Graduate seminar, WS 2020:
Transfer operator and semiclassical analysis methods.

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Description. Problems in equilibrium statistical mechanics are often formulated in terms of an infinite family of probability measures $N \mapsto \mu_N$ on Ω^N of the form

$$d\mu_N(\varphi_1, \ldots, \varphi_N) = \frac{1}{Z_N} e^{-\beta S(\varphi_1,\ldots,\varphi_N)} \prod_{j=1}^{N} d\rho(\varphi_j),$$

where Ω describes the 'spin domain', ρ is some reference measure, the function $S(\varphi_1, \ldots, \varphi_N) \in \mathbb{R}$ has the role of an the 'energy' for the configuration $(\varphi_1, \ldots, \varphi_N)$, $\beta \geq 0$ is a parameter and Z_N a normalization constant.

The most famous examples are

- $\Omega = \mathbb{R}^n$, $n \geq 1$, (unbounded spin) with $\rho =$ Lebesgue measure on $\mathbb{R}^n$,
- $\Omega = \{-1, +1\}$ (Ising model) with $\rho = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$,
- $\Omega = \mathcal{S}^{n-1}$, $n \geq 2$ ($O(n)$ model) with $\rho =$ Hausdorff mass.

The main problem is to understand properties of this measures in the limit $N \to \infty$, in particular the following three quantities:

- $\lim_{N\to\infty} \frac{1}{N} \ln Z_N$ (pressure),
- $\lim_{N\to\infty} \mathbb{E}_N[\varphi_j]$ (average), where $\mathbb{E}_N[\cdot] := \int \cdot \, d\mu_N$,
- $\lim_{N\to\infty} \mathbb{E}_N[\varphi_j\varphi_k] - \mathbb{E}_N[\varphi_j]\mathbb{E}_N[\varphi_k]$ (covariance).

A powerful tool to study this kind of problems is the so-called transfer operator approach. When applicable, it allows to reformulate the above averages in terms of expressions of the form

$$(w_l, T^N w_r)$$

where T is a linear operator acting on some Hilbert space $\mathcal{H}$ (for example $L^2(\mathbb{R}^n)$) with inner product $(\, , \,)$ and w_l, w_r are two vectors in $\mathcal{H}$. The problem is then reduced to study high powers of the operator T. If the operator is 'nice enough' T^N will converge in an appropriate sense to the projection on a single eigenvector v for the top eigenvalue λ. Except in a few cases, v cannot be computed explicitely. Nevertheless for large parameter β one can use an extension of Laplace method to 'expand' v in the neighborhood of the limit $\beta \to \infty$ (the so-called semiclassical limit).

In this seminar we plan to learn this method and see a few applications and generalizations. We will mostly read parts of the book

Semiclassical analysis, Witten Laplacians and statistical mechanics,
by B. Helffer (World Scientific).

We will start by considering the Ising model (chapter 3). In this case the transfer operator reduces to a matrix and many computations can be performed exactly. We will then proceed with continuous spin models and see how to construct approximate explicit formulas by semicalssical analysis (mostly using chapter 4 and 5). According to time/interest we will see some generalizations to less standard situations.

Prerequisites. Functional analysis. Some basic knowledge in probability/physics may be useful but is not necessary.