Prof. Dr. M. Disertori
Raphael Winter
Institut für Angewandte Mathematik
Universität Bonn

PDE and Modelling

Sheet 10, due July 8

Problem 1 (*3+5+2 points*)

Consider the Lagrangian $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times d}$ given by

$$f(x, z, P) = \det P.$$

(a) Derive the Euler-Lagrange equation corresponding to f.

(b) Show that f is a null Lagrangian, that is, every $u \in C^2(\Omega)$ solves the Euler-Lagrange equation corresponding to f.

(c) Show that, for $u \in C^2$, the volume of $u(\Omega)$ only depends on the boundary values of u at $\partial\Omega$.

Problem 2 (*10 points*)

Suppose that $\Omega \subset \mathbb{R}^d$ is open and bounded $T > 0$, and let $\varepsilon > 0$. Find $f : \Omega \times (0, T) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ such that the Euler-Lagrange equation for the functional

$$I[u] = \int_0^T \int_\Omega f(x, t, u, u_t, Du)\, \mathrm{d}x\, \mathrm{d}t$$

is

$$u_t - \Delta u - \varepsilon u_{tt} = 0 \qquad \text{in } \Omega \times (0, T).$$

Hint: Try a term containing $e^{-t/\varepsilon}$.

Problem 3 (*2+2+2+2+2 points*)

According to Fermat's principle, a ray of light traveling between two points will follow the path that can be traversed in the shortest time.

Let $\gamma : [-1, 1]^2 \to (0, \infty)$ be a function denoting the speed of light at a point $(x, y) \in [-1, 1]^2$, and consider a piecewise C^1 curve $x \mapsto (x, u(x))$.

(a) Show that the time needed to traverse the path is given by

$$I[u] = \int_{-1}^{1} \frac{\sqrt{1 + u'(x)^2}}{\gamma(x, u(x))} \, \mathrm{d}x$$

(b) Suppose that γ is smooth. Compute the Euler-Lagrange equation corresponding to I.

(c) Suppose that $\gamma(x, y) = \gamma_0$ for $x < 0$ and $\gamma(x, y) = \gamma_1$ for $x \geq 0$. Show that a continuous minimizer u of I satisfies $u''(x) = 0$ for $x \neq 0$.

(d) Under the same assumptions, derive an equation for the left and ride side derivatives of u at $x = 0$ in terms of γ_0 and γ_1.

(e) Derive Snell's law:

$$\frac{\sin \alpha_1}{\sin \alpha_0} = \frac{\gamma_1}{\gamma_0}$$

where α_0 denotes the angle of incidence and α_1 the angle of refraction of u, that is, α_0 and α_1 are the angles that u makes with the line $y = u(0)$ on the left and right, respectively.