

where

$$f(k) = \frac{\cos[k(x-1/2)]}{2\beta(1-\cos(k))+m^2}.$$

Now

$$f(k_{n+1}) - f(k_n) = f'(k^*)\delta k, \qquad k_n \leqslant k^* \leqslant k_{n+1}, \quad \delta k = \frac{2\pi}{2L+1}.$$

There exist constants $C_1(x,\beta), C_2(x,\beta)$ such that

$$|f'(k)| \leqslant \frac{|\sin(k(x-1/2))|}{[2\beta(1-\cos(k))+m^2]} + \frac{|\cos(k(x-1/2))||\sin(k)|}{[2\beta(1-\cos(k))+m^2]^2}$$

$$\leqslant \begin{cases} C_1(x,\beta) & \forall \frac{\pi}{10|x|} \leqslant k \leqslant \pi \\ C_2(x,\beta)\left[\frac{k}{2\beta\rho k^2+m^2} + \frac{k}{[2\beta\rho k^2+m^2]^2}\right] & \forall 0 \leqslant k \leqslant \frac{\pi}{10|x|} \end{cases}$$

Inserting these bounds in the sum above we obtain

$$\tfrac{1}{|\Lambda|}\sum_{n=1}^{L}|f(k_{n+1})-f(k_n)| \leqslant C_1(x,\beta)\delta k + C_2(x,\beta)\delta k\int_0^{\pi/10|x|}\left[\tfrac{k}{2\beta\rho k^2+m^2}+\tfrac{k}{[2\beta\rho k^2+m^2]^2}\right]dk$$
$$= O(L^{-1}) + O(L^{-1})|\ln m| + O\left(\tfrac{1}{m^2 L}\right) = \tfrac{1}{m}O\left(\tfrac{1}{mL}\right).$$

This proves the result.

■

3.1.5 Infinite volume limit for periodic boundary conditions

When $L \to \infty$ the Riemann sums become integrals

$$\tfrac{1}{2L+1}\ln Z_\Lambda^{(per)} = \ln\sqrt{\pi} - \tfrac{\ln m}{2L+1} - \tfrac{1}{2L+1}\sum_{n=1}^{L}\ln\mu(k_n)$$
$$\to_{L\to\infty} \ln\sqrt{\pi} - \tfrac{1}{2\pi}\int_0^{\pi}\ln[2\beta(1-\cos k)+m^2]dk.$$
$$\mathbb{E}_\Lambda^{per}[\phi_x^2] = \tfrac{1}{2m|\Lambda|} + \tfrac{1}{|\Lambda|}\sum_{n=1}^{L}\frac{1}{\mu(k_n)} \to_{L\to\infty} \tfrac{1}{2\pi}\int_0^{\pi}\frac{1}{2\beta(1-\cos k)+m^2}dk$$
$$\mathbb{E}_\Lambda^{per}[\phi_x\phi_y] = \tfrac{1}{2m|\Lambda|} + \tfrac{1}{|\Lambda|}\sum_{n=1}^{L}\frac{\cos(k_n(x-y))}{\mu(k_n)}$$
$$\to_{L\to\infty} \tfrac{1}{2\pi}\int_0^{\pi}\frac{\cos(k(x-y))}{2\beta(1-\cos k)+m^2}dk = \tfrac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{ik(x-y)}}{2\beta(1-\cos k)+m^2}dk$$

Lemma 3 *The limits obtained above coincide with the results we obtained by transfer matrix approach. In particular the two-point correlations are given by*

$$\lim_{L\to\infty}\mathbb{E}_\Lambda^{per}[\phi_x^2] = \frac{1}{4\alpha}$$
$$\lim_{L\to\infty}\mathbb{E}_\Lambda^{per}[\phi_x\phi_y] = \frac{1}{4\alpha}z_1^{|x|}$$

where

$$z_1 = \left(1+\tfrac{m^2}{2\beta}\right) - \sqrt{\left(1+\tfrac{m^2}{2\beta}\right)^2 - 1}.$$

Proof. The two point function is symmetric under exchange of x and y so we can always choose $x - y \geqslant 0$.

$$\lim_{L\to\infty}\mathbb{E}_\Lambda^{per}[\phi_x\phi_y] = \tfrac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{ik(x-y)}}{2\beta(1-\cos k)+m^2}dk = \tfrac{-i}{4\pi}\int_C\frac{z^{x-y}}{\beta(2-z-z^{-1})+m^2}\frac{dz}{z}$$
$$= \tfrac{-i}{4\pi}\int_C\frac{z^{x-y}}{\beta(2z-z^2-1)+m^2 z}dz = \tfrac{-i}{4\pi\beta}\int_C\frac{z^{x-y}}{(z-z_1)(z_2-z)}dz$$

where $C = \{z = e^{i\theta} \in \mathbb{C} | \theta \in [0, 2\pi[\}$ is the circle of radius 1 and

$$z_1 = \left(1 + \tfrac{m^2}{2\beta}\right) - \sqrt{\left(1 + \tfrac{m^2}{2\beta}\right)^2 - 1} = \left(1 + \tfrac{m^2}{2\beta}\right) - \tfrac{\alpha}{\beta} < 1,$$
$$z_2 = \left(1 + \tfrac{m^2}{2\beta}\right) + \sqrt{\left(1 + \tfrac{m^2}{2\beta}\right)^2 - 1} = \left(1 + \tfrac{m^2}{2\beta}\right) + \tfrac{\alpha}{\beta} > 1$$

and $\alpha = \sqrt{(\beta + m^2/2)^2 - \beta^2}$ was introduced in Chapter 2. Since $x - y \geqslant 0$ the function inside the integral is holomorphic on the whole plane $\mathbb{C}$ except at the two points $z = z_1, z_2$ where it has a simple pole. Therefore

$$\lim_{L \to \infty} \mathbb{E}_\Lambda^{per}[\phi_x \phi_y] = \frac{-i2\pi i}{4\pi\beta} \frac{z^{x-y}}{z_2 - z_1} = \frac{1}{4\alpha} z_1^{|x-y|}$$

■

3.2 Gaussian integrals is $d \geqslant 1$.

3.2.1 The harmonic cristal in $d \geqslant 1$.

For $d \geqslant 1$ we consider the cube $\Lambda_L = \{-L, \ldots, L\}^d$. The set of possible configurations is now $\Omega_\Lambda = \{\phi : \Lambda \to \mathbb{R}\}$. The energy associated to a configuration is

$$\beta H_\Lambda^{(bc)}(\phi) = \sum_{j \sim k \in \Lambda} \beta(\phi_j - \phi_k)^2 + \sum_{j \in \Lambda} m^2 \phi_j^2 + F^{(b.c)}(\phi)$$

where $j \sim k$ is $\|j - k\| = 1$ (with the euclidian norm $\|x\|^2 = \sum_{\rho=1}^d x_\rho^2$) and

$$F^{(b.c)}(\phi) = \begin{cases} \sum_{\substack{z, z' \in \partial\Lambda \\ \|z - z'\| > 1 \|z-z'\|_p = 1}} \beta(\phi_z - \phi_{z'})^2 & \text{periodic b.c.} \\ \sum_{\substack{z \in \partial\Lambda, z' \in \Lambda^c \\ \|z - z'\| = 1}} \beta(\phi_z - \phi_{z'})^2_{\phi_{z'} = 0} = \sum_{\substack{z \in \partial\Lambda, z' \in \Lambda^c \\ \|z - z'\| = 1}} \beta\phi_z^2 & \text{Dirichlet b.c.} \\ 0 & \text{Neuman b.c.} \end{cases}$$

where $\|z - z'\|_p$ is the norm on the periodic torus $\mathbb{Z}^d/\Lambda_L$. All these expressions can be written as quadratic forms

$$\beta H_\Lambda^{(bc)}(\phi) = (\phi, A_\Lambda^{(b.c.)}\phi), \qquad A_\Lambda^{(b.c.)} = -\beta\Delta_\Lambda^{(b.c.)} + m^2 Id_\Lambda.$$

where $-\Delta_\Lambda$ is the generalization of the discrete Laplacian to dimension $d \geqslant 1$. The formulas for Gaussian integrals generalize directly to any dimension. In particular

$$\mathbb{E}_\Lambda^{(bc)}[\phi_x \phi_y] = \tfrac{1}{2}(A_\Lambda^{(bc)})^{-1}_{xy}.$$

3.2.2 Periodic boundary conditions

In the case of periodic boundary conditions we can apply discrete Fourier transform (as in $d = 1$) to prove

$$\mathbb{E}_\Lambda^{(per)}[\phi_x\phi_y] = \tfrac{1}{2m^2|\Lambda|} + \tfrac{1}{2|\Lambda|}\sum_{\substack{n\in\Lambda\\ n\neq 0}} \frac{e^{i(k_n,(x-y))}}{2\beta\sum_{\rho=1}^d(1-\cos k_n^\rho)+m^2}$$

where $n = (n_1, \ldots n_d) \in \Lambda$, $k_n = (k_n^1, \ldots, k_n^d)$ and $k_n^\rho = \frac{2\pi n_\rho}{2L+1}$. By the same arguments we used in $d = 1$ we can show that for small m

$$\left|(A_\Lambda^{(per)})^{-1}_{xy} - \tfrac{1}{m^2|\Lambda|}\right| = \begin{cases} O\left(\frac{1}{m}\right) & d=1\\ O\left(|\ln m|\right) & d=2\\ O(1) & d\geqslant 3\end{cases} \tag{3.2.5}$$

the main reason being that for small n ie $\|n\| \leqslant L/10$ the Fourier sum can be approximated by the integral

$$\tfrac{1}{|\Lambda|}\sum_{\|n\|\leqslant L/10}\frac{1}{2\beta\sum_{\rho=1}^d(1-\cos k_n^\rho)+m^2} \sim \int_{\|k\|\leqslant \pi/10}\frac{1}{\|k\|^2+m^2}d^dk = C_d\int_0^{\pi/10}\frac{k^{d-1}}{k^2+m^2}dk.$$

This integral is linearly divergent in $d = 1$, log divergent in $d = 2$ and bounded in $d \geqslant 3$.

Infinite volume

As in $d = 1$ when $L \to \infty$ and m is kept fixed the Riemann sum converges to an integral

$$\lim_{\Lambda\to\mathbb{Z}^d}\mathbb{E}_{\Lambda,m}^{(per)}[\phi_x\phi_y] = \frac{1}{2(2\pi)^d}\int_{[-\pi,\pi]^d}\frac{e^{i(k,(x-y))}}{2\beta\sum_{\rho=1}^d(1-\cos k^\rho)+m^2}d^dk.$$

With some extra work one can show that the limit exists also if we let $m \to 0$ and $L \to \infty$ simultaneously as long as $mL \to \infty$. Precisely we have

$$\lim_{\substack{m\to 0, L\to\infty\\ mL\to\infty}} c(m)\mathbb{E}_{\Lambda,m}^{(per)}[\phi_x\phi_y] = \lim_{m\to 0}\lim_{L\to\infty} c(m)\mathbb{E}_{\Lambda,m}^{(per)}[\phi_x\phi_y]$$

where

$$c(m) = \begin{cases} m & d=1\\ |\ln m| & d=2\\ 1 & d\geqslant 3\end{cases}$$

To prove this result one has to compare the Riemann sum with the integral. The difference can be expressed as sum over gradients $f(k) - f(k_n)$ which in turn give some decay improvement by the same arguments we used to prove eq. (3.1.4).

3.2.3 Existence and uniqueness of the thermodynamic limit.

Theorem 1 *The thermodynamic limit for the 2 point correlation function* $\mathbb{E}^{(per)}_{\Lambda,m}[\phi_x\phi_y]$ *exists* $\forall d \geqslant 1$ *and is independent of the boundary conditions:*

$$\lim_{L\to\infty}\left[\mathbb{E}^{(per)}_{\Lambda,m}[\phi_x\phi_y] - \mathbb{E}^{D}_{\Lambda,m}[\phi_x\phi_y]\right] = \lim_{L\to\infty}\left[\mathbb{E}^{(per)}_{\Lambda,m}[\phi_x\phi_y] - \mathbb{E}^{N}_{\Lambda,m}[\phi_x\phi_y]\right] = 0,$$

for any fixed $m > 0$. *This remains true also is we let* $m \to 0$ *and* $L \to \infty$ *simultaneously with* $mL \to \infty$. *Precisely*

$$\lim_{\substack{m\to 0,\, L\to\infty \\ mL\to\infty}} c(m)\left[\mathbb{E}^{(per)}_{\Lambda,m}[\phi_x\phi_y] - \mathbb{E}^{D}_{\Lambda,m}[\phi_x\phi_y]\right] = \lim_{\substack{m\to 0,\, L\to\infty \\ mL\to\infty}} c(m)\left[\mathbb{E}^{(per)}_{\Lambda,m}[\phi_x\phi_y] - \mathbb{E}^{N}_{\Lambda,m}[\phi_x\phi_y]\right] = 0.$$

Proof. Existence follows directly from the results of the previous section in the case of periodic boundary conditions. To prove uniqueness let $M^D = -\beta\Delta^D_\Lambda + m^2$ and $M^N = -\beta\Delta^N_\Lambda + m^2$ and $M = -\beta\Delta^{per}_\Lambda + m^2$ the matrices corresponding to Dirichlet, Neuman and periodic boundary conditions. We remark that M^D and M differ only on the boundary of Λ. The same is true for M^N. Precisely

$$M^D = M + X, \qquad M^N = M + \tilde{X}$$

where

$$X_{xy} = \sum_{\substack{z,z'\in\partial\Lambda \\ \|z-z'\|>1 \|z-z'\|_p=1}} \beta[\delta_{x,z}\delta_{y,z'} + \delta_{x,z'}\delta_{y,z}],$$

$$\tilde{X}_{xy} = \sum_{\substack{z,z'\in\partial\Lambda \\ \|z-z'\|>1 \|z-z'\|_p=1}} \beta[\delta_{x,z}\delta_{y,z'} + \delta_{x,z'}\delta_{y,z} - \delta_{x,z}\delta_{y,z} - \delta_{x,z'}\delta_{y,z'}].$$

For any two matrices A and B (with A and $A+B$ invertible) we have

$$(A+B)^{-1} - A^{-1} = -(A+B)^{-1}BA^{-1}.$$

Applying the relation above

$$\begin{aligned}
(M^D)^{-1}_{xy} - M^{-1}_{xy} &= (M+X)^{-1}_{xy} - M^{-1}_{xy} = -\sum_{zz'}(M+X)^{-1}_{xz}X_{zz'}M^{-1}_{z'y} \\
&= -\sum_{\substack{z,z'\in\partial\Lambda \\ \|z-z'\|>1 \|z-z'\|_p=1}} \beta\left[(M^D)^{-1}_{xz}M^{-1}_{z'y} + (M^D)^{-1}_{xz'}M^{-1}_{zy}\right] \\
(M^N)^{-1}_{xy} - M^{-1}_{xy} &= (M+\tilde{X})^{-1}_{xy} - M^{-1}_{xy} = -\sum_{zz'}(M+\tilde{X})^{-1}_{xz}\tilde{X}_{zz'}M^{-1}_{z'y} \\
&= -\sum_{\substack{z,z'\in\partial\Lambda \\ \|z-z'\|>1 \|z-z'\|_p=1}} \beta\left[(M^N)^{-1}_{xz} - (M^N)^{-1}_{xz'}\right]\left[M^{-1}_{zy} - M^{-1}_{z'y}\right]
\end{aligned}$$

Case of fixed mass. By Combes-Thomas estimate (see the next subsection) there exists a constant $\mu_{m,d}$ that depends only on m and d such that, for any boundary condition

$$\left|[M^{bc}]^{-1}_{xy}\right| \leqslant \tfrac{2}{m^2} e^{-|x-y|\mu_{m,d}}$$

uniformly in the volume. Then

$$\left|(M^D)^{-1}_{xy} - M^{-1}_{xy}\right| \leqslant \tfrac{CL^{d-1}}{m^4} e^{-\mu_{m,d}L} \to_{L\to\infty} 0$$

for some constant C. The same holds for M^N.

Case of vanishing mass with Dirichlet b.c. For m small the factor $\mu_m \simeq m^2$ so the Combes-Thomas estimate gives a decay in Lm^2 which is not enough to prove the convergence. By matrix-tree thorem (see the next subsection) we can prove

$$(M^D)^{-1}_{xy} \geqslant 0 \quad \forall x, y \in \Lambda, \text{ and } \quad \sum_{z\in\partial\Lambda} (M^D)^{-1}_{xz} \leqslant \tfrac{1}{\beta} \ \forall x \in \Lambda.$$

Moreover, by Fourier analysis (see eq. (3.2.5) above) one can show that

$$c(m)\left|M^{-1}_{xy}\right| \to_{\substack{m\to 0, L\to\infty \\ mL\to\infty}} 0$$

Putting together these estimates

$$c(m)\left|(M^D)^{-1}_{xy} - M^{-1}_{xy}\right| \leqslant \beta \sum_{z\in\partial\Lambda} (M^D)^{-1}_{xz}\ c(m) \sup_{z\in\partial\Lambda} \left|M^{-1}_{zy}\right| \to_{\substack{m\to 0, L\to\infty \\ mL\to\infty}} 0.$$

■

3.2.4 Combes-Thomas estimate

Theorem 2 (Combes-Thomas) *Let Γ be a finite or countable set, $M = T + U$ a self-adjoint operator on $l^2(\Gamma)$, with U an arbitrary diagonal operator and T an off-diagonal operator. Let $|x-y|$ the distance in Γ. If there exists a parameter $\eta > 0$ such that*

$$\sup_{x\in\Gamma} \sum_{y\in\Gamma} |T_{xy}| e^{\eta|x-y|} = S < \infty$$

then for any E outside the spectrum of M with dist$\{M, E\} = \Delta > 0$

$$\left|(M-E)^{-1}_{xy}\right| \leqslant \frac{2}{\Delta} e^{-\mu|x-y|}, \qquad \textit{with } \mu = \frac{\Delta\eta}{\Delta + 2S}.$$

Proof. Let $e_x \in l^2(\Gamma)$ the function defined by $e_x(y) = \delta_{x=y}$, then

$$(M-E)^{-1}_{xy} = (e_x, (M-E)^{-1} e_y).$$

Let $R : l^2(\Gamma) \to l^2(\Gamma)$ the multiplication operator defined by

$$[Rf](y) = e^{\mu|x-y|_N} f(y), \qquad \text{where } |x-y|_N = \min\{|x-y|, N\}.$$

The parameter N makes R a bounded operator also when Γ is a countable set. At the end of the proof, we will take N to infinity. Then

$$(M-E)^{-1}_{xy}\ e^{\mu|x-y|_N} = (e_x, (M-E)^{-1} e_y)\ e^{\mu|x-y|_N} = (R^{-1}e_x,\ (M-E)^{-1}\ R\ e_y)$$
$$= (e_x,\ R^{-1}(M-E)^{-1}R\ e_y) = (e_x, \left[R^{-1}(M-E)R\right]^{-1} e_y)$$

Then

$$\left|(M-E)^{-1}_{xy}\right|\ e^{\mu|x-y|_N} \leqslant \left\| \frac{1}{R^{-1}(M-E)R} \right\| = \left\| \frac{1}{[R^{-1}TR - T] + [M-E]} \right\|$$

where we used $R^{-1}UR = U$. The kernel of $[R^{-1}TR - T]$ is given by $k(x,y)$

$$[R^{-1}TR - T]f(y) = \sum_z T_{yz} \left[e^{\mu(|z-x|_N - |y-x|_N)} - 1 \right] f(z)$$
$$= \sum_z k(y,z) f(z).$$

Since $\big||z-x|_N - |y-x|_N\big| \leqslant |y-z|_N$ we have

$$\left| e^{\mu(|z-x|_N - |y-x|_N)} - 1 \right| \leqslant \max\left[(e^{\mu|z-y|_N} - 1), (1 - e^{-\mu|z-y|_N}) \right]$$
$$= e^{\mu|z-y|_N} - 1.$$

Then the kernel $k(y,z)$ satisfies

$$\sup_y \sum_z |k(y,z)| = \sup_z \sum_y |k(y,z)| \leqslant \sup_y \sum_z |T_{yz}| \left(e^{\mu|z-y|_N} - 1 \right)$$
$$\leqslant \left[\sup_u e^{-\eta|u|} \left(e^{\mu|u|} - 1 \right) \right] \sup_y \sum_z |T_{yz}| e^{\eta|z-y|_N}$$
$$\leqslant S \tfrac{\mu}{\eta-\mu} \left(\tfrac{\eta-\mu}{\eta} \right)^{\frac{\eta}{\mu}} \leqslant S \tfrac{\mu}{\eta-\mu}.$$

since $\mu < \eta$. Then by the Schur's bound we have

$$\|[R^{-1}TR - T]\| \leqslant S \tfrac{\mu}{\eta-\mu} = \tfrac{\Delta}{2} \quad \text{since } \mu = \tfrac{\Delta\eta}{\Delta+2S}.$$

On the other hand

$$\|[M-E]f\| \geqslant \Delta \|f\| \quad \forall f \in l_2(\Gamma).$$

With these bounds we obtain

$$\left\| \tfrac{1}{[R^{-1}TR-T]+[M-E]} \right\| = \frac{1}{\inf_f \frac{\|[R^{-1}TR-T]f + [M-E]f\|}{\|f\|}} \leqslant \tfrac{2}{\Delta}$$

since

$$\|[R^{-1}TR - T]f + [M-E]f\| \geqslant \big| \|[R^{-1}TR-T]f\| - \|[M-E]f\| \big|.$$

These bounds do not depend on the N, so we can take $N \to \infty$. This completes the proof. ■

Application of Combes-Thomas: bound on the two point function. Let $A_\Lambda = -\beta\Delta_\Lambda^{(b.c)} + m^2 I_\Lambda$. For any choice of the boundary conditions we can write

$$A_\Lambda = T + U, \quad \text{where } |T_{x,y}| = \beta\delta_{|x-y|=1},$$

where $|x-y|$ is the euclidean norm in $\mathbb{Z}^d$. In the case of periodic boundary conditions $|x-y|$ is the euclidean norm on the torus $\mathbb{Z}^d/\Lambda_L$. Moreover $\|A\| \geqslant m^2$ and

$$\sum_y \sum_z |T_{z,y}| e^{\eta|z-y|} \leqslant 2d\beta e^\eta = S < \infty$$

for any choice of $\eta > 0$ and for any choice of the boundary conditions. Then we can apply Combes-Thomas estimate with $E = 0$

$$\left|\mathbb{E}_\Lambda^{(b.c.)}[\phi_x\phi_y]\right| = \frac{1}{2}|(A_\Lambda^{-1})_{xy}| \leqslant \tfrac{1}{m^2} e^{-\mu_m|x-y|}$$

where

$$\mu_m = \frac{m^2\eta}{m^2 + 4d\beta e^\eta} \tag{3.2.6}$$

and $\eta > 0$ is arbitrary. This bound holds uniformly in the volume Λ and for **any dimension** $d \geqslant 1$.

3.2.5 Matrix-tree theorem

Let Λ be a finite set of points. Let $E_\Lambda = \{(i,j)|\ i,j \in \Lambda, i \neq j\}$ be the set un **undirected edges** $e = (i,j) = (j,i)$ on Λ. For each edge $e \in E_\Lambda$ we denote its endpoints by i_e, j_e.

Definition 2 *A subset $E \subset E_\Lambda$ of edges forms a loop (cycle) if we can order its edges $E = (e_1, \ldots, e_n)$ such that $i_{e_l} = j_{e_{l-1}}$, $\forall l = 2, \ldots n$ and $i_{e_1} = j_{e_n}$.*

Definition 3 *A **forest** F on Λ is a subset of E_Λ with no cycle. Let $\mathcal{F}[\Lambda]$ be the set of forests on Λ.*

Definition 4 *A **spanning tree** T on Λ is a forest on Λ such that for each pair $x, y \in \Lambda$ there exists a path in T connecting x to y. Precisely there exists a subset $\gamma_{xy}^T = (e_1, \ldots e_n) \subset T$ such that $i_{e_1} = x$, $j_{e_n} = y$ and $i_{e_l} = j_{e_{l-1}}$ $\forall l = 2, \ldots, n$.*

Characterization of a forest. A forest F can be uniquely determined by the following information.

1. We fix a partition P of the set Λ.

2. Inside each element X of the partition we choose a spanning tree.

The forest is then obtained taking the union over the spanning trees. Note that this implies there is no edge connecting points in different elements of the partition. On the contrary any two points inside $X \in P$ are connected by a path in the forest. The elements $X \in P$ are also called *connected components* of the forest. For each forest F we denote by $P(F)$ the corresponding partition.

Theorem 3 (matrix-tree) *Let M be a $N \times N$ symmetric invertible matrix (not necessarily positive or real). Let $\Lambda = \{1, \dots, N\}$. Then*

$$\det M = \sum_{F \in \mathcal{F}[\Lambda]} \prod_{e \in F} [-M_{i_e j_e}] \prod_{X \in P(F)} [\sum_{r \in X} B_r]$$
$$M^{-1}_{xy} \det M = \sum_{F \in \mathcal{F}_{xy}[\Lambda]} \prod_{e \in F} [-M_{i_e j_e}] \prod_{X \in P(F), x \notin X} [\sum_{r \in X} B_r]$$

where

$$B_r = \sum_{j \in \Lambda} M_{rj}$$

and $\mathcal{F}_{xy}[\Lambda]$ is the set of forests such that x and y belong to the same connected component. Alternatively one may write

$$\det M = \sum_{P \in \mathcal{P}[\Lambda]} \prod_{X \in P} \left\{ \sum_{T \in \mathcal{T}[X]} \prod_{e \in T} [-M_{i_e j_e}][\sum_{r \in X} B_r] \right\}$$
$$M^{-1}_{xy} \det M = \sum_{P \in \mathcal{P}_{xy}[\Lambda]} \left[\sum_{T \in \mathcal{T}[X_x]} \prod_{e \in T} [-M_{i_e j_e}] \right] \prod_{\substack{X \in P \\ x \notin X}} \left\{ \sum_{T \in \mathcal{T}[X]} \prod_{e \in T} [-M_{i_e j_e}][\sum_{r \in X} B_r] \right\}$$

where $\mathcal{P}[\Lambda]$ is the set partitions of Λ, $\mathcal{T}[X]$ the set of spanning trees on X, and finally $\mathcal{P}_{xy}[\Lambda]$ is the set of partitions such that x and y belong to the same element of the partition: this special element of the partition is denoted by X_x.

Remark. The general matrix-tree theorem applies also to non-symmetric and non invertible matrices, with a slight modification in the definitions.

With these definitions we can prove the following result.

Lemma 4 *Let $\Lambda = \{-L, \dots, L\}^d$ and $A_\Lambda = -\beta \Delta^D_\Lambda + m^2 \mathbb{I}_\Lambda$ a matrix on $\Lambda \times \Lambda$, where $-\Delta^D_\Lambda$ is the discrete Laplacian with Dirichlet boundary conditions. Then*

$$0 \leqslant (A^{-1}_\Lambda)_{xy} \qquad \forall x, y \in \Lambda \tag{3.2.7}$$

and

$$\sum_{z \in \partial\Lambda} (A^{-1}_\Lambda)_{xz} \leqslant \frac{1}{\beta} \qquad \forall x \in \Lambda. \tag{3.2.8}$$

Proof. Applying the matrix-tree theorem we can write

$$(A^{-1}_\Lambda)_{xy} = \frac{\sum_{P \in \mathcal{P}_{xy}[\Lambda]} \left[\sum_{T \in \mathcal{T}[X_x]} \prod_{e \in T} [-A_{i_e j_e}] \right] \prod_{\substack{X \in P \\ x \notin X}} \left\{ \sum_{T \in \mathcal{T}[X]} \prod_{e \in T} [-A_{i_e j_e}][\sum_{r \in X} B_r] \right\}}{\sum_{P \in \mathcal{P}[\Lambda]} \prod_{X \in P} \left\{ \sum_{T \in \mathcal{T}[X]} \prod_{e \in T} [-A_{i_e j_e}][\sum_{r \in X} B_r] \right\}}$$

Note that $-A_{i_e j_e} = \beta$ when $i_e \sim j_e$, i.e $|i_e - j_e| = 1$ and zero otherwise (since $i_e \neq j_e$ for any edge e in the forest). Then only nearest neighbor edges give a non zero contribution. Let $\mathcal{P}^c[\Lambda]$ the set of partitions of Λ into *connected components* and $\tilde{\mathcal{T}}[X]$ the set of trees on X made only of nearest neighbor pairs $i \sim j$. Finally note that

$$B_r = \sum_{j\in\Lambda} A_{jr} = m^2 + \beta d_r$$

where

$$d_r = \#\{j \in \Lambda^c |\ |j-r| = 1\} \quad \text{so} \quad \begin{cases} d_r = 0 & \text{if } r \in \Lambda \backslash \partial\Lambda \\ d_r \in \{1,\dots,d\} & \text{if } r \in \partial\Lambda. \end{cases}$$

Then $\sum_{r\in X} B_r = m^2|X| + \beta d_X$, where

$$d_X = \sum_{r\in X} d_r, \qquad \text{hence} \quad |X \cap \partial\Lambda| \leqslant d_X \leqslant d|X\cap\partial\Lambda|.$$

Inserting all this we obtain

$$(A_\Lambda^{-1})_{xy} = \frac{\sum_{P\in\mathcal{P}^c_{xy}[\Lambda]} \left[\sum_{T\in\tilde{\mathcal{T}}[X_x]} \beta^{|T|}\right] \prod_{\substack{X\in P \\ x\notin X}} \left\{\sum_{T\in\tilde{\mathcal{T}}[X]} \beta^{|T|}\ [m^2|X| + \beta d_X]\right\}}{\sum_{P\in\mathcal{P}^c[\Lambda]} \prod_{X\in P} \left\{\sum_{T\in\tilde{\mathcal{T}}[X]} \beta^{|T|}\ [m^2|X| + \beta d_X]\right\}}$$

This expression is manifestly positive hence (3.2.7). Let

$$\omega(X) = [m^2|X| + \beta d_X] \sum_{T\in\tilde{\mathcal{T}}[X]} \beta^{|T|}.$$

Then

$$\rho(P) = \frac{\prod_{X\in P}\omega(X)}{\sum_{P\in\mathcal{P}^c[\Lambda]}\prod_{X\in P}\omega(X)}$$

is a probability measure on $\mathcal{P}^c[\Lambda]$ and $(A_\Lambda^{-1})_{xy}$ can be expressed as an average

$$(A_\Lambda^{-1})_{xy} = \sum_{P\in\mathcal{P}^c_{xy}[\Lambda]} \rho(P) \tfrac{1}{[m^2|X_x| + \beta d_{X_x}]}$$

To prove (3.2.8) we replace y by z and sum over all $z \in \partial\Lambda$

$$\begin{aligned}\sum_{z\in\partial\Lambda} (A_\Lambda^{-1})_{xz} &= \sum_{z\in\partial\Lambda} \sum_{P\in\mathcal{P}^c_{xz}[\Lambda]} \rho(P) \tfrac{1}{[m^2|X_x| + \beta d_X]} \\ &= \sum_{P\in\mathcal{P}^c_{x\partial\Lambda}[\Lambda]} \sum_{z\in X_x\cap\partial\Lambda} \rho(P) \tfrac{1}{[m^2|X_x| + \beta d_{X_x}]} = \sum_{P\in\mathcal{P}^c_{x\partial\Lambda}[\Lambda]} \rho(P) \tfrac{|X_x\cap\partial\Lambda|}{[m^2|X_x| + \beta d_{X_x}]} \leqslant \frac{1}{\beta}\end{aligned}$$

since $d_{X_x} \geqslant |X_x \cap \partial\Lambda|$. This ends the proof. ■

3.3 Perturbation around a gaussian integral

3.3.1 The $O(n)$ model

Let $\Lambda = \{-L, \dots, L\}^d$ a cube inside $\mathbb{Z}^d$. To each lattice point j we associate a spin $S_j \in \mathcal{S}_n$ taking values in the unit $n-$dimensional sphere. The three main examples are

1. $n = 1$: in this case $S_j = \pm 1$ and we obtain the Ising model;
2. $n = 2$: the spin takes values on the unit circle. This is the so called XY (or rotator) model;
3. $n = 3$: the spin takes value on the sphere. This is the so called Heisenberg model.

The space of configurations is $\Omega_\Lambda = \{S : \Lambda \to \mathcal{S}_n\}$ and the corresponding Gibbs measure is

$$d\mu^{\beta,h}_{\Lambda,n}(S) = \prod_{j\in\Lambda} d\Omega_n(S_j)\; e^{\frac{\beta}{2}\sum_{j,k\Lambda} J_{jk}(S_j,S_k)} e^{(h,\sum_{j\in\Lambda} S_j)} \tag{3.3.9}$$

where $(.,.)$ is the euclidean scalar product in $\mathbb{R}^n$, $h \in \mathbb{R}^n$ is the magnetic field and J_{jk} is a collection of real interaction constants such that

$$J_{jk} = J_{kj} \geqslant 0 \;\forall\; j,k \in \Lambda$$

and there exists a constant $c > 0$ independent of the volume Λ such that

$$0 \leqslant \sum_{k\in\Lambda} J_{jk} \leqslant c \qquad \forall j \in \Lambda.$$

One can understand this constraint by regarding J_{jk} as the probability to jump from j to k. Then $\sum_{k\in\Lambda} J_{jk} = 1$ since it is the probability of jumping to *any* point. Finally $d\Omega_n$ is the invariant measure on the sphere $\mathcal{S}_n$, normalized to 1. In particular

1. for the Ising model the measure is discrete: $\int d\Omega_1 = \frac{1}{2}\sum_{\sigma=\pm1}$;
2. for $n = 2$ we can parametrize the circle by one angle: $\int d\Omega_2 = \frac{1}{2\pi}\int_0^{2\pi} d\theta$;
3. for $n = 3$ we can parametrize the sphere by two angles: $\int d\Omega_3 = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta$.

Phenomenology and symmetries.

Since $J_{jk} \geqslant 0$ the interaction favors the configurations with spins *aligned* (we have a so called " *ferromagnetic interaction*").

When $h = 0$ the Gibbs measure is invariant under global rotation

$$S_j \to US_j \;\forall j \qquad U^*U = \mathrm{Id}_{\mathbb{R}^n} \tag{3.3.10}$$

for any $n \geqslant 2$. In particular it is invariant under *flip* $S_j \to -S_j$ $\forall j$ (this is true also for $n = 1$). Then

$$\mathbb{E}_\Lambda^{\beta,h=0}[S_j] = -\mathbb{E}_\Lambda^{\beta,0}[S_j] \Rightarrow \ \mathbb{E}_\Lambda^{\beta,0}[S_j] = 0 \ \forall j \in \Lambda, \ \forall n \geqslant 1, \ \forall d \geqslant 1,$$

and we say that the average magnetization is zero (the spins are not aligned). For general h, the finite volume magnetization $\mathbb{E}_\Lambda^{\beta,h}[\frac{1}{|\Lambda|}\sum_{j\in\Lambda} S_j]$ is a smooth function in each component of the vector h hence

$$\lim_{\Lambda\to\mathbb{Z}^d} \lim_{h\to 0} \mathbb{E}_\Lambda^{\beta,h} \left[\frac{1}{|\Lambda|}\sum_{j\in\Lambda} S_j\right] = 0.$$

If we invert the limits we may have two results:

$$\lim_{h\to 0} \mathbb{E}_\Lambda^{\beta,h} \lim_{\Lambda\to\mathbb{Z}^d} \mathbb{E}_\Lambda^{\beta,h} \left[\frac{1}{|\Lambda|}\sum_{j\in\Lambda} S_j\right] = \begin{cases} 0 \\ M \neq 0. \end{cases}$$

In the first case there is no magnetization. This means the infinite volume measure $\lim_{\Lambda\to\mathbb{Z}^d} d\mu_{\Lambda,\beta,h}$ recovers the flip symmetry when $h \to 0$. In this case we say *the symmetry is restored*. In the second case we have magnetization. Then the infinite volume measure $\lim_{\Lambda\to\mathbb{Z}^d} d\mu_{\Lambda,\beta,h}$ does not recover the symmetry when $h \to 0$. Then we say we have *spontaneous symmetry breaking*.

One can show that at high enough temperature (i.e. β small) there is never a magnetization, since the thermal fluctuations are too strong. On the contrary at low temperature (i.e. β large) the forces trying to align the spins may be strong enough to create a magnetization. In this case we say we have a *phase transition*.

Mermin-Wagner: low dimensional systems.

Phase transitions are harder to observe in low dimensions. This is the content of the so called *Mermin-Wagner theorem* (also known a Mermin-Wagner-Hohenberg theorem or Coleman theorem). It is a series of papers that can be summarized in the following statement:

Continuous symmetries cannot be spontaneously broken at finite temperature in systems with sufficiently short-range interactions in dimensions $d \leqslant 2$.

Application to $O(n)$ with short range interaction.

Let us consider the $O(n)$ model defined above with $J_{jk} = 1$ when $|j-k| = 1$ and $J_{jk} = 0$ otherwise. Then

$$d\mu(S) = \prod_{j\in\Lambda} d\Omega_n(S_j) \ e^{\beta\sum_{j\sim k}(S_j,S_k)} e^{(h,\sum_{j\in\Lambda} S_j)},$$

where $j \sim k$ means the two points are nearest neighbors in $\mathbb{Z}^d$. For $n \geqslant 2$ this measure has a continuous symmetry at $h = 0$, so by Mermin-Wagner theorem we cannot expect a magnetization (hence a phase transition) in $d \leqslant 2$. The theorem does not apply to $n = 1$ (Ising model) since there the symmetry is discrete ($\sigma \to -\sigma$).

In $d = 2$ one may still observe a softer version of phase transition known as *Kosterlitz-Thouless transition* that corresponds to a change in the decay rate of two point correlations. More precisely

$$\lim_{\Lambda \to \mathbb{Z}^d} \mathbb{E}_\Lambda^{\beta,0} \ [S_0 S_x] \ = \begin{cases} c_1 e^{-\frac{|x|}{\xi}} & T \gg 1 (\text{i.e.} \beta \ll 1) \\ \frac{c_2}{|x|^\eta} & T \ll 1 (\text{i.e.} \beta \gg 1) \end{cases} \qquad \text{as } |x| \gg 1.$$

for some constants $c_1, c_2, \xi, \eta > 0$.

3.3.2 A first example of perturbation around a Gaussian measure: the $O(2)$ model in $d = 2$

Let $\Lambda = \mathbb{Z}^2/\{-L, \dots, L\}^2$ a cube in $\mathbb{Z}^2$ with periodic boundary conditions. The space of configurations is $\Omega_\Lambda = \{S : \Lambda \to \mathcal{S}_2\}$ and we consider the Gibbs measure

$$d\mu(S) = \prod_{j \in \Lambda} d\Omega_2(S_j) \ e^{\beta \sum_{j \sim k} (S_j, S_k)}$$

where $j \sim k$ are pairs at distance one in the torus. For this model one can prove a Kosterlitz-Thouless transition. More precisely we have

Theorem 1 [Mc Bryan, Spencer (1977)]. For any $0 < \epsilon < 1$ there exists a $\beta_0(\epsilon) > 0$ such that for all $\beta \geqslant \beta_0(\epsilon)$

$$\lim_{\Lambda \to \mathbb{Z}^d} |\mathbb{E}_\Lambda^{\beta,0} [S_0 S_x]| \ \leqslant \frac{1}{|x|^{\frac{1-\epsilon}{2\pi\beta}}} \tag{3.3.11}$$

Theorem 2 [Fröhlich, Spencer (1981)]. There exists a a $\beta_0 > 0$ and a constant $c > 0$ such that for all $\beta \geqslant \beta_0$

$$\lim_{\Lambda \to \mathbb{Z}^d} |\mathbb{E}_\Lambda^{\beta,0} [S_0 S_x]| \ \geqslant \frac{c}{|x|^{\frac{1}{2\pi\beta}}}.$$

Theorem 3. There exists a a $\beta_0 > 0$ such that for all $\beta \leqslant \beta_0$

$$\lim_{\Lambda \to \mathbb{Z}^d} |\mathbb{E}_\Lambda^{\beta,0} [S_0 S_x]| \ \leqslant C_\beta e^{-\frac{|x|}{\xi_\beta}}$$

In this section we will review the proof of Theorem 1. This is based on two steps. The first is non rigorous and consists in approximating the measure by a Gaussian integral. The second step is rigorous and consists in mimicking some of the operations we did to compute the (non-rigorous) Gaussian approximation in a rigorous context. The key step is a complex deformation.

Proof of Theorem 1 (based on [Mc Bryan, Spencer]) Using polar coordinates the average above can be written as

$$\mathbb{E}_\Lambda^{\beta,0}\left[S_0 S_x\right] = \frac{1}{Z_\Lambda}\int_{[0,2\pi]^{|\Lambda|}} e^{\beta\sum_{j\sim j'}\cos(\theta_j-\theta_{j'})}\cos(\theta_x-\theta_0)\prod_{j\in\Lambda} d\theta_j$$
$$= \frac{1}{2Z_\Lambda}(I_+ + I_-)$$

where the partition function is

$$Z_\Lambda = \int_{[0,2\pi]^{|\Lambda|}} e^{\beta\sum_{j\sim j'}\cos(\theta_j-\theta_{j'})}\prod_{j\in\Lambda} d\theta_j$$

and we defined

$$I_\sigma = \int_{[0,2\pi]^{|\Lambda|}} e^{\beta\sum_{j\sim j'}\cos(\theta_j-\theta_{j'})} e^{i\sigma(\theta_x-\theta_0)}\prod_{j\in\Lambda} d\theta_j, \quad \sigma = \pm 1.$$

Preliminary heuristic arguments. Using some *non rigorous arguments* we establish what kind of behavior we expect from the integrals above. Since $1 \geqslant \cos(\theta_j - \theta_{j'}) \geqslant -1$ and $\beta \gg 1$, the function $\exp[\beta\cos(\theta_j-\theta_{j'})]$ is exponentially small unless $\theta_j - \theta_{j'} \simeq 0$ or 2π. Inspired by this fact we perform two approximations.

a). We take the Taylor expansion up to order 2 and neglect the remainder. Then

$$e^{\beta\sum_{j\sim j'}\cos(\theta_j-\theta_{j'})} \simeq e^{-\beta \mathrm{C}(\Lambda)} e^{-\frac{\beta}{2}\sum_{j\sim j'}(\theta_j-\theta_{j'})^2} = e^{-\beta \mathrm{C}(\Lambda)} e^{-\frac{\beta}{2}(\theta,-\Delta_\Lambda\theta)}$$

where $C(\Lambda) = \sum_{j\sim j'} 1$ is a constant independent of θ and $-\Delta_\Lambda$ is the discrete Laplacian on Λ with periodic boundary conditions.

b). We replace the interval $[0, 2\pi]$ by $\mathbb{R}$ in the integral, for each $j \in \Lambda$. Inserting these two approximations both in the numerator and in the partition function above we obtain

$$\frac{I_\sigma}{Z_\Lambda} \simeq \frac{\int_{\mathbb{R}^{|\Lambda|}} e^{-\frac{\beta}{2}(\theta,-\Delta_\Lambda\theta)} e^{i\sigma(\theta_x-\theta_0)}\prod_{j\in\Lambda} d\theta_j}{\int_{\mathbb{R}^{|\Lambda|}} e^{-\frac{\beta}{2}(\theta,-\Delta_\Lambda\theta)}\prod_{j\in\Lambda} d\theta_j}$$

where the normalization is

$$\mathcal{N} = \int_{\mathbb{R}^{|\Lambda|}} e^{-\frac{\beta}{2}(\theta,-\Delta_\Lambda\theta)}\prod_{j\in\Lambda} d\theta_j.$$

These two integrals are ill defined since $-\Delta_\Lambda$ is not invertible! One may give a sensible definition of a Gaussian measure even in this situation, but since here we are doing non rigorous arguments we ignore the problem. We introduce now the two functions

$$\begin{array}{ll} v: & \Lambda \to \mathbb{R} \\ & j \to v_j = \delta_{jx} - \delta_{j0} \end{array}, \qquad \begin{array}{ll} \alpha: & \Lambda \to \mathbb{R} \\ & j \to \alpha_j = [(-\beta\Delta_\Lambda)^{-1} v]_j. \end{array} \tag{3.3.12}$$

Note that $\sum_j v_j = (1, v) = 0$, $v \in \ker(-\Delta_\Lambda)^\perp$, therefore the function α is *well defined*, even if $(-\Delta_\Lambda)$ is not invertible. Then $(\theta_x - \theta_0) = (v, \theta)$ and

$$I_\sigma = e^{-\frac{1}{2}(\alpha,(-\beta\Delta_\Lambda)\alpha)} \int_{\mathbb{R}^{|\Lambda|}} e^{-\frac{1}{2}((\theta - i\sigma\alpha), -\beta\Delta_\Lambda(\theta - i\sigma\alpha))} \prod_{j\in\Lambda} d\theta_j = e^{-\frac{1}{2}(\alpha,(-\beta\Delta_\Lambda)\alpha)} \mathcal{N}$$

where in the last step we perform the complex traslation

$$\theta_j \to \theta_j + i\sigma\alpha_j, \qquad \forall j \in \Lambda.$$

Now inserting the definition of α

$$\begin{aligned}
(\alpha, (-\beta\Delta_\Lambda)\alpha) &= (v, (-\beta\Delta_\Lambda)^{-1} v) \\
&= \frac{1}{\beta}\left[(-\Delta_\Lambda)^{-1}_{00} - (-\Delta_\Lambda)^{-1}_{0x} + (-\Delta_\Lambda)^{-1}_{x0} - (-\Delta_\Lambda)^{-1}_{xx}\right] \\
&= \frac{1}{\beta(2L+1)^2} \sum_{n\in\Lambda_L\setminus 0} \frac{2(1-\cos(k_n x))}{2(1-\cos(k_{n1})) + 2(1-\cos(k_{n2}))} \\
&= 2\frac{1}{2\pi\beta} \ln\|x\| \left[1 + O\left(\tfrac{1}{\ln\|x\|}\right)\right] \sim 2\frac{1}{2\pi\beta}\ln\|x\| \qquad |x| \gg 1
\end{aligned}$$

With these approximations we would obtain

$$\mathbb{E}^{\beta,0}_\Lambda [S_0 S_x] = \frac{I_+ + I_-}{2Z_\Lambda} \simeq \frac{1}{|x|^{\frac{1}{2\pi\beta}}}, \qquad |x| \gg 1.$$

Step 2. Inspired by the non rigorous arguments above we perform the following complex translation in the integral I_σ:

$$\theta_j \to \theta_j + i\sigma\alpha_j, \qquad \forall j \in \Lambda,$$

where α_j is defined in (3.3.12). Remember that the definitions given in (3.3.12) make sense even though $(-\Delta)$ is not invertible. The integral becomes

$$I_\sigma = e^{-(\alpha_x - \alpha_0)} \int_{[0,2\pi]^{|\Lambda|}} e^{\beta\sum_{j\sim j'}\cos(\theta_j - \theta_{j'} + i(\alpha_j - \alpha_{j'}))} e^{i\sigma(\theta_x - \theta_0)} \prod_{j\in\Lambda} d\theta_j.$$

In order to close the contour in the complex plane we need to add the integrals along the paths $y_j = iz_j,\ z_j \in [0, \sigma\alpha_j]$ and $y_j = 2\pi + iz_j,\ z_j \in [0, \sigma\alpha_j]$. By periodicity they cancel each other. Since

$$\cos(\theta_j - \theta_{j'} + i(\alpha_j - \alpha_{j'})) = \cos(\theta_j - \theta_{j'})\cosh(\alpha_j - \alpha_{j'}) - i\sin(\theta_j - \theta_{j'})\sinh(\alpha_j - \alpha_{j'})$$

after inserting absolute values we have

$$\begin{aligned}
|I_\sigma| &\leqslant e^{-(\alpha_x - \alpha_0)} \int_{[0,2\pi]^{|\Lambda|}} e^{\beta\sum_{j\sim j'}\cos(\theta_j - \theta_{j'})\cosh(\alpha_j - \alpha_{j'})} \prod_{j\in\Lambda} d\theta_j \\
&\leqslant e^{-(\alpha_x - \alpha_0)} e^{\beta\sum_{j\sim j'}[\cosh(\alpha_j - \alpha_{j'}) - 1]} \int_{[0,2\pi]^{|\Lambda|}} e^{\beta\sum_{j\sim j'}\cos(\theta_j - \theta_{j'})} \prod_{j\in\Lambda} d\theta_j \\
&= Z_\Lambda e^{-(\alpha_x - \alpha_0)} e^{\beta\sum_{j\sim j'}[\cosh(\alpha_j - \alpha_{j'}) - 1]}
\end{aligned}$$

where in the second line we use

$\cos(\theta_j - \theta_{j'})[\cosh(\alpha_j - \alpha_{j'}) - 1] + \cos(\theta_j - \theta_{j'}) \leqslant [\cosh(\alpha_j - \alpha_{j'}) - 1] + \cos(\theta_j - \theta_{j'})$.

Now

$$|\alpha_j - \alpha_{j'}| \leqslant \frac{1}{\beta} |[(-\Delta)^{-1}_{j0} - (-\Delta)^{-1}_{j'0}] + [(-\Delta)^{-1}_{jx} - (-\Delta)^{-1}_{j'x}]| \leqslant \frac{K}{\beta}$$

for some constant K independent of x and Λ. This last inequality can be obtained by the same kind of arguments in the Fourier sum we used to prove the estimate (3.1.2) in Lemma 2. Since β is large we can make $|\alpha_j - \alpha_{j'}|$ as small as we want. To complete the argument note that for any $0 < \epsilon < 1$ there exists a $\delta(\epsilon) > 0$ such that

$$\cosh(t) - 1 \leqslant \frac{1 + \epsilon/2}{2} t^2 \qquad \forall |t| \leqslant \delta,$$

where the factor $1/2$ in front of ϵ is just a convenient choice to control some additional error terms later in the proof. From the bound above there exists a β_0 such that $|\alpha_j - \alpha_{j'}| \leqslant \delta$ for all $j \sim j'$ and for any $\beta \geqslant \beta_0$. Inserting this in our estimate we obtain

$$\begin{aligned}
|\mathbb{E}^{\beta,0}_{\Lambda}\left[S_0 S_x\right]| &\leqslant \frac{|I_+| + |I_-|}{2Z_\Lambda} \leqslant e^{-(\alpha_x - \alpha_0)} e^{\beta \sum_{j \sim j'} \frac{1+\epsilon/2}{2} (\alpha_j - \alpha_{j'})^2} \\
&= e^{-(\alpha_x - \alpha_0)} e^{\frac{1+\epsilon/2}{2} (\alpha, -\beta \Delta_\Lambda \alpha)} \\
&= e^{-\frac{1-\epsilon/2}{2} (v, (-\beta\Delta_\Lambda)^{-1} v)} = e^{-\frac{1-\epsilon/2}{2\pi\beta} \ln \|x\| \left[1 + O\left(\frac{1}{\ln \|x\|}\right)\right]} \\
&\leqslant e^{-\frac{1-\epsilon}{2\pi\beta} \ln \|x\|} = \frac{1}{|x|^{\frac{1-\epsilon}{2\pi\beta}}}
\end{aligned}$$

where in the last line we use $(\alpha_x - \alpha_0) = (v, \alpha)$, $\alpha = (-\beta\Delta_\Lambda)^{-1} v$ and we take $\|x\|$ large enough to ensure

$$\left[1 + O\left(\tfrac{1}{\ln \|x\|}\right)\right] \geqslant (1 - \epsilon/2).$$

This concludes the proof.

3.3.3 An example of phase transition: the mean field case

In this section we consider the $O(n)$ model defined in (3.3.9) with non zero magnetic field $h \in \mathbb{R}^n$ and with interaction parameter

$$J_{jk} = \frac{1}{|\Lambda|} \qquad \forall i, j \in \Lambda.$$

With this choice

$$0 \leqslant \sum_{k \in \Lambda} J_{jk} \leqslant 1 \qquad \forall j \in \Lambda.$$

Note that in this case we have *long range interactions* since J_{jk} is constant for any pair $jk \in \Lambda$. Then the Mermin-Wagner theorem does not apply and one may have a phase transition also in $d = 2$.

Duality

The partition function in the mean field $O(n)$ model can be reformulated as an integral over n real variables

Lemma 5 *For any dimension $d \geqslant 1$ and any $n \geqslant 1$ we have*

$$Z^{\beta}_{\Lambda,n}(h) = \int d\mu^{\beta,h}_{\Lambda,n}(S) = \frac{1}{\mathcal{N}_{\Lambda,n,\beta}} \int_{\mathbb{R}^n} d^n x \; e^{-|\Lambda| F_{n,\beta}(x,h)}, \qquad \textit{with } \mathcal{N}_{\Lambda,n,\beta} = \left(\frac{2\pi\beta}{|\Lambda|}\right)^{|\Lambda|},$$

$$\begin{array}{llll} F_{n,\beta}: & \mathbb{R}^n \times \mathbb{R}^n & \to \mathbb{R} & \\ & (x,h) & \to F_{n,\beta}(x,h) = \frac{\|(x-h\|^2}{2\beta} - \ln J_n(\|(x\|) & \end{array}$$

and

$$\begin{array}{llll} J_n: & \mathbb{R}^+ & \to \mathbb{R}^+ & \\ & t & \to J_n(t) & = \cosh t \qquad \textit{if } n = 1 \\ & & & = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} (\sin\theta)^{n-2} \cosh[t\cos\theta] d\theta \qquad \textit{if } n \geqslant 2 \end{array}$$

Proof Since $J_{ij} = |\Lambda|^{-1}$ $\forall i,j$ we can write

$$e^{\frac{\beta}{2}\sum_{j,k\Lambda} J_{jk}(S_j,S_k)} = e^{\frac{\beta}{2|\Lambda|}\|\sum_{j\in\Lambda} S_j\|^2} = \frac{1}{\mathcal{N}_{\Lambda,n,\beta}} \int_{\mathbb{R}^n} d^n x \; e^{-\frac{|\Lambda|}{2\beta}\|x\|^2} e^{(x,\sum_{j\in\Lambda} S_j)}$$

Exchanging the integrals we obtain

$$\begin{aligned} Z^{\beta}_{\Lambda,n}(h) &= \frac{1}{\mathcal{N}_{\Lambda,n,\beta}} \int_{\mathbb{R}^n} d^n x \; e^{-\frac{|\Lambda|}{2\beta}\|x\|^2} \left[\int d\Omega_n(S) \; e^{(x+h,S)}\right]^{|\Lambda|} \\ &= \frac{1}{\mathcal{N}_{\Lambda,n,\beta}} \int_{\mathbb{R}^n} d^n x \; e^{-\frac{|\Lambda|}{2\beta}\|x-h\|^2} \left[\int d\Omega_n(S) \; e^{(x,S)}\right]^{|\Lambda|} \end{aligned}$$

When $n = 1$ we have

$$\int d\Omega_1(S) \; e^{(x,S)} = \frac{1}{2}\sum_{\sigma=\pm 1} e^{x\sigma} = \cosh(x) = \cosh(|x|).$$

When $n = 2$ we have

$$\int d\Omega_2(S)\, e^{(x,S)} = \frac{1}{2\pi}\int_0^{2\pi} e^{\|x\|\cos\theta} d\theta = \frac{1}{\pi}\int_0^{\pi} e^{\|x\|\cos\theta} d\theta = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cosh(\|x\|\cos\theta) d\theta,$$

where in the first passage we perform a rotation in order to have x parallel to the vertical axis, then go to polar coordinates. Similarly for $n > 2$ we have

$$\int d\Omega_n(S)\, e^{(x,S)} = \frac{1}{\pi}\int_0^{\pi} (\sin\theta)^{n-2} e^{\|x\|\cos\theta} d\theta = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} (\sin\theta)^{n-2} \cosh(\|x\|\cos\theta) d\theta.$$

□

Remarks. The duality reduces the problem to the study of a n variable integral (compared to $n^{|\Lambda|}$ variables in the initial representation). Moreover, for large $|\Lambda|$ the integral will be concentrated around the minimal with respect to x of the function $F_n(x,h)$, therefore a saddle point analysis is possible.

Generating function

Using the dual representation above the average magnetization at finite volume can be expressed as

$$\mathbb{E}_\Lambda^{\beta,h}\left[\tfrac{1}{|\Lambda|}\sum_{j\in\Lambda} S_j\right] = \tfrac{1}{|\Lambda|}\partial_h \ln Z^\beta_{\Lambda,n}(h) = \tfrac{1}{\beta}\frac{\int_{\mathbb{R}^n} d^n x\ (x-h)e^{-|\Lambda|F_{n,\beta}(x,h)}}{\int_{\mathbb{R}^n} d^n x\ e^{-|\Lambda|F_{n,\beta}(x,h)}} \qquad (3.3.13)$$

Phase transition

Theorem The $O(n)$ model in the mean field case has a phase transition in any $d \geqslant 1$. Precisely

$$\lim_{h\to 0+}\lim_{\Lambda\to\mathbb{Z}^d}\mathbb{E}_\Lambda^{\beta,h}\left[\frac{1}{|\Lambda|}\sum_{j\in\Lambda} S_j\right] = \begin{cases} 0 & \text{if } \beta<1 \text{ (high temperature)} \\ M_{d,\beta,n}>0 & \text{if } \beta>1 \text{ (low temperature)} \end{cases}$$

Proof In the following we set $h>0$. By (3.3.13) the problem can be reduced to the rigorous saddle analysis of a n variable integral. For simplicity we will restrict here to the case $n=1$. Then

$$F_1(x,h) = \frac{(x-h)^2}{2\beta} - \ln\cosh x$$

and the equations for the first and second derivative are

$$\partial_x F_1(x,h) = \tfrac{(x-h)}{\beta} - \tanh x, \qquad \partial_x^2 F_1(x,h) = \tfrac{1}{\beta} - \tfrac{1}{(\cosh x)^2}.$$

Note that

$$\partial_x^2 F_1(x,h) \leqslant \frac{1}{\beta} \qquad \forall x,h. \qquad (3.3.14)$$

Case 1: $\beta<1$ (high temperature). In this case F_1 is a convex function in x

$$\partial_x^2 F_1(x,h) \geqslant \tfrac{(1-\beta)}{\beta}\ \forall x,h \qquad (3.3.15)$$

therefore F_1 has only one minimum at the point $x_0(h)$ satisfying

$$\tfrac{(x_0-h)}{\beta} = \tanh x_0.$$

At $h=0$ $x_0=0$ is a solution of this equation, therefore $\lim_{h\to 0} x_0(\beta,h)=0$. By a Taylor expansion with integral remainder

$$F_1(x,h) = F_1(x_0,h) + (x-x_0)^2\int_0^1 (1-t)\partial_x^2 F_1(x_0+t(x-x_0),h)\ dt.$$

Inserting (3.3.14) and (3.3.15) we obtain $\forall x, h$

$$F_1(x_0,h) + \tfrac{1}{2\beta}(x-x_0)^2 \geqslant F_1(x,h) \geqslant F_1(x_0,h) + \tfrac{(1-\beta)}{2\beta}(x-x_0)^2. \qquad (3.3.16)$$

Now we can reexpress (3.3.13) as

$$\mathbb{E}_\Lambda^{\beta,h}\left[\tfrac{1}{|\Lambda|}\sum_{j\in\Lambda} S_j\right] = \tfrac{1}{\beta}\frac{\int_{\mathbb{R}} dx\ (x-h)e^{-|\Lambda|F_{1,\beta}(x,h)}}{\int_{\mathbb{R}} dx\ e^{-|\Lambda|F_{1,\beta}(x,h)}} = \tfrac{x_0(\beta,h)-h}{\beta} + R(\beta,h,|\Lambda|),$$

where

$$R(\beta,h,|\Lambda|) = \frac{\int_{\mathbb{R}} dx\ (x-x_0)e^{-|\Lambda|F_{n,\beta}(x,h)}}{\int_{\mathbb{R}} dx\ e^{-|\Lambda|F_{n,\beta}(x,h)}}.$$

Inserting absolute values, and the upper and lower estimates from (3.3.16) we obtain

$$\begin{aligned}|R(\beta,h,|\Lambda|)| &\leqslant \frac{\int_{\mathbb{R}} dx\ |x-x_0| e^{-\frac{|\Lambda|(1-\beta)}{2\beta}(x-x_0)^2}}{\int_{\mathbb{R}} dx\ e^{-\frac{|\Lambda|}{2\beta}(x-x_0)^2}} \\ &= \frac{2\int_0^\infty dx\ x\ e^{-\frac{|\Lambda|(1-\beta)}{2\beta}x^2}}{\int_{\mathbb{R}} dx\ e^{-\frac{|\Lambda|}{2\beta}x^2}} = \tfrac{1}{\sqrt{|\Lambda|}}\tfrac{2\sqrt{\beta}}{\sqrt{2\pi}(1-\beta)} \to_{|\Lambda|\to\infty} 0\end{aligned}$$

Finally

$$\lim_{h\to 0+}\lim_{\Lambda\to\mathbb{Z}^d}\mathbb{E}_\Lambda^{\beta,h}\left[\tfrac{1}{|\Lambda|}\sum_{j\in\Lambda} S_j\right] = \lim_{h\to 0+}\tfrac{x_0(\beta,h)-h}{\beta} = 0.$$

Case 2: $\beta > 1$ (low temperature). In this case the function $F_1(x,h)$ has two minimum points $x_1(h), x_2(h)$ satisfying

$$x_1(h) < 0, \quad x_2(h) > 0, \quad \lim_{h\to 0} x_2(h) = -\lim_{h\to 0} x_1(h) = x_0(\beta) > 0.$$

At $h = 0$ F_1 is symmetric in x so the two minimums are at the same height

$$F_1(-x_0(\beta),0) = F_1(x_0(\beta),0) = F_m.$$

To see what is the approximate value of the two minimum points at $h \neq 0$, we expand near $h = 0$ (remember that at the end we will take the limit $h \to 0$)

$$x_j(h) = \sigma_j x_0 + \delta_j h + O(h^2), \quad \sigma_1 = -1,\ \sigma_2 = 1.$$

Inserting this relation in the saddle point equation we obtain

$$\begin{aligned}0 &= \partial_x F_1(x_j(h),h) \\ &= \partial_x F_1(x_j(0),0)\ +\ \partial_x^2 F_1(x_j(0),0)\ \delta_j h\ +\ \partial_h\partial_x F_1(x_j(0),0)\ h\ +\ O(h^2) \\ &= h\left(\partial_x^2 F_1(x_j(0)\ \delta_j + \partial_h\partial_x F_1(x_j(0),0)\right) + O(h^2)\end{aligned}$$

since $\partial_x F_1(x_j(0), 0) = 0$. Note that

$$\partial_x^2 F_1(x_j(0), 0) = \tfrac{1}{\beta} - \tfrac{1}{(\cosh x_0(\beta))^2} = H(\beta) > 0, \quad \partial_h \partial_x F_1(x_j(0), 0) = -\frac{1}{\beta}$$

are independent of j then

$$\delta_1 = \delta_2 = \delta = \tfrac{1}{\beta H(\beta)} > 0.$$

Inserting these results in the expression for F_1 and expanding around $h = 0$ we obtain

$$\begin{aligned} F_1(x_j(h), h) &= F_1(x_j(0), 0) + \partial_x F_1(x_j(0), 0)\delta h + \partial_h F_1(x_j(0), 0)h + O(h^2) \\ &= F_1(x_j(0), 0) - \frac{x_j(0)}{\beta} h + O(h^2) = F_m - \sigma_j \frac{x_0(\beta)}{\beta} h + O(h^2) \end{aligned}$$

Then

$$F_1(x_1(h), h) - F_1(x_2(h), h) = \frac{2hx_0(\beta)}{\beta} > 0, \quad \text{since } h > 0,$$

and F_1 has a *global minimum* at $x_2(h)$. As in the case $\beta < 1$ we extract the contribution of the minimum

$$\mathbb{E}_\Lambda^{\beta,h}\left[\tfrac{1}{|\Lambda|}\sum_{j\in\Lambda} S_j\right] = \tfrac{x_2(h)-h}{\beta} + R(\beta, h, |\Lambda|)$$

where

$$|R(\beta, h, |\Lambda|)| \leqslant \frac{\int_{\mathbb{R}} dx\ |x - x_2| e^{-|\Lambda|[F_1(x,h)-F_m]}}{\int_{\mathbb{R}} dx\ e^{-|\Lambda|[F_1(x,h)-F_m]}} = \frac{N}{D}.$$

To estimate the integral in the numerator we distinguish three regions

$$I_1 = \{x|\ |x - x_2(h)| < \epsilon\}, \quad I_2 = \{x|\ |x| > M\}, \quad I_3 = \{x|\ |x| \leqslant M, |x - x_2(h)| \geqslant \epsilon\}$$

where ϵ and are chosen in order to have $I_2 \cap I_1 = \varnothing$,

$$\partial_x^2 F_1(x, h) > c_1 > 0 \quad \forall x \in I_1, \quad \text{and} \quad [F_1(x, h) - F_m] \geqslant \frac{c_2}{2} x^2\ \forall x \in I_3,$$

for some constant c_1, c_2. It is not difficult to see that such regions exist for the function F_1. Then

$$\int_{I_1} dx\ |x - x_2| e^{-|\Lambda|[F_1(x,h)-F_m]} \leqslant \int_{I_1} dx\ |x - x_2| e^{-\frac{|\Lambda| c_1}{2}(x-x_2)^2} \leqslant \int_{\mathbb{R}} dx\ |x - x_2| e^{-\frac{|\Lambda| c_1}{2}(x-x_2)^2} = \tfrac{2}{|\Lambda| c_1}$$

$$\begin{aligned} \int_{I_2} dx\ |x - x_2| e^{-|\Lambda|[F_1(x,h)-F_m]} &\leqslant \int_{I_2} dx\ |x - x_2| e^{-\frac{|\Lambda| c_2}{2} x^2} \\ &\leqslant e^{-\frac{|\Lambda| c_2 M^2}{4}} \int_{\mathbb{R}} dx\ |x - x_2| e^{-\frac{|\Lambda| c_2}{4} x^2} = e^{-\frac{|\Lambda| c_2 M^2}{4}} O\left(\tfrac{1}{\sqrt{|\Lambda|}}\right) \end{aligned}$$

$$\int_{I_3} dx\ |x - x_2| e^{-|\Lambda|[F_1(x,h)-F_m]} \leqslant 2M \sup_{x\in I_3}\left[|x - x_2| e^{-|\Lambda|[F_1(x,h)-F_m]}\right] \leqslant e^{-|\Lambda| c(h,\epsilon,M)}.$$

In the third line we used $|x - x_2(h)| \geqslant \epsilon > 0\ \forall x \in I_3$ and for some constant $\epsilon > 0$, since $F(x,h)$ is at a finite distance from the minimum. Putting all these bounds together we obtain an upper bound for the numerator

$$N = O\left(\tfrac{1}{|\Lambda|}\right).$$

To estimate the denominator note that

$$\partial_x^2 F_1(x,h) \leqslant \frac{1}{\beta} \quad \forall x, h$$

then

$$\int_{\mathbb{R}} dx\ e^{-|\Lambda|[F_1(x,h)-F_m]} \geqslant \int_{\mathbb{R}} dx\ e^{-\frac{|\Lambda|}{2\beta}(x-x_2)^2} = \sqrt{\tfrac{2\pi\beta}{|\Lambda|}}$$

hence

$$|R(\beta,h,|\Lambda|)| \leqslant \sqrt{\tfrac{|\Lambda|}{2\pi\beta}} O\left(\tfrac{1}{|\Lambda|}\right) \to_{|\Lambda|\to\infty} 0$$

Finally

$$\lim_{h\to 0+} \lim_{\Lambda\to\mathbb{Z}^d} \mathbb{E}_\Lambda^{\beta,h}\left[\tfrac{1}{|\Lambda|}\sum_{j\in\Lambda} S_j\right] = \lim_{h\to 0+} \tfrac{x_2(\beta,h)-h}{\beta} = \tfrac{x_0(\beta)}{\beta} > 0.$$

This concludes the proof. □