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PDE and Modelling

Sheet 5, due May 27

Problem 1 *(10 Points)*

Consider the heat flux q in an isotropic material and assume that

$$q = \hat{q}(\varrho, \theta, \nabla\theta).$$

Show that $\hat{q}$ must be of the form

$$\hat{q}(\varrho, \theta, x) = -\hat{k}(\varrho, \theta, |x|)x \qquad \forall x \in \mathbb{R}^d,$$

where $\hat{k}$ is a real valued function.

Problem 2 *(10 points)*

Suppose $W : \mathrm{GL}_+(d) \to \mathbb{R}$ satisfies

$$W(QF) = W(F) = W(FQ)$$

for all $Q \in \mathrm{SO}(d)$.

(a) Show that there exists a function $g : (0, \infty)^n \to \mathbb{R}$ such that

$$W(F) = g(\lambda_1(F), \ldots, \lambda_n(F)),$$

where $\lambda_i(F)$ are the singular values of F.

(b) Argue that g is invariant under permutation of the arguments, that is,

$$g(z_1, \ldots z_n) = g(z_{i(1)}, \ldots, z_{i(d)})$$

if i is a permutation.

Problem 3 *(10 points)*

Let $d = 3$, and assume that $\mathcal{L} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}_{\text{sym}}$ is linear, and

$$\mathcal{L}\left(QFQ^T\right) = Q\mathcal{L}(F)Q^T$$

for all $F \in \mathbb{R}^{d\times d}$ and all $Q \in \mathrm{SO}(d)$.

(a) Let $F = e_1 \otimes e_2 - e_2 \otimes e_1$, and show that $F\mathcal{L}(F) = \mathcal{L}(F)F$.

Hint: Consider $Q_\tau = e^{\tau F} \in \mathrm{SO}(d)$.

(b) Conclude from (a) that

$$\mathcal{L}(F) = \begin{pmatrix} a & c & 0 \\ c & d & 0 \\ 0 & 0 & b \end{pmatrix}$$

for some $a, b, c \in \mathbb{R}$.

(c) Use (a) again to show that the matrices

$$\begin{pmatrix} a & c \\ c & d \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

commute, and argue that $c = 0$ and $a = d$ using Problem 4 of sheet 3.

(d) Use the hypothesis for $\mathcal{L}$ for an appropriate Q to obtain $a = b = 0$.

(e) Argue that $\mathcal{L}(F) = 0$ for every skew matrix F, and that (for general F) $\mathcal{L}(F)$ only depends on $\frac{1}{2}(F + F^T)$, that is, $\mathcal{L}(F) = \mathcal{F}(F + F^T))$.

(f) Argue that the arguments from parts (a)-(c) can be repeated to show

$$\mathcal{L}(e_3 \otimes e_3) = \lambda \,\mathrm{Id} + \mu e_3 \otimes e_3,$$

for some $\lambda, \mu \in \mathbb{R}$, and conclude that

$$\mathcal{L}(e_j \otimes e_j) = \lambda \,\mathrm{Id} + \mu e_j \otimes e_j,$$

for $j = 1, 2, 3$.

(g) Show that

$$\mathcal{L}(F) = \mu \frac{F + F^T}{2} + \lambda (\operatorname{tr} F)\,\mathrm{Id}$$

for all F.

Hint: Assume that F is diagonal at first.