Chapter 2

One dimensional problems

When $d = 1$ our finite region Λ is a finite chain of points $\Lambda = (-L, -L + 1, \dots 0, 1, \dots, L)$. The techniques applying to 1d systems can be generalized to quasi-one dimensional systems, such as strips of finite width. The material in this chapter is mostly based on the lecture notes by A. Kupiainen [Kup] (for the Ising model part) and on the book by B. Helffer [Hel02] (for the part on integral operators).

2.1 Ising model

We will define the model in general dimension and later specialize to $d = 1$. Let $\bar{L} = (L_1, \dots L_d) \in \mathbb{N}^d$ and $\Lambda = [-L_1, \dots, L_1] \times \cdots [-L_d, \dots L_d]$ a rectangle in $\mathbb{Z}^d$ centered around the origin. To each site $x \in \Lambda$ we associate a spin (the analog of $\varphi(x)$ in the cristal example) taking only two values $-1, +1$. The configuration space is then $\Omega_\Lambda = \{1, -1\}^\Lambda$ and a configuration of the finite system is

$$\begin{aligned} \sigma : \quad \Lambda &\to \{1, -1\}^\Lambda \\ x &\to \sigma(x), \end{aligned}$$

where $\sigma(x)$ is called the "spin" at site x. Let $\Omega = \{1, -1\}^{\mathbb{Z}^d}$ be the set of spin configurations on the whole lattice.

The energy for a configuration $\sigma \in \Omega_\Lambda$, is given by the finite volume Ising Hamiltonian $H_\Lambda^{\bar{\sigma}} : \Omega_\Lambda \to \mathbb{R}$

$$H_\Lambda^I(\sigma) = -J \sum_{x \sim y \in \Lambda} \sigma_x \sigma_y - \sum_{x \in \Lambda} h \sigma_x, \qquad J > 0, \; h \in \mathbb{R}$$

The first term in H^I represents an interaction between nearest neighbor sites and the parameter J is called the coupling constant. The last term is a sum of independent contributions at each site. The parameter h is called the external magnetic field.

Phenomenology The coupling term in H^I is minimum when all spins σ_x have the same orientation $\sigma_x = +1 \ \forall x$, or $\sigma_x = -1 \ \forall x$: in both cases the coupling contribution is $-J \sum_{x \sim y \in \Lambda} 1$. On the other hand, the second sum in H^I is minimum when all spins have the same sign as h, hence for $h \neq 0$ the only spin configuration minimizing the energy is $\sigma_x = \mathrm{sign}(h)$ $\forall x \in \Lambda$. In this sense, h plays the role of an external magnetic field for a ferromagnetic material: when $h = 0$ the spins try to align, but since they do not know which direction to take ($+1$ or -1) they end up being half $+1$ and half -1 so the average orientation is zero. When an external field h is present, the spins align with it.

Note that when $J < 0$ nearest neighbor spin pairs try to take *opposite* spin orientations. This is called paramagnetic behavior.

History The Ising model was introduced to describe ferromagnetic materials, but it proved to be relevant in a wide variety of problems, from lattice gases, to biology, economics and image analysis.

2.1.1 Boundary conditions

Let $\bar{\sigma} \in \{1, -1\}^{\mathbb{Z}^d}$ a *fixed* configuration on the *infinite* lattice.

Definition 1 The boundary of Λ is defined by

$$\partial\Lambda = \{x \in \Lambda | \, \exists y \in \Lambda^c \text{with } \|x - y\| = 1\}$$

Definition 2 The Ising Hamiltonian with $\bar{\sigma}$ boundary conditions is $H_\Lambda^{\bar{\sigma}}$: $\Omega_\Lambda \to \mathbb{R}$

$$H_\Lambda^{\bar{\sigma}}(\sigma) = H^I(\sigma) - J \sum_{x \in \partial\Lambda} \sum_{y \in \Lambda^c, y \sim x} \sigma_x \bar{\sigma}_y$$

where $\bar{\sigma} \in \Omega$ is some fixed infinite volume spin configuration.

The Ising Hamiltonian with periodic boundary conditions is $H_\Lambda^{per} : \Omega_\Lambda \to \mathbb{R}$

$$H_\Lambda^{per}(\sigma) = -J \sum_{x \sim y \in \mathbb{T}_{\bar{\mathbb{L}}}} \sigma_x \sigma_y - h \sum_{x \in \Lambda} \sigma_x$$

where $T_{\bar{L}} = \mathbb{Z}/L_1 \times \cdots \times \mathbb{Z}/L_d$ is a torus.

Finally The Ising Hamiltonian with free boundary conditions is

$$H_\Lambda^{free}(\sigma) = H_\Lambda^I(\sigma).$$

2.1.2 Probability measure and thermodynamic limit

Let $H^{(bc)_\Lambda}$ be the finite volume Ising energy with some fixed boundary conditions. We define a probability measure on Ω_Λ by

$$\mu_{\Lambda,\beta}^{(bc)}(\sigma) = \frac{e^{-\beta H_\Lambda^{(bc)}(\sigma)}}{Z_{\Lambda,\beta}^{(bc)}}$$

where the normalization factor

$$Z_{\Lambda,\beta}^{(bc)} = \sum_{\sigma\in\Omega_\Lambda} e^{-\beta H_\Lambda^{(bc)}(\sigma)}$$

is called the partition function. Let $\{\Lambda_n\}_{n\in\mathbb{N}}$ be a growing sequence of regions s.t. $\Lambda_n \subset \Lambda_{n+1}$ $\forall n$ and $\lim_{n\to\infty}\Lambda_n = \mathbb{Z}^d$.

We denote by $\mathcal{F}_n$ the sigma algebra on Ω_{Λ_n} and by $\mathcal{F}$ the (infinite volume) sigma algebra on Ω. Then each measure $\mu_{\Lambda_n,\beta}^{(bc)}$ on $\mathcal{F}_n$ can be extended to a measure $\tilde{\mu}_n$ on $\mathcal{F}$ with the following definition

$$\begin{aligned} \tilde{\mu}_n(A) &= 0 && \text{if } A\cap\Omega_{\Lambda_n} = \emptyset \\ &= \mu_{\Lambda_n,\beta}^{(bc)}(A\cap\Omega_{\Lambda_n}) && \text{otherwise.} \end{aligned}$$

In order to study the thermodynamic limit we will consider the following class of functions.

Definition: local functions. A function $f : \Omega \to \mathbb{R}$ is local if it depends only on the spin value on a finite set of lattice points. Precisely, f is local if $\exists$ a set $X \subset \mathbb{Z}^d$ with $|X_f| < \infty$ and a function $F : \Omega_X \to \mathbb{R}$ s.t.

$$f(\sigma) = F(\sigma_X) \qquad \forall\sigma\in\Omega,$$

where $\sigma_X = \{\sigma_x\}_{x\in X}$ is the restriction of the configuration σ to the set X.

Example The functions $f_1(\sigma) = \sigma_{x_1}$ and $f_2(\sigma) = \sigma_{x_1}\sigma_{x_2}$ (where x_1, x_2 are fixed lattice points) are both local functions with $X = \{x_1\}$, $\{x_1, x_2\}$ respectively. We will see below that all local functions can be obtained from functions of this form.

Lemma For any function $f : \Omega \to \mathbb{R}$ depending only on spins inside the finite set X, there exists a family of real parameters $\{a_A\}_{A\subseteq X}$ associated to each subset of X satisfying

$$f(\sigma) = \sum_{A\subseteq X} a_A \sigma_A$$

where

$$\sigma_A = \prod_{x\in A} \sigma_x.$$

Proof. Let $\mathbf{1}_+(\sigma_x) = \mathbf{1}_{\{\sigma_x=1\}}(\sigma_x)$ and $\mathbf{1}_-(\sigma_x) = \mathbf{1}_{\{\sigma_x=-1\}}(\sigma_x)$. This can be written in the more condensed form

$$\mathbf{1}_{\sigma'_x}(\sigma_x) = \delta_{\sigma_x,\sigma'_x} = \mathbf{1}_{\sigma_x}(\sigma'_x), \qquad \sigma_x, \sigma'_x = \pm 1.$$

Let $\chi_1 = (\mathbf{1}_+ + \mathbf{1}_-)/2$ and $\chi_2 = (\mathbf{1}_+ - \mathbf{1}_-)/2$. Then

$$\mathbf{1}_\sigma = \chi_1 + \sigma\chi_2, \qquad \text{with } \sigma = \pm 1.$$

and the function

$$\begin{aligned}\mathbf{1}_{\sigma'}(\sigma) &= \prod_{x\in X} \mathbf{1}_{\sigma'_x}(\sigma_x) = \prod_{x\in X} \mathbf{1}_{\sigma_x}(\sigma'_x) = \prod_{x\in X} \left[\chi_1(\sigma'_x) + \sigma_x\chi_2(\sigma'_x)\right] \\ &= \sum_{A\subseteq X} \prod_{x\in A} \sigma_x \prod_{x\in A} \chi_2(\sigma'_x) \prod_{x\in x\setminus A} \chi_2(\sigma'_x)\end{aligned}$$

equals 1 when $\sigma = \sigma'$ and equals 0 otherwise. Then

$$\begin{aligned}f(\sigma) &= \sum_{\sigma'} \mathbf{1}_{\sigma'}(\sigma) f(\sigma) = \sum_{\sigma'} \mathbf{1}_{\sigma'}(\sigma) f(\sigma') \\ &= \sum_{\sigma'} f(\sigma') \prod_{x\in X} \left[\chi_1(\sigma'_x) + \sigma_x\chi_2(\sigma'_x)\right] \\ &= \sum_{A\subseteq X} \prod_{x\in A} \sigma_x \left\{ \sum_{\sigma'} f(\sigma') \prod_{x\in A} \chi_2(\sigma'_x) \prod_{x\in x\setminus A} \chi_2(\sigma'_x) \right\} \\ &= \sum_{A\subseteq X} \prod_{x\in A} \sigma_x a_X\end{aligned}$$

where a_X is a constant independent of the configuration σ. ■

Definition: thermodynamic limit We say that the sequence of measures $\mu^{(bc)}_{\Lambda_n,\beta}$ converges to a measure μ on Ω if

$$\mathbb{E}_{\tilde{\mu}_n}[f] = \sum_\sigma \tilde{\mu}_n(\sigma) f(\sigma) \to_{n\to\infty} \sum_\sigma \mu(\sigma) f(\sigma) = \mathbb{E}_\mu[f]$$

for all local functions $f : \Omega \to \mathbb{R}$.

By the lemma above, it is enough to prove the existence of the limit for $\mathbb{E}_\mu[\sigma_X]$ for any subset X with $|X| < \infty$.

2.2 Transfer matrix for the Ising model in one dimension

Let $\Lambda = [-L, \ldots, L]$. The finite volume Ising Hamiltonian in $d = 1$ can be written

$$H^I_\Lambda(\sigma) = -J \sum_{x=-L}^{L-1} \sigma_x \sigma_{x+1} - h \sum_{x\in\Lambda} h\sigma_x, \qquad J > 0,\ h \in \mathbb{R}$$

The boundary is reduced to two points $\partial\Lambda = \{-L, L\}$, therefore the Hamiltonian with $\bar{\sigma}$ (resp. periodic, free) boundary conditions is

$$\begin{aligned} H^{\bar{\sigma}}_\Lambda(\sigma) =& H^I(\sigma) - J\left[\sigma_{-L}\bar{\sigma}_{-L-1} + \sigma_L \bar{\sigma}_{L+1}\right] \\ H^{per}_\Lambda(\sigma) =& H^I(\sigma) - J\sigma_L \sigma_{-L} \\ H^{free}_\Lambda(\sigma) =& H^I_\Lambda(\sigma). \end{aligned}$$

where $\bar{\sigma} \in \Omega$ is some fixed infinite volume spin configuration.

2.2.1 Partition function

Let $Z^{(bc)}_{\Lambda,\beta}$ be the partition function at finite volume with some fixed boundary conditions. Then we have

Lemma 1 *The limit as $L \to \infty$ of $|\Lambda|^{-1} \ln Z^{(bc)}_{\Lambda,\beta}$ is finite and independent of the boundary conditions*

Proof We will prove this result for $\bar{\sigma}$, periodic and free boundary conditions. In the case of $\bar{\sigma}$ and periodic boundary conditions, the partition function can be written as

$$Z^{(bc)}_{\Lambda,\beta} = \sum_{\sigma\in\Omega_\Lambda} e^{-\beta H^{(bc)}_\Lambda(\sigma)} = \sum_{\sigma\in\Omega_\Lambda} F^{left}_h(\sigma_{-L}) \left[\prod_{x=-L}^{L-1} T_h(\sigma_x,\sigma_{x+1})\right] F^{right}_h(\sigma_L)$$
$$= \left(F^{left}_h, T^{2L}_h F^{right}_h\right)$$

where T_h is a 2×2 matrix

$$T_h = \begin{pmatrix} e^{\beta+h\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta-h\beta} \end{pmatrix}, \qquad T_h(\sigma,\sigma') = e^{\frac{\beta h\sigma}{2}} e^{\beta\sigma\sigma'} e^{\frac{\beta h\sigma'}{2}}, \qquad (2.2.1)$$

while $F^{left/right}_h$ are 2 component vectors encoding the boundary conditions

$$F^{left}_h(\sigma) = e^{\beta\sigma\bar\sigma_{-L-1}} e^{\frac{\beta h\sigma}{2}}, \qquad F^{right}_h(\sigma) = e^{\frac{\beta h\sigma}{2}} e^{\beta\sigma\bar\sigma_{L+1}} \qquad \text{for } \bar\sigma \text{ b.c.},$$
$$F^{left}_h(\sigma) = F^{right}_h(\sigma) = e^{\frac{\beta h\sigma}{2}} \qquad \text{for free b.c.} \qquad (2.2.2)$$

Finally (,) denotes the real euclidean scalar product . In the case of periodic boundary conditions

$$Z^{(per)}_{\Lambda,\beta} = \sum_{\sigma\in\Omega_\Lambda} e^{-\beta H^{(per)}_\Lambda(\sigma)} = \sum_{\sigma\in\Omega_\Lambda} \left[\prod_{x=-L}^{L-1} T_h(\sigma_x,\sigma_{x+1})\right] T_h(\sigma_L,\sigma_{-L})$$
$$= \text{Tr } T^{2L+1}_h$$

To study the large volume properties of the partition function then, we have to study a 2×2 matrix, reducing the problem from 2^{2L+1} to 2 spins only. The matrix T_h is real symmetric hence diagonalisable. The eigenvalues are

$$\lambda_1 = e^\beta\cosh(\beta h) + \sqrt{[e^\beta\sinh(\beta h)]^2 + e^{-2\beta}},$$
$$\lambda_2 = e^\beta\cosh(\beta h) - \sqrt{[e^\beta\sinh(\beta h)]^2 + e^{-2\beta}},\ 0<\lambda_2<\lambda_1.$$

Let v_1, v_2 the corresponding normalized eigenvectors and P_1, P_2 are 2×2 matrices corresponding to the orthogonal projections on v_1, v_2:

$$P_1(\sigma,\sigma') = v_1(\sigma)v_1(\sigma'), \qquad P_1(v) = (v_1,v)v_1,\ \forall v\in\mathbb{R}^2.$$

The definition for P_2 is similar. Since they are orthogonal projections P_1, P_2 satify

$$P_1^2 = P_1,\ P_2^2 = P_2,\ P_1P_2 = P_2P_1 = 0.$$

Moreover, the eigenvector v_1 for the largest eigenvalue has the following additional property, that will be crucial for our proof:

$$v_1(\sigma) > 0 \ \forall \sigma.$$

Indeed let $v_1 = (x_1, y_1)$. Then we obtain

$$y_1 = x_1 C_1 \qquad \text{where } C_1 = e^{\beta}\left[\sqrt{[e^{\beta}\sinh(\beta h)]^2 + e^{-2\beta}} - [e^{\beta}\sinh(\beta h)]\right]. \tag{2.2.3}$$

Since $C_1 > 0$ for any choice of β, h the two components x_1 and y_1 must have the same sign. Inserting the spectral decomposition $T = \lambda_1 P_1 + \lambda_2 P_2$ in the expression for Z we have

$$\begin{aligned} Z_{\Lambda,\beta}^{(bc)} = \left(F_h^{left}, T_h^{2L} F_h^{right}\right) = \lambda_1^{2L}\left[\left(F_h^{left}, P_1 F_h^{right}\right) + \left(\frac{\lambda_2}{\lambda_1}\right)^{2L}\left(F_h^{left}, P_2 F_h^{right}\right)\right] \\ = \lambda_1^{2L}\left[\left(F_h^{left}, v_1\right)\left(v_1, F_h^{right}\right) + \left(\frac{\lambda_2}{\lambda_1}\right)^{2L}\left(F_h^{left}, P_2 F_h^{right}\right)\right] \end{aligned}$$

To complete the proof we need two ingredients

- the first term in the parenthesis is strictly positive. Indeed $(F_h^{left}, v_1) = \sum_\sigma F_h^{left}(\sigma) v_1(\sigma) > 0$ since $v_1(\sigma) > 0$ and $F^{left}(\sigma) > 0$ for all σ. For the same reason $(v_1, F_h^{right}) > 0$.
- the second term in the parenthesis disappears in the limit $L \to \infty$. This holds since $|\lambda_2| < \lambda_1$.

Using these two ingredients we obtain

$$\begin{aligned} \frac{\ln Z_{\Lambda,\beta}^{(bc)}}{2L+1} = & \tfrac{2L}{2L+1}\ln\lambda_1 + \tfrac{1}{2L+1}\ln\left[(F_h^{left}, v_1)(v_1, F_h^{right}) + \left(\tfrac{\lambda_2}{\lambda_1}\right)^{2L}(F_h^{left}, P_2 F_h^{right})\right] \\ & \to_{L\to\infty} \ln\lambda_1 = \ln\left[e^{\beta}\cosh(\beta h) + \sqrt{[e^{\beta}\sinh(\beta h)]^2 + e^{-2\beta}}\right] \end{aligned}$$

Since the boundary conditions appear only in $F^{left/right}$, the result is the same for free, or for any choice of $\bar{\sigma}$ boundary conditions.

In the case of periodic boundary conditions

$$Z_{\Lambda,\beta}^{(per)} = \mathrm{Tr}\ T_h^{2L+1} = \lambda_1^{2L+1}\mathrm{Tr}\ P_1 + \lambda_2^{2L+1}\mathrm{Tr}\ P_2 = \lambda_1^{2L+1}\left[1 + \left(\frac{\lambda_2}{\lambda_1}\right)^{2L+1}\right]$$

Therefore

$$\frac{\ln Z^{(per)}_{\Lambda,\beta}}{2L+1} = \ln \lambda_1 + \tfrac{1}{2L+1} \ln \left[1 + \left(\tfrac{\lambda_2}{\lambda_1}\right)^{2L+1} \right] \to_{L\to\infty} \ln \lambda_1.$$

The limit exists for any choice of β, h and coincides with the result obtained with free or $\bar{\sigma}$ boundary conditions. □

2.2.2 Average magnetization

The finite volume average magnetization at position x is defined by

$$\mathbb{E}_\Lambda[\sigma_x] = \frac{\sum_{\sigma\in\Omega_\Lambda} e^{-\beta H^{(bc)}_\Lambda(\sigma)} \sigma_x}{\sum_{\sigma\in\Omega_\Lambda} e^{-\beta H^{(bc)}_\Lambda(\sigma)}}$$

We have the following result

Lemma 2 *The average magnetization has a limit*

$$\mathbb{E}_\Lambda[\sigma_x] \to_{L\to\infty} M_\beta(h) = \frac{1-C_1^2}{1+C_1^2} \tag{2.2.4}$$

where C_1 *is given in* (2.2.3)*. The limit* $M_\beta(h)$ *is independent of* x *and the boundary conditions, is a smooth increasing function of* h *satisfying*

$$\begin{aligned} &-1 < M_\beta(h) < +1 \ \forall h \in \mathbb{R}, \\ &\lim_{h\to\infty} M_\beta(h) = +1, \ \lim_{h\to-\infty} M_\beta(h) = -1, \end{aligned}$$

and has the same sign as h*. In particular* $M_\beta(0) = 0$*.*

Remark 1 This result is consistent with the physical intuition saying that the spins try to align with the magnetic field h. When h becomes very large all spins align hence the magnetization becomes $+1$ (resp. -1) depending if $h > 0$ or $h < 0$.

Remark 2 The function $M : \mathbb{R} \to]-1, 1[$ is invertible, so we could use the magnetization M as a parameter in our measure instead of h: $\mu_{\beta,h(M)}$,

Proof For simplicity we consider $x > 0$. The same arguments then hold for $x \leq 0$.

As in the case of the partition function we can express $\mathbb{E}_\Lambda[\sigma_x]$ in terms of the transfer matrix T_h:

$$\mathbb{E}_\Lambda[\sigma_x] = \frac{\left(F_h^{left}, T_h^{L+x}\ \Sigma\ T_h^{L-x} F_h^{right}\right)}{\left(F_h^{left}, T_h^{2L} F_h^{right}\right)}$$

where $T_h, F_h^{left}, F_h^{right}$ were defined in (2.2.1) and (2.2.2) above. The 2×2 matrix Σ encodes the new term σ_x

$$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Sigma_{\sigma,\sigma'} = \delta_{\sigma\sigma'}\sigma.$$

Inserting the spectral decomposition $T = \lambda_1 P_1 + \lambda_2 P_2$ we get

$$\begin{aligned}
\mathbb{E}_\Lambda[\sigma_x] =& \frac{\left(F_h^{left}, \left[P_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^{L+x} P_2\right]\ \Sigma\ \left[P_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^{L-x} P_2\right] F_h^{right}\right)}{\left(F_h^{left}, \left[P_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^{2L} P_2\right] F_h^{right}\right)} \\
&\to_{L\to\infty} \frac{\left(F_h^{left}, P_1\ \Sigma\ P_1 F_h^{right}\right)}{\left(F_h^{left}, P_1 F_h^{right}\right)} = \frac{(F_h^{left}, v_1)(v_1,\ \Sigma v_1)(v_1, F_h^{right})}{(F_h^{left}, v_1)(v_1 F_h^{right})} \\
&= (v_1,\ \Sigma v_1) = \sum_\sigma v_1(\sigma)^2\sigma = v_1(+1)^2 - v_1(-1)^2,
\end{aligned}$$

where we used as before $(F_h^{left}, v_1) > 0$, $(F_h^{right}, v_1) > 0$ and $|\lambda_2| < \lambda_1$. Using (2.2.3) we see that

$$v_1(1)^2 - v_1(-1)^2 = \frac{1 - C_1^2}{1 + C_1^2}$$

where $C_1 > 0$ is a smooth function of h and satisfies

$$\begin{aligned}
&C_1 < e^\beta[e^{-\beta}] = 1 \qquad \text{when } h > 0 \\
&C_1 > e^\beta[e^{-\beta}] = 1 \qquad \text{when } h < 0 \\
&C_1 = 1 \qquad \text{when } h = 0.
\end{aligned}$$

Therefore $M(h)$ has the same sign as h and $M(0) = 0$. Moreover

$$C_1' = e^{2\beta}\beta\cosh(\beta h)\left[\frac{e^\beta\sinh(\beta h)}{\sqrt{(e^\beta\sinh(\beta h))^2 + e^{-2\beta}}} - 1\right] < 0 \qquad \forall h,$$

then $M'(h) = -\frac{4C_1C_1'}{(1+C_1^2)^2} > 0\ \forall h$. Finally

$$C_1(h) = e^{2\beta}\sinh(\beta h)\left[\sqrt{1+\frac{e^{-4\beta}}{\sinh^2(\beta h)}}-1\right] = O\left(\frac{1}{\sinh(\beta h)}\right) \to_{h\to\infty} 0$$
$$C_1(h) = e^{2\beta}|\sinh(\beta h)|\left[2+O\left(\frac{1}{\sinh^2(\beta h)}\right)\right] \to_{h\to-\infty} +\infty$$

hence $\lim_{h\to\pm\infty} M(h) = \pm 1$. This completes the proof. □

2.2.3 Spin-spin correlation

The two spin correlation is defined by

$$C_{xy}^{\Lambda} = \mathbb{E}_{\Lambda}[\sigma_x\sigma_y] - \mathbb{E}_{\Lambda}[\sigma_x]\mathbb{E}_{\Lambda}[\sigma_y].$$

This quantity is zero when σ_x and σ_y are independent. We have the following result

Lemma 3 *The infinite volume limit for C_{xy}^{Λ} exists, is independent of the boundary conditions and satisfies*

$$\lim_{L\to\infty} C_{xy}^{\Lambda} = C_{xy} \ = \ Ke^{-\frac{|x-y|}{\xi}}$$

where $\xi > 0$, $K \geq 0$ are constants independent of x and y. The parameter ξ gives the distance where the spin correlation starts to become small and is called the localization distance*.*

Proof As in the previous subsections we use the transfer matrix representation. Without loss of generality we can consider $y > x$. Then

$$\mathbb{E}_{\Lambda}[\sigma_x\sigma_y] = \frac{\left(F_h^{left}, T_h^{L+x}\ \Sigma\ T_h^{y-x}\ \Sigma\ T_h^{L-y}F_h^{right}\right)}{\left(F_h^{left}, T_h^{2L}F_h^{right}\right)}$$

where $F_h^{left/right}$, T_h and Σ are defined above. Inserting the spectral decomposition $T = \lambda_1 P_1 + \lambda_2 P_2$ we get

$$\mathbb{E}_\Lambda[\sigma_x \sigma_y] = \frac{\left(F_h^{left}, \left[P_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^{L+x} P_2\right] \Sigma \left[P_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^{y-x} P_2\right] \Sigma \left[P_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^{L-y} P_2\right] F_h^{right}\right)}{\left(F_h^{left}, \left[P_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^{2L} P_2\right] F_h^{right}\right)}$$

$$\to_{L\to\infty} \frac{\left(F_h^{left}, P_1 \ \Sigma \ \left[P_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^{y-x} P_2\right] \ \Sigma \ P_1 F_h^{right}\right)}{\left(F_h^{left}, P_1 F_h^{right}\right)}$$

$$= \frac{(F_h^{left}, P_1 \ \Sigma \ P_1 \ \Sigma \ P_1 F_h^{right})}{(F_h^{left}, P_1 F_h^{right})} + \left(\frac{\lambda_2}{\lambda_1}\right)^{y-x} \frac{(F_h^{left}, P_1 \ \Sigma \ P_2 \ \Sigma \ P_1 F_h^{right})}{(F_h^{left}, P_1 F_h^{right})}$$

The first term in this sum gives

$$\frac{(F_h^{left}, P_1 \ \Sigma \ P_1 \ \Sigma \ P_1 F_h^{right})}{(F_h^{left}, P_1 F_h^{right})} = \frac{(F_h^{left}, v_1)(v_1, \Sigma v_1)(v_1, \Sigma v_1)(v_1, F_h^{right})}{(F_h^{left}, v_1)(v_1, F_h^{right})}$$
$$= (v_1, \Sigma v_1)^2 = M_\beta(h)^2 = \lim_{L\to\infty} \mathbb{E}_\Lambda[\sigma_x] \ \mathbb{E}_\Lambda[\sigma_y].$$

Therefore $\lim_{L\to\infty} C_{xy}^\Lambda = K e^{-|x-y|/\xi}$ with

$$K = \frac{(F_h^{left}, P_1 \ \Sigma \ P_2 \ \Sigma \ P_1 F_h^{right})}{(F_h^{left}, P_1 F_h^{right})} = (v_1, \Sigma v_2)^2, \qquad \xi = \frac{1}{\ln \frac{\lambda_1}{\lambda_2}}.$$

The values of K and ξ do not depend on F_h so the result is the same for all boundary conditions. Similar arguments hold in the case of periodic boundary conditions.

Comparison with the case of no interaction If we set $J = 0$ instead of $J = 1$ in the Ising Hamiltonian we obtain a product measure on Ω_Λ

$$\mu_{J=0}(\sigma) = \prod_{j=-L}^{L-1} e^{\beta h \sigma_x},$$

and all correlation functions are easy to compute. In particular

$$
\begin{aligned}
Z_{\Lambda}^{J=0} &= \prod_{j=-L}^{L-1} \sum_{\sigma_x \in \{1,-1\}} e^{\beta h \sigma_x} = [2\cosh(\beta h)]^{2L+1} \\
\mathbb{E}_{\Lambda, J=0}[\sigma_x] &= \frac{\sum_{\sigma_x \in \{1,-1\}} e^{\beta h \sigma_x} \sigma_x}{\sum_{\sigma_x \in \{1,-1\}} e^{\beta h \sigma_x}} = \frac{\sinh(\beta h)}{\cosh(\beta h)} = M(h) \\
C_{\Lambda, J=0}^{xy} &= 0 \qquad\qquad \forall x, y, \ \forall \Lambda.
\end{aligned}
$$

As in the $J = 1$ case the magnetization $M(h)$ is invertible, and we can define h as a function of M, i.e. the magnetization we want to obtain: $h(M) = \frac{1}{\beta} \tanh^{-1}(M)$. All correlation functions are zero because the measure is factored over a product of local measures. The infinite volume measure exists and is given by

$$
\mu_{\beta,M}^{J=0}(\sigma) = \prod_{x \in \mathbb{Z}} e^{\beta h_0(M) \sigma_x}, \qquad h_0(M) = \tfrac{1}{\beta} \tanh^{-1}(M).
$$

In the case $J = 1$, we have see that two point correlations decay exponentially and one can show the same result for all correlation functions. This means that the infinite volume measure $\mu_{\beta,J=1}$ is "approximately" the product measure (in a sense to be made precise)

$$
\mu_{\beta,J=1} \sim \prod_{x \in \mathbb{Z}} e^{\beta h_1(M) \sigma_x}, \qquad h_1(M) = M_{\beta,J=1}^{-1}(M),
$$

where the magnetization h_1 is now fixed by the function (2.2.4). Therefore the measure "looks like" what we get in the $J = 0$ case, with a modified parameter h. We say the magnetic field parameter has been "renormalized".

2.2.4 Generalizations: transfer matrix in a strip

Let $\Lambda = \{-L, \ldots, L\} \times \{1, \ldots, W\}$. When $L \to \infty$ this becomes an infinite strip. Its properties are similar to 1d chain, hence this is called a "quasi-one dimensional" problem. A point $\vec{x} \in \Lambda$ is identified by two coordinates $\vec{x} = (x, y)$ with $x \in \{-L, \ldots, L\}$, $y \in \{1, \ldots, W\}$. The space of configurations is $\Omega_\Lambda = \{1, -1\}^\Lambda$ and the Ising Hamiltonian on the strip is

$$
\begin{aligned}
H^I(\sigma) &= -J \sum_{\vec{x} \sim \vec{y} \in \Lambda} \sigma_{\vec{x}} \sigma_{\vec{y}} - h \sum_{\vec{x} \in \Lambda} \sigma_{\vec{x}} \\
&= -J \sum_{x=-L}^{L-1} \left[\sum_{y=1}^{W} \sigma_{x,y} \sigma_{x+1,y} \right] - \sum_{x=-L}^{L} \left[J \sum_{y=1}^{W-1} \sigma_{x,y} \sigma_{x,y+1} + h \sum_{y=1}^{W} \sigma_y \right]
\end{aligned}
$$

where in the first term we have the (horizontal) interactions between spins at the same height y, and in the second term we put together all terms involving only spins on the same vertical line corresponding to x. To make the transfer matrix easier to see, we define

$$X_x(y) = \sigma_{x,y}, \; y \in \{1, \ldots, W\}$$

the vector made with all spins on the vertical line x. The configuration σ can be written in terms of X

$$\sigma = \{\sigma_{x,y}\}_{(x,y)\in\Lambda} = \{X_x\}_{x=-L}^{L}$$

and we can write H^I as

$$H^I(\sigma) = H^I(X) = \sum_{x=-L}^{L-1} I(X_x, X_{x+1}) + \sum_{x=-L}^{L} D(X_x)$$

where the interaction I and the diagonal D terms are

$$I(X, X') = -J\sum_{y=1}^{W} X(y)X'(y), \qquad D(X) = -J\sum_{y=1}^{W-1} X(y)X(y+1) - h\sum_{y=1}^{W} X(y).$$

Then the partition function can be written as

$$Z_\Lambda = \sum_{\sigma\in\Omega_\Lambda} e^{-\beta H^I(\sigma)} = \sum_{X(-L),\ldots X(L)} F^{left}(X_{-L}) \left[\prod_{x=-L}^{L-1} T(X_x, X_{x+1})\right] F^{right}$$
$$= (F^{left}, T^{2L} F^{right}).$$

where $F^{left}(X) = F^{right}(X) = e^{-\frac{\beta}{2}D(X)}$ and

$$T(X, X') = e^{-\frac{\beta}{2}D(X)} e^{-\beta I(X,X')} e^{-\frac{\beta}{2}D(X')}.$$

Instead of a 2×2 matrix this time we have a $2^W \times 2^W$ matrix and computing the eigenvalues and eigenvectors may become cumbersome. To avoid doing the explicit we apply the following result

Theorem 1 (Perron-Frobenius) *[without proof] Let T be a $N \times N$ real matrix with $T_{ij} > 0$ $\forall i, j$. Then*

1. *$\lambda = \|T\|$ is an eigenvalue of T*
2. *for any eigenvalue $\lambda' \neq \lambda$ we have $|\lambda'| < \lambda$,*

3. λ is simple and the corresponding eigenvector can be chosen so that $v_j > 0 \; \forall j$.

4. let v be an eigenvector for $\lambda' \neq \lambda$. Then v must have some negative or zero components.

In our case T is a real symmetric matrix, hence there exists a orthonormal basis of eivenvectors. Morevoer $T(X, X') > 0 \; \forall X, X'$ so the theorem ensures that the top eigenvalue (in absolute value) λ_1 is positive, simple and the corresponding eigenvector v_1 satisfies $v_1(j) > 0 \; \forall j$. Then

$$T = \sum_{j=1}^{2^W} \lambda_j P_j, \qquad T^{2L} = \lambda_1^{2L} \left[P_1 + \sum_{j=2}^{2^W} \left(\frac{\lambda_j}{\lambda_1} \right)^{2L} P_j \right]$$

where P_j are orthogonal projections and $|\lambda_j|/\lambda_1 < 1 \; \forall j \geq 2$. Then

$$\frac{1}{|\Lambda|} \ln Z_\Lambda = \frac{2L}{W(2L+1)} \ln \lambda_1 + \frac{1}{|\Lambda|} \left[(F^{left}, P_1 F^{right}) + \sum_{j=2}^{2^W} \left(\frac{\lambda_j}{\lambda_1} \right)^{2L} (F^{left}, P_j, F^{right}) \right]$$

$$\to_{L\to\infty} \frac{\ln \lambda_1}{W}$$

since $(F^{left}, P_1 F^{right}) = (F^{left}, v_1)(v_1, F^{right}) > 0$. The magnetization and correlation functions can be studied in a similar way.

Remark The argument works since W is kept fixed while $L \to \infty$. If we try to send W to infinity at the same time several problems appear. Among them: *(a)* the ratio $|\lambda_j|/\lambda_0$ depends on W and may converge to 1, *(b)* the size of the matrix T diverges and we have to ensure the sum over orthogonal projections remains well defined. Far from being just a nuisance, these problems signal that something fundamentally different may happen in higher dimensions.

2.3 Transfer matrix for continuous spin

Let us now go back to the first example we gave in Ch. 1, namely the deformations inside a perfect cristal.

Let $\Lambda = \{-L, \ldots, L\}$ as before. The spin $\sigma_x = \pm 1$ at the position $x \in \Lambda$ is now replaced by the atom displacement $\phi_x \in \mathbb{R}$. The finite volume set of spin configurations $\{\sigma \in \{1, -1\}^\Lambda\}$ becomes now

$$\Omega_\Lambda = \mathbb{R}^\Lambda = \{\phi | \phi : \Lambda \to \mathbb{R}\}$$

We consider the energy functional

$$H_\Lambda(\phi) = \sum_{j=-L}^{L-1} [\phi_j - \phi_{j+1}]^2 + \frac{m^2}{\beta} \sum_{j=-L}^{L} \phi_j^2$$

This corresponds to the hamiltonian (1.1.1) for a cristal in one dimension, with an additional term $m^2 \sum_x \phi^2$, favoring configurations with ϕ_x near zero for each x. Intuitively, this means that each atom wants to remain near to its equilibrium position on the lattice, independently of what the other atoms do. The parameter $m > 0$ is called the mass, and we rescaled by β in order to simplify the formulas.

We will consider first the case of **free boundary conditions**: $H_\Lambda^{(free)} = H_\Lambda$. We define a probability measure

$$d\mu_\Lambda(\phi) = \frac{e^{-\beta H_\Lambda}}{Z_\Lambda} d\phi$$

where $d\phi = \prod_{j=-L}^{L} d\phi_j$ is the product Lebesgue measure and

$$Z_\Lambda = \int_{\mathbb{R}^{2L+1}} e^{-\beta H_\Lambda} d\phi = \int_{\mathbb{R}^{2L+1}} e^{-\beta \sum_{j=-L}^{L-1} [\phi_j - \phi_{j+1}]^2} e^{-m^2 \sum_{j=-L}^{L} \phi_j^2} d\phi$$

is the normalization constant. The integrand inside Z is strictly positive, so $Z > 0$. Moreover $[\phi_j - \phi_{j+1}]^2 \geq 0$ for any choice of ϕ then

$$0 < Z_\Lambda \leq \prod_{j=-L}^{L} \int_{\mathbb{R}} e^{-m^2 \phi_j^2} d\phi_j = \left(\sqrt{\frac{\pi}{m^2}} \right)^{2L+1} < \infty.$$

Hence the measure is well defined.

As we did in the Ising model, we start by studying $\ln Z/|\Lambda|$ as $\Lambda \to \mathbb{Z}$. Our goal is to mimick the strategy we developed in the Ising model. We can write Z_Λ as

$$Z_\Lambda = \int_{\mathbb{R}^{2L+1}} F^{left}(\phi_{-L}) \prod_{j=-L}^{L-1} k(\phi_j, \phi_{j+1}) \; F^{right}(\phi_L) \tag{2.3.5}$$

where

$$k(\phi, \phi') = e^{-\frac{m^2}{2}\phi^2} e^{-\beta(\phi - \phi')^2} e^{-\frac{m^2}{2}\phi'^2} \qquad F^{left}(\phi) = F^{right}(\phi) = e^{-\frac{m^2}{2}\phi^2}. \tag{2.3.6}$$

This expression is identical to what we obtained in the Ising case, but sums are now replaced by integrals and the arguments we applied to not generalize automatically.

2.3.1 From matrices to integral kernels: transfer operator

In the Ising case we defined the transfer operator as

$$\begin{array}{lll} T: & \mathbb{R}^2 & \to \mathbb{R}^2 \\ & v & \to [Tv](\sigma) = \sum_{\sigma'} \; T_{\sigma\sigma'} \; v(\sigma') \end{array}$$

where T is a 2×2 matrix acting on $\mathbb{R}^2$ endowed with the norm $\|v\|^2 = \sum_\sigma v(\sigma)^2$. The natural generalization in this context is the integral operator

$$\begin{array}{lll} K: & L_2(\mathbb{R}) & \to L_2(\mathbb{R}) \\ & f & \to [Kf](\phi) = \int d\phi' \; k(\phi,\phi') \; f(\phi') \end{array} \tag{2.3.7}$$

where

$$\begin{array}{lll} k: & \mathbb{R}\times\mathbb{R} & \to \mathbb{R} \\ & (x,y) & \to k(x,y) \end{array}$$

is called the *integral kernel*. While the matrix operator T was trivially well defined, here we need to check that: *(a)* the function $k(\phi,\phi')f(\phi')$ is integrable and *(b)* the function kf is still in $L_2(\mathbb{R})$.

A simple criterion is given by the Schur's bound below.

Lemma 4 *[Schur's bound.] Let $k : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ satisfy the two bounds*

$$\begin{aligned} M_1 = & \sup_x \int_{\mathbb{R}} |k(x,y)| dy \; < \infty \\ M_2 = & \sup_y \int_{\mathbb{R}} |k(x,y)| dx \; < \infty. \end{aligned} \tag{2.3.8}$$

Then $Kf(x) = \int k(x,y)f(y)dy$ defines a bounded linear operator from $L_2(\mathbb{R})$ to $L_2(\mathbb{R})$, with

$$\|K\| \;\leq\; \sqrt{M_1 M_2} \tag{2.3.9}$$

Proof Let $f \in L_2(\mathbb{R})$. By Cauchy-Schwartz inequality

$$\begin{aligned} [Ff](x)^2 \leq & \left[\int |k(x,y)||f(y)|dy\right]^2 = \int \sqrt{|k(x,y)|}\sqrt{|k(x,y)|}|f(y)|dy \\ \leq & \left[\int |k(x,y)|dy\right] \left[\int |k(x,y)|f(y)^2 dy\right] \leq \; M_1 \int |k(x,y)|f(y)^2 dy \end{aligned}$$

Using Fubini's theorem we have

$$\int dx \; \int dy |k(x,y)| f(y)^2 \; = \int dy \; f(y)^2 \int dx \; |k(x,y)| \leq M_2 \|f\|^2 < \infty.$$

As a consequence $\int |k(x,y)|\, f(y)^2 dy$ and hence also $\int |k(x,y)|\, |f(y)| dy$ exist for all x, (except eventually on sets of measure zero). Then $[Kf](x)$ is well defined and

$$\|Kf\|^2 \le M_1 M_2 \|f\|^2$$

so $Kf \in L_2(\mathbb{R})$ and $\|K\| \le \sqrt{M_1 M_2}$. □

Symmetric kernels When the kernel satisfies (2.3.8) and has the additional property $k(x,y) = k(y,x)$ we can write for any $f, g \in L_2(\mathbb{R})$

$$(f, Kg) = (Kf, g), \qquad \text{where } (f,g) = \int f(x) g(x) dx$$

is the real scalar product on $L_2(\mathbb{R})$.

In the case of the cristal the kernel given by (2.3.6)

$$k(x,y) = e^{-\frac{m^2}{2}x^2} e^{-\beta(x-y)^2} e^{-\frac{m^2}{2}y^2}$$

is symmetric and satisfies

$$M_1 = M_2 = \sup_x \int |k(x,y)| dy \le \int e^{-\frac{m^2}{2}y^2} dy = \sqrt{\frac{2\pi}{m^2}} < \infty$$

Then K defines a symmetric bounded linear operator on $L_2(\mathbb{R})$ and we can write the partition function as

$$Z_\Lambda = (F^{left}, K^{2L} F^{rigth}). \tag{2.3.10}$$

2.3.2 Expanding in a sum of projections

In the Ising case we used the expansion $T = \lambda_1 P_1 + \lambda_2 P_2$, where P_1, P_2 are orthogonal projections. For an integral operator this decomposition in general does not exist. An integral operator "looks like" a finite matrix when it is compact. Precisely

Definition: compact operator. An operator $K : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is compact if it is the limit in norm of a sequence of finite rank operators, i.e. there exists a sequence $\{K_N\}_{N\in\mathbb{N}}$ such that $K_N : L_2(\mathbb{R}) \to L_2(\mathbb{R})$, its image has finite dimension for each N and

$$\lim_{N\to\infty} \|K - K_N\| = 0.$$

There is an easy criterion to check if an operator is compact.

Criterion for compactness. If K is Hilbert-Schmidt then it is compact.

Definition: Hilbert-Schmidt operator. An operator $K : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is called Hilbert-Schmidt if the kernel satisfies

$$\int_{\mathbb{R}\times\mathbb{R}} |k(x,y)|^2 dxdy < \infty$$

In our example

$$\int_{\mathbb{R}\times\mathbb{R}} |k(x,y)|^2 dxdy = \int_{\mathbb{R}\times\mathbb{R}} e^{-m^2x^2} e^{-2\beta(x-y)^2} e^{-m^2y^2} dxdy$$
$$\leq \int_{\mathbb{R}} e^{-m^2x^2} dx \int_{\mathbb{R}} e^{-m^2y^2} dy = \frac{\pi}{m^2} < \infty.$$

Then K is a compact operator.

The following theorem gives conditions to ensure we can write K a a linear combination of orthogonal projections.

Theorem 2 *[without proof] Let $K : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ be compact, symmetric and injective. Then*

1. *there exists a decreasing (in modulus) sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ of eigenvalues $|\lambda_j| \geq |\lambda_{j+1}|$ with $\lim_{j\to\infty} \lambda_j = 0$.*
2. *There exists a corresponding sequence of eigenvectors $v_j \in L_2(\mathbb{R})$ such that $\{v_j\}_{j\in\mathbb{N}}$ forms an orthonormal basis of $L_2(\mathbb{R})$.*
3. *Let $K_N = \sum_{j=0}^{N} \lambda_j P_j$, where*

$$[P_j f](x) = v_j(x)\ (v_j, f) = \int v_j(x) v_j(y)\ f(y)$$

is the orthogonal projections on $Vect(v_j)$. Then

$$\lim_{N\to\infty} \|K - K^N\| = 0 \qquad \equiv \qquad K = \sum_{j=0}^{\infty} \lambda_j P_j.$$

In our case we already checked that K is compact and symmetric. It remains to verify that K is injective. We will prove the following stronger result.

Lemma 5 *Let $K : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ be defined by the kernel $k(x,y)$ given by (2.3.6). Then $K > 0$ as a quadratic form i.e. $(f, Kf) > 0$ for any function $f \in L_2(\mathbb{R})$ except the zero function $f(x) = 0\ \forall x$.*

Proof

$$\begin{aligned}(f, Kf) &= \int_{\mathbb{R}\times\mathbb{R}} f(x)k(x,y)f(y)dxdy \\ &= \int_{\mathbb{R}\times\mathbb{R}} g(x)e^{-\beta(x-y)^2}g(y)dxdy = \int_{\mathbb{R}} g(x)[F*g](x)dx\end{aligned}$$

where we defined

$$g(x) = f(x)e^{-\frac{m^2}{2}x^2}, \qquad F(x) = e^{-\beta x^2}.$$

The exponential factor ensures that $g \in L_2(\mathbb{R}) \cap L_1(\mathbb{R})$ so the Fourier transform of g is well defined and

$$\int_{\mathbb{R}} g(x)[F*g](x)dx = \int_{\mathbb{R}} \overline{\hat{g}(k)}\ \widehat{[F*g]}(k)dk = \int_{\mathbb{R}} |\hat{g}(k)|^2\hat{F}(k)dk$$

where we used

$$\hat{g}(k) = \tfrac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} g(x)e^{-ikx}dx, \qquad \widehat{[F*g]}(k) = \hat{F}(k)\hat{g}(k).$$

Finally

$$\hat{F}(k) = \tfrac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\beta x^2}e^{ikx}dx = \tfrac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{4\beta}}\int_{\mathbb{R}} e^{-\beta\left(x-\frac{ik}{2\beta}\right)^2}dx = e^{-\frac{x^2}{4\beta}}\tfrac{1}{\sqrt{\beta\pi}} > 0.$$

To perform the last integral we deform the contour in the complex plane and use the fact that $e^{-\beta z^2}$ is analytic, hence the integral over any closed path equals zero. Putting this results together we see that

$$(f, Kf) = \int_{\mathbb{R}} |\hat{g}(k)|^2\hat{F}(k)dk \geq 0$$

Since $\hat{F}(k) > 0\ \forall k$, then $(f, Kf) = 0$ iff $\hat{g}(k) = 0\ \forall k$, iff $g(x) = 0\ \forall x$, iff $f(x) = 0\ \forall x$. □

Consequences. Since $K > 0$ we have $0 = (f, 0) = (f, Kf) > 0$ for any $f \in \ker K$. Then $\ker K = \{0\}$, hence K is injective and the theorem above applies. Moreover, $K > 0$ implies that all eigenvalues of K must be stricly positive.

As a conclusion, in the case of our example, there exists a decreasing sequence of positive eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ and a correspoding sequence eigenvectors $\{v_j\}_{j\in\mathbb{N}}$ forming an orthonormal basis such that

$$\lim_{N\to\infty} \|K - K_N\| = 0 \qquad \text{where} \qquad K_N = \sum_{j=0}^{N} \lambda_j P_j.$$

As a consequence $\lim_{N\to\infty} |(u, Kw) - (u, K_N w)| = 0$ for all $u, w \in L_2(\mathbb{R})$ and (2.3.10) becomes

$$Z_\Lambda = (F^{left}, K^{2L} F^{right}) = \lim_{N\to\infty} \sum_{j=0}^{N} \lambda_j^{2L} (F^{left}, P_j F^{right})$$

2.3.3 Infinite volume limit

In the Ising case we needed two additional ingredients to control the limit as $L \to \infty$: *(a)* the largest eigenvalue is simple and *(b)* the corresponding eigenvector has strictly positive components. Since the elements of T are striclty positive Perron-Frobenius theorem ensures that both *(a)* and *(b)* are verified. Here we need a generalization of Perron-Frobenius result to integral operators.

Definition. An operator K on $L_2(\mathbb{R})$ with integral kernel $k(x, y)$ is said to have **strictly positive kernel** if for any function $f \in L_2(\mathbb{R})$ such that $f(x) \geq 0\ \forall x$ and $f > 0$ on a set of positive Lebesgue measure, then $[Kf](x) > 0\ \forall x$, almost surely (i.e. except eventually on a set of measure zero). This means in particular that $k(x, y) > 0\ \forall x, y$ a.s.

Theorem 3 (Krein-Rutman) *Let K be a bounded compact symmetric operator on $L_2(\mathbb{R})$ with strictly positive kernel. Let $\lambda = \|K\|$. Then*

1. *λ is the largest eigenvalue (in absolute value) of K,*
2. *there exists an eigenvector v for λ such that $v(x) > 0\ \forall x \in \mathbb{R}$,*
3. *λ has multiplicity one.*
4. *for any eigenvalue $|\lambda'| < \lambda$, let w be an eigenvector. Then there are two sets I_1 and I_2 in $\mathbb{R}$ of positive Lebesgue measure such that $w(x) > 0$ $\forall x \in I_1$ and $w(x) < 0\ \forall x \in I_2$.*

Proof Since K is compact and symmetric, then the largest eigenvalue (in absolute value) λ_0 satisfies $\|A\| = |\lambda_0| > 0$. We suppose now $\lambda_0 > 0$. We will see at the end that this must always be the case. Let v be a normalized eigenvector for λ_0. Since K is symmetric we can take v real. Then

$$\begin{aligned} 0 <& (v, Kv) = |(v, Kv)| = \left| \int v(x) k(x, y) v(y)\ dxdy \right| \\ &\leq \int |v(x)|\ k(x, y)\ |v(y)|\ dxdy = (|v|, K|v|). \end{aligned} \qquad (2.3.11)$$

where $|v|(x) = |v(x)|$, in the first passage we used $K > 0$ (as a quadratic form) and in the last one we used $k(x,y) > 0$ (pointwise). Since v is an eigenvector for λ_0 we also have

$$\lambda_0 \|v\|^2 = (v, Kv) \leq (|v|, K|v|) \leq \|K\| \; \| \; |v| \; \|^2 = \|K\| \; \|v\|^2. \tag{2.3.12}$$

But $\lambda_0 = \|K\|$ then $(v, Kv) = (|v|, K|v|)$. Now let $v(x) = v_+(x) - v_-(x)$ where

$$v_+(x) = v(x)\mathbf{1}_{v(x)>0}, \quad v_-(x) = -v(x)\mathbf{1}_{v(x)\leq 0}$$

hence $v_\pm(x) \geq 0$ for all x, $|v| = v_+ + v_-$ and

$$(v_+, Kv_-) = (v_-, Kv_+) = \int v_+(x)k(x,y)v_-(x)dx \geq 0$$

since all integrands are non negative. Inserting these expressions inside $(v, Kv) = (|v|, K|v|)$ we get

$$\begin{aligned} 0 < (v, Kv) &= (v_+, Kv_+) + (v_-, Kv_-) - (v_+, Kv_-) - (v_-, Kv_+) \qquad (2.3.13) \\ &= (v_+, Kv_+) + (v_-, Kv_-) + (v_+, Kv_-) + (v_-, Kv_+) = (|v|, K|v|) \\ &\Rightarrow \; (v_+, Kv_-) + (v_-, Kv_+) = 0. \end{aligned}$$

Therefore

$$0 = (v_+, Kv_-) = (v_-, Kv_+) = \int v_-(x)[Kv_+](x)dx.$$

We remember that $v_+(x) \geq 0$ and $v_-(x) \geq 0$. We have two possible cases: *(a)* $v_+ > 0$ on a set of positive measure, then $[Kv_+](x) > 0 \; \forall x$, then the integral above equals zero only of $v_-(x) = 0 \; \forall x$, hence $v(x) = v_+(x) \geq 0 \; \forall x$. The second possibility *(b)* is that $v_+(x) = 0 \; \forall x$, then $v(x) = -v_-(x) \leq 0$ $\forall x$. We conclude that v can be chosen to be non negative $v(x) = |v|(x) \geq 0$ $\forall x$. To prove strict positivity $v(x) > 0$ we observe that $\lambda_0 > 0$ then

$$v(x) = \tfrac{1}{\lambda_0}[Kv](x) = \tfrac{1}{\lambda_0} \int k(x,y)v(y)dy > 0$$

since $v(y) \geq 0$ and there is a set I of non zero measure such that $v(y) > 0$ $\forall y \in I$. To prove that the eigenvalue λ_0 is simple, suppose λ_0 is not simple and let v' be another eigenvector. Then we can always choose v' such that $(v, v') = 0$. Applying the arguments above to v' we conclude that $v'(x) > 0$ $\forall x$. But then

$$0 = (v, v') = \int v(x)v'(x)dx > 0,$$

that is impossible. Then λ_0 is simple. Finally, let w an eigenvector for $|\lambda'| < \lambda_0$. Since K is symmetric we must have

$$0 = (w, v_0) = \int v_0(x) w(x) dx.$$

Since $v_0(x) > 0 \; \forall x$, w must take both positive and negative values to ensure the integral is zero.

It remains to prove that $\lambda_0 > 0$. Suppose $\lambda_0 < 0$. Then repeating the same arguments as in (2.3.12) we find $-(v, Kv) = |(v, Kv)| = (|v|, K|v|)$. Then (2.3.13) becomes $(v_+, Kv_+) + (v_-, Kv_-) = 0$, hence using strict positivity of the kernel $v_+(x) = v_-(x) = 0 \; \forall x$. This ends the proof. □

Using the results above we can prove the following lemma.

Lemma 6 *Let $K : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ be defined by the kernel $k(x, y)$ given by (2.3.6). Let λ_0 be the largest eigenvalue $\lambda_1 < \lambda_0$ the next eigenvalue. Let v_0 be the normalized eigenvector for λ_0 with $v_0(x) > 0 \; \forall x$ and P_0 the corresponding orthogonal projector. Then*

$$K = \lambda_0 P_0 + K_1$$

where $K_1 P_0 = P_0 K_1$ and $\|K_1\| = \lambda_1$.

Proof By Th. 2 and 3 we have

$$K_1 = \sum_{j \geq 1} \lambda_j P_j = K_{\mathcal{H}_1} : \mathcal{H}_1 \to \mathcal{H}_1$$

where $\mathcal{H}_1 = v_0^\perp$ is the subspace orthogonal to v_0 and $0 < \lambda_j \leq \lambda_1 < \lambda_0$ for all j. The result follows. □

2.3.4 Partition function and moments

Partition function Using the results of the previous sections we can write

$$Z_\Lambda = (F^{left}, K^{2L} F^{right}) = (F^{left}, K^{2L} F^{right}) = \lambda_0^{2L} \left[(F^{left}, P_0 F^{right}) + (F^{left}, \tfrac{K_1^{2L}}{\lambda_0^{2L}} F^{right}) \right]$$

Since

$$|(F^{left}, \tfrac{K_1^{2L}}{\lambda_0^{2L}} F^{right})| \leq \|F^{left}\| \|F^{right}\| \left[\tfrac{\|K_1\|}{\lambda_0} \right]^{2L} = \|F^{left}\| \|F^{right}\| \left[\tfrac{\lambda_1}{\lambda_0} \right]^{2L} \to_{L \to \infty} 0$$
$$(F^{left}, P_0 F^{right}) = (F^{left}, v_0) \; (v_0, F^{right}) > 0,$$

we can write

$$\lim_{L \to \infty} \frac{\ln Z_\Lambda}{|\Lambda|} = \ln \lambda_0.$$

Magnetization Contrary the the Ising model here by symmetry we have

$$\mathbb{E}_\Lambda[\phi_j] = 0 \quad \forall j,\ \forall\Lambda.$$

To get some non trivial result we must consider ϕ^2. In the Ising case $\mathbb{E}_\Lambda[\sigma_x^2] = 1$ trivially since $\sigma^2 = 1$. On the contrary here

$$\mathbb{E}_\Lambda[\phi_j^2] = \frac{(F^{left}, K^{L+j}\Sigma^2 K^{L-j} F^{right})}{(F^{left}, K^{2L} F^{right})}$$

where we suppose $j > 0$ and we defined $[\Sigma^2 f](x) = x^2 f(x)$. Note that $[\Sigma^2 f] \notin L^2(\mathbb{R})$ in general. Let

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) |\ \sup_x |x|^q |f^{(p)}(x)| < \infty\ \forall q, p \geq 0\}$$

be the Schwartz space on $\mathbb{R}$. Then $\Sigma^2 : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ Moreover $Kf \in \mathcal{S}(\mathbb{R})$ for any $f \in L^2(\mathbb{R})$ (as long as $m > 0$). Then for each finite volume Λ the expression above is finite and

$$\lim_{L\to\infty} \mathbb{E}_\Lambda[\phi_j^2] = \frac{(F^{left}, P_0\Sigma^2 P_0 F^{right})}{(F^{left}, P_0 F^{right})} = (v_0, \Sigma^2 v_0) = \int x^2 v_0^2(x) dx \quad (2.3.14)$$

Here comes a new problem: though in the discrete case the final expression was obviously finite, here the information $v_0 \in L_2(\mathbb{R})$ is not enough to garantee that the integral is finite. We will need to determine more precisely the properties of $v_0(x)$. This will be done in the next subsection.

Two point correlation Let us suppose now $(v_0, \Sigma^2 v_0) < \infty$ and consider the correlation

$$C_\Lambda^{jl} = \mathbb{E}_\Lambda[\phi_j \phi_l] = \frac{(F^{left}, K^{L+j}\Sigma K^{l-j} \Sigma K^{L-l} F^{right})}{(F^{left}, K^{2L} F^{right})}$$

where we set $0 < j < l$ and $[\Sigma f](x) = x f(x)$. As in the case of Σ^2 we have $\Sigma : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ and $Kf \in \mathcal{S}(\mathbb{R})$ for any $f \in L^2(\mathbb{R})$, then

$$\begin{aligned}
&\lim_{L\to\infty} \mathbb{E}_\Lambda[\phi_j \phi_l] = \frac{(F^{left}, P_0 \Sigma \frac{K^{l-j}}{\lambda_0^{l-j}} \Sigma P_0 F^{right})}{(F^{left}, P_0 F^{right})} = (v_0, \Sigma \frac{K^{l-j}}{\lambda_0^{l-j}} \Sigma v_0) \\
&= (v_0, \Sigma v_0)(v_0, \Sigma v_0) + (v_0, \Sigma \frac{K_1^{l-j}}{\lambda_0^{l-j}} \Sigma v_0) = (v_0, \Sigma \frac{K_1^{l-j}}{\lambda_0^{l-j}} \Sigma v_0) \\
&\leq \|v_0 \Sigma\|^2 \left(\frac{\lambda_1}{\lambda_0}\right)^{l-j} = (v_0, \Sigma^2 v_0)\ e^{-\frac{|l-j|}{\xi}}
\end{aligned}$$

where $\xi = [\ln \frac{\lambda_0}{\lambda_1}]^{-1}$.

2.3.5 Eigenvalues and eigenvectors of K

Top eigenvalue

Since the kernel $k(x, y)$ is written as product of gaussians, we can try to find an eigenvector with a gaussian form.

Lemma 7 *The function $g_\alpha(x) = e^{-\alpha x^2}$, $\alpha > 0$ is an eigenvector of K iff α*

$$\alpha = \sqrt{\beta m^2 + m^4/4}. \tag{2.3.15}$$

The corresponding eigenvalue is

$$\mu_\alpha = \sqrt{\tfrac{\pi}{\alpha+(\beta+m^2/2)}}. \tag{2.3.16}$$

Proof If we apply K to g_a we obtain

$$\begin{aligned}
[Kg_\alpha](x) &= e^{-x^2(\beta+m^2/2)} \int e^{-y^2\left[\alpha+(\beta+m^2/2)\right]} e^{+2\beta xy} dy \\
&= \sqrt{\tfrac{\pi}{\alpha+(\beta+m^2/2)}} e^{-x^2(\beta+m^2/2)} e^{\frac{4\beta^2 x^2}{4[\alpha+(\beta+m^2/2)]}} \\
&= \sqrt{\tfrac{\pi}{\alpha+(\beta+m^2/2)}} e^{-x^2\left[(\beta+m^2/2)-\frac{\beta^2}{\alpha+(\beta+m^2/2)}\right]}
\end{aligned}$$

Then $[Kg_\alpha](x) = \mu g_\alpha(x)$ iff $\mu = \mu_\alpha$ and

$$\alpha = (\beta + m^2/2) - \frac{\beta^2}{\alpha + (\beta + m^2/2)} \text{ iff } \alpha^2 = (\beta + m^2/2)^2 - \beta^2.$$

□

Note that $g_\alpha(x) > 0$ $\forall x$ then by Krein-Rutman theorem μ_α must the top eigenvalue $\mu_\alpha = \lambda_0 = \|K\|$. Let

$$v_0(x) = \left(\tfrac{2\alpha}{\pi}\right)^{\frac{1}{4}} g_\alpha(x) \tag{2.3.17}$$

be the corresponding normalized eigenvector. Then the expression $(v_0, \Sigma^2 v_0)$ in (2.3.14) is

$$(v_0, \Sigma^2 v_0) = \sqrt{\tfrac{2\alpha}{\pi}} \int x^2 e^{-2\alpha x^2} dx = \frac{1}{2\alpha} < \infty,$$

the $\lim_{L\to\infty} \mathbb{E}_\Lambda[\phi_j^2]$ is finite and $\lim_{L\to\infty} \mathbb{E}_\Lambda[\phi_j \phi_k] \leq Ce^{-|j-k|/\xi}$.

Other eigenvalues

In order to estimate the localization lenght ξ in $\mathbb{E}_\Lambda[\phi_j\phi_k]$ we need to know also the second eigenvalue λ_1. In our example, we can actually find all eigenvalues and the corresponding eigenvectors.

Lemma 8 *The eigenvalues of K are the sequence*

$$\lambda_j = \mu_\alpha \lambda_\alpha^j, j \in \mathbb{N}, \qquad \lambda_\alpha = \frac{\beta}{(\beta + m^2/2 + \alpha)}.$$

Each eigenvalue is simple and the corresponding eigenvector v_j can be written as

$$v_j(x) = (\mathrm{a}^*)^j v_0(x), \qquad \textit{where} \qquad \mathrm{a}^* = -\frac{d}{dx} + 2\alpha x,$$

and v_0 is given in (2.3.17) *above.*

Proof We remark that g_α is the solution of $g'_\alpha(x) + 2\alpha x g_\alpha(x) = 0$. Let $\mathrm{a} = \frac{d}{dx} + 2\alpha x$ and

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) | \sup_x |x|^q |f^{(n)}(x)| < \infty \ \forall q, p \geq 0\}$$

be the Schwartz space on $\mathbb{R}$. Then $\mathrm{a} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ and

$$(f, \mathrm{a}g) = (\mathrm{a}^* f, g) \qquad \forall f, g \in \mathcal{S}(\mathbb{R}).$$

Since $v_0 \in \mathcal{S}(\mathbb{R})$, $u_j = \mathrm{a}^* v_0 \in L_2(\mathbb{R})\ \forall j > 0$. Morevover for any $f \in \mathcal{S}(\mathbb{R})$

$$\begin{aligned}
[\mathrm{a}^* K f](x) &= \left[-\frac{d}{dx} + 2\alpha x\right] \int k(x,y) f(y) dy \\
&= 2 \int \left[x\left(\beta + m^2/2 + \alpha\right) - y\beta\right] k(x,y) f(y) dy \\
[K \mathrm{a}^* f](x) &= \int k(x,y) \left[-\frac{d}{dy} + 2\alpha y\right] f(y) dy \\
&= 2 \int \left[x\beta - y\left(\beta + m^2/2 - \alpha\right)\right] k(x,y) f(y) dy \\
&= \frac{\beta}{(\beta + m^2/2 + \alpha)} [\mathrm{a}^* K f](x)
\end{aligned}$$

where we used $\alpha^2 = (\beta + m^2/2)^2 - \beta^2$. Taking $f = v_0$ we obtain immediately that v_j is a sequence of eigenvectors for the eigenvalues λ_j. Since $\lambda_j \neq \lambda_k$ $\forall j \neq k$ and $K^* = K$ the eigenvectors are orthogonal

$$\lambda_j (v_j, v_k) = (K v_j, v_k) = (v_j, K v_k) = \lambda_k (v_j, v_k).$$

More precisely, using $[a, a^*] = 4\alpha \mathrm{Id}$ and

$$[a, (a^*)^k] = [a, a^*](a^*)^{k-1} + a^*[a, (a^*)^{k-1}] = 4\alpha(a^*)^{k-1} + a^*[a, (a^*)^{k-1}] = 4\alpha k(a^*)^{k-1}$$

we obtain

$$(v_j, v_k) = \delta_{jk} \frac{(4\alpha)^j}{j!}.$$

Finally we remark that $v_j(x) = p_j(x)e^{-\alpha x^2}$ where $p_j(x)$ is a polynomial of order j and

$$\int e^{-2\alpha x^2} p_j(x) p_k(x) = c_j \delta_{jk},$$

where c_j is some positive constant. Precisely

$$\|g_\alpha\| \; p_j(x) = e^{\alpha x^2}(a^*)^j e^{-\alpha x^2} = e^{2\alpha x^2} \left(-\tfrac{d}{dx}\right)^j e^{-2\alpha x^2} = (2\alpha)^{j/2} H_j(x\sqrt{2\alpha})$$

where we used

$$\left(-\tfrac{d}{dx} + 2\alpha x\right) e^{\alpha x^2} = e^{\alpha x^2} \left(-\tfrac{d}{dx}\right)$$

and

$$H_j(x) = e^{+x^2} \left(-\tfrac{d}{dx}\right)^j e^{-x^2} = e^{+\frac{x^2}{2}} \left(-\tfrac{d}{dx} + x\right)^j e^{-\frac{x^2}{2}}$$

is the Hermite polynomial of order j. Since Hermite polynomials span $L_2(\mathbb{R})$, by Th. 2 above the family $\{v_j\}_{j \in \mathbb{N}}$ contains all eigenvectors. □

2.4 Conclusions, remarks

In this chapter we have considered the one dimensional version of two models: the Ising model and the harmonic cristal. In both cases we have applied the transfer matrix approach to study the infinite volume limit. Below is a summary of the results we obtained.

2.4.1 Hamiltonians

The starting hamiltonians for the Ising (resp. harmonic cristal) model are

$$H^I_\Lambda(\sigma) = -\sum_{j=-L}^{L-1} \sigma_j \sigma_{j+1} - \frac{h}{\beta} \sum_{j=-L}^{L} \sigma_j, \qquad \sigma \in \Omega_\Lambda = \{1, -1\}^\Lambda$$

$$H^{har}_\Lambda(\phi) = \sum_{j=-L}^{L-1} (\phi_j - \phi_{j+1})^2 + m^2 \sum_{j=-L}^{L} \phi_j^2, \qquad \phi \in \Omega_\Lambda = \mathbb{R}^\Lambda$$

Boundary conditions. In the Ising case we have considered three types of boudary conditions:

$$\begin{aligned}
\bar\sigma: &\quad H_\Lambda^{\bar\sigma}(\sigma) = H^I(\sigma) - J\left(\sigma_{-L}\bar\sigma_{-L-1} + \sigma_L\bar\sigma_{L+1}\right)\\
\text{periodic:} &\quad H_\Lambda^{per}(\sigma) = H^I(\sigma) - J\sigma_L\sigma_{-L}\\
\text{free:} &\quad H_\Lambda^{free}(\sigma) = H^I(\sigma).
\end{aligned}$$

The corresponding boundary conditions in the case of the harmonic cristal are

$$\begin{aligned}
\text{Dirichlet:} &\quad H_\Lambda^{D}(\phi) = H_\Lambda^{har}(\phi) + \phi_{-L}^2 + \phi_L^2 &&\to \phi_{L+1} = \phi_{-L-1} = 0\\
\text{periodic:} &\quad H_\Lambda^{per}(\phi) = H_\Lambda^{har}(\phi) + (\phi_L - \phi_{-L})^2 &&\\
\text{Neuman:} &\quad H_\Lambda^{N}(\phi) = H^{har}(\phi) &&\to [\nabla\phi]_{\partial\Lambda} = 0.
\end{aligned}$$

2.4.2 Partition function

In both models we wrote the partition function in terms of a transfer operator. As a result

$$\begin{aligned}
\lim_{L\to\infty} \frac{\ln Z_\Lambda^I}{|\Lambda|} &= \ln\lambda_1 = \ln[e^\beta \cosh h + \sqrt{(e^\beta \sinh h)^2 + e^{-2\beta}}]\\
\lim_{L\to\infty} \frac{\ln Z_\Lambda^{har}}{|\Lambda|} &= \ln\lambda_0 = \tfrac{1}{2}\ln\frac{\pi}{\alpha + (\beta + m^2/2)}, \quad \alpha = \sqrt{m^2\beta + \tfrac{m^4}{4}},
\end{aligned}$$

where λ_1 (resp. λ_0) is the largest eigenvalue of the transfer matrix T (resp. the transfer operator K). These limits are independent from the boundary conditions.

2.4.3 Magnetization

For the magnetization we obtained

$$\begin{aligned}
&\lim_{L\to\infty} \mathbb{E}_\Lambda[\sigma_j] = (v_1, \Sigma v_1) = M(h) \to \begin{cases} \pm 1 & h \to \pm\infty \\ 0 & h \to 0 \end{cases}\\
&\lim_{L\to\infty} \mathbb{E}_\Lambda[\phi_j] = 0\\
&\lim_{L\to\infty} \mathbb{E}_\Lambda[\sigma_j^2] = 1\\
&\lim_{L\to\infty} \mathbb{E}_\Lambda[\phi_j^2] = (v_0, \Sigma^2 v_0) = \tfrac{1}{4\alpha} \to \begin{cases} 0 & m \to \infty \\ +\infty & m \to 0 \end{cases}
\end{aligned}$$

In both cases the result is independent from the position j and from the boundary conditions. Note that though the averages spin is always finite, the average ϕ^2 diverges as $m \to 0$, reflecting the fact that ϕ_j is an unbounded variable and the fluctuations become very large when m is small.

2.4.4 Correlations

We have considered only two point correlations functions:

$$\lim_{L\to\infty} (\mathbb{E}_\Lambda[\sigma_i\sigma_j] - \mathbb{E}_\Lambda[\sigma_i]\mathbb{E}_\Lambda[\sigma_j]) = Ce^{-\frac{|i-j|}{\xi}}, \xi = \frac{1}{\ln\frac{\lambda_1}{\lambda_2}} \to \begin{cases} 0 & h \to \pm\infty \\ \frac{1}{\ln\frac{\cosh\beta}{\sinh\beta}} & h \to 0 \end{cases}$$

$$\lim_{L\to\infty} (\mathbb{E}_\Lambda[\phi_i\phi_j] - \mathbb{E}_\Lambda[\phi_i]\mathbb{E}_\Lambda[\phi_j]) \leq \tfrac{1}{4\alpha}e^{-\frac{|i-j|}{\xi}}, \xi = \frac{1}{\ln\frac{\beta+m^2/2+\alpha}{\beta}} \to \begin{cases} 0 & m \to \infty \\ +\infty & m \to 0 \end{cases}$$

Note that the correlation length ξ is always finite in the Ising model (unless $\beta \to \infty$). On the contrary, ξ diverges as $m \to 0$ in the harmonic cristal. Since the prefactor $1/\alpha$ also diverges it is better to consider the expression

$$\lim_{L\to\infty} \frac{(\mathbb{E}_\Lambda[\phi_i\phi_j] - \mathbb{E}_\Lambda[\phi_i]\mathbb{E}_\Lambda[\phi_j])}{\sqrt{\mathbb{E}_\Lambda[\phi_i^2]\mathbb{E}_\Lambda[\phi_j^2]}} \leq e^{-\frac{|i-j|}{\xi}}.$$

It is important to remark that the divergent quantities in the harmonic cristal appear *for any choice* of the boundary conditions.

2.4.5 Generalizations

The transfer matrix approach may be applied to much more general situations. One may for example replace the quadratic potential $m^2\phi^2$ by some function $V(\phi)$ such that

- $V(\phi) \to \infty$ as $|\phi| \to \infty$
- $V(0) = 0$ and V has a unique minimum at $\phi = 0$.

These conditions garantee that $V(\phi) = m^2\phi^2 + O(\phi^3)$ near $\phi = 0$. Then when β is large the transfer matrix is well approximated (see [Hel02, Ch. 5] for more details) by the harmonic transfer matrix we already studied. Some examples of such potential are $V(\phi) = \phi^4$ or $V(\phi) = \ln(1+\phi^2)$. Note that in the second example we cannot study high order correlation functions since $\mathbb{E}_\Lambda[\phi_i^n]$ since the log-potential does not garantee that the integral remains finite. More work is needed when the potential $V(\phi)$ has several minima.

When the transfer matrix is real but not symmetrix, or complex but not self-adjoint, then most of the theorems we used do not apply! Situations when one can still do something are

- the transfer operator K is real with (non strictly) positive kernel (not necessarily symmetric) such that *some power of* K has strictly positive kernel.
- the transfer operator is complex and normal.

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