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# PDE and Modelling

Sheet 4, due Mai 13

## Problem 1 (*10 points*)

Suppose that conservation of mass, momentum and energy (with $r \equiv 0$) hold, i.e.

$$\begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho v) &= 0 \\ \varrho(\partial_t v + (v \cdot \nabla) v) &= \operatorname{div} \sigma + f \\ \varrho \frac{D\varepsilon}{Dt} &= \sigma : Dv - \operatorname{div} q, \end{aligned}$$

where we defined $A : B = tr A^T B$. Furthermore assume the following relations: $\varepsilon = a(\theta, \varrho) + \theta\eta$, where $\theta(t,x)$ denotes the temperature at $(t,x)$ and

$$\eta = -\frac{\partial a}{\partial \theta}.$$

Deduce that

$$\partial_t(\varrho\eta) + \operatorname{div}\left(\frac{q}{\theta} + \varrho\eta v\right) = \frac{1}{\theta}(\sigma + p\,\mathrm{Id}) : Dv + q \cdot \nabla \frac{1}{\theta},$$

where $p = \varrho^2 \frac{\partial a}{\partial \varrho}$.

## Problem 2 (*10 points*)

Let $\Omega$ be a bounded domain in $\mathbb{R}^d$ and $x(t,X)$ be a motion. Let $\hat{f} : \Omega \times GL_+(d) \to \mathbb{R}$ be a given function. We assume the Piola-Kirchoff $S$ stress is of the form $S(t,X) = \hat{S}(X, Dx(t,X))$, where $\hat{S}(X,F) := \frac{\partial \hat{f}}{\partial F}$. Assume that $x$ satisfies the equation of motion (in material coordinates)

$$\varrho_0 \partial_t^2 x = (\mathrm{DIV}_X\, \hat{S})(X, Dx(t,X)),$$

where $\varrho_0 : \Omega \to (0,\infty)$ is the reference mass density. Show that the total energy

$$\int_\Omega \left(\frac{1}{2}\varrho_0(X)|\partial_t x(t,X)|^2 + \hat{f}(X, Dx(t,X))\right) dX$$

is independent of t. Hint: Multiply the equation of motion by $\partial_t x$.

**Problem 3** (*2+2+4+2 points*)

Suppose that $\hat{\sigma} : \mathrm{GL}_+(d) \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ satisfies

(i) $\hat{\sigma}(QF) = Q\hat{\sigma}(F)Q^T$ for all $Q \in \mathrm{SO}(d)$ (frame indifference),

(ii) $\hat{\sigma}(FG) = \hat{\sigma}(F)$ for all $G \in \mathrm{SL}(d)$ (isotropy group of a fluid).

(a) Use *(ii)* to show that $\hat{\sigma}(F) = \hat{\sigma}(\sqrt[d]{\det F}\ \mathrm{Id})$.

(b) Show that $Q\hat{\sigma}(F)Q^T = \hat{\sigma}(F)$.

(c) Let $M$ be a symmetric matrix such that $M = QMQ^T$ for all $Q \in \mathrm{SO}(d)$. Show that $M = \alpha\,\mathrm{Id}$ for some $\alpha \in \mathbb{R}$.

*Hint:* You may consider a diagonal matrix $M$ and permutation matrices $Q$ first. Then consider the general case. Alternatively you may prove that $M$ cannot have two distinct eigenvalues.

(d) Prove that there exists a function $\hat{p} : (0,\infty) \to \mathbb{R}$ such that $\hat{\sigma}(F) = -\hat{p}(\frac{1}{\det F})\,\mathrm{Id}$.

**Problem 4** (*5+5 points*)

Suppose $\hat{W} : \mathrm{GL}_+(n) \to \mathbb{R}$ is $C^1$, and let $g \subset \mathrm{SO}(n)$ be a group. Prove that i) $\Rightarrow$ ii) $\Leftrightarrow$ iii), where

(i) $\hat{W}(F) = \hat{W}(FG)$ for all $G \in g$ and $F \in \mathrm{GL}_+(n)$

(ii) $\hat{S}(FG)G^t = \hat{S}(F)$ for all $G \in g$ and $F \in \mathrm{GL}_+(n)$

(iii) $\hat{\sigma}(FG) = \hat{\sigma}(F)$ for all $G \in g$ and all $F \in \mathrm{GL}_+(n)$