Chapter 3

Low temperature region

(Notes under construction)

3.1 Ferromagnetic order: Ising model with long range interaction

Let $\Lambda = (\mathbb{Z}/2L+1)^d$ the cube $\{-L,\ldots,L\}^d$ with periodic boundary conditions. The finite volume configuration set is

$$\Omega_\Lambda = \{1,-1\}^\Lambda = \{\sigma : \Lambda \to \{1,-1\}\}, \qquad \sigma(x) = \sigma_x\}.$$

The energy functional is

$$H_\Lambda(\sigma) = -\tfrac{1}{2} \sum_{x,y\in\Lambda} J_{xy}\sigma_x\sigma_y - \frac{1}{\beta}\sum_{x\in\Lambda} h_x\sigma_x$$

where $J_{xy} = J_{yx} \geq 0$ encodes the interaction between spins at position x and y, $\beta = 1/T$, and h_x plays the role of a local magnetic field. The corresponding finite volume partition function is

$$Z_\Lambda(\beta, \mathbf{h}_\Lambda) = \sum_\sigma \; e^{\frac{\beta}{2}\sum_{x,y\in\Lambda} J_{xy}\sigma_x\sigma_y} e^{\sum_{x\in\Lambda} h_x\sigma_x}.$$

In the following we consider the interaction

$$J_{xy} = (-W^2\Delta + \mathbf{1}_\Lambda)^{-1}_{xy}$$

where $W \gg 1$ is a parameter, $\mathbf{1}_\Lambda$ is the identity matrix and Δ_Λ is the discrete Laplacian with periodic boundary conditions defined by

$$-(\Delta f)(x) = \sum_{|e|=1} [f(x) - f(x+e)] \quad \forall x \in \Lambda.$$

When $x+e$ is not inside Λ we take the corresponding projection (by periodicity) inside Λ. Note that the corresponding quadratic form can be written as

$$(f, -\Delta f) = \sum_{xy\in\Lambda} f(x)(-\Delta)_{xy} f(y) = \sum_{x,y\in\Lambda, |x-y|=1} (f(x)-f(y))^2 \geq 0.$$

Once can show that $J^{-1} > 0$ as a quadratic form and that $J_{xy} > 0$ pointwise (for each x, y).

3.1.1 Duality transformation

Lemma 1

$$Z_\Lambda(\beta, \mathbf{h}_\Lambda) = \frac{2^{|\Lambda|}}{\mathcal{N}_J} \int \prod_{x\in\Lambda} d\phi_x e^{-\frac{1}{2}(\phi, J^{-1}\phi)} \prod_x \cosh(\sqrt{\beta}\phi_x + h_x)$$

where $\phi_x \in \mathbb{R}$ for each $x \in \Lambda$, $d\phi_x$ is the corresponding Lebesgue measure,

$$(\phi, J^{-1}\phi) = \sum_{xy} \phi_x J^{-1}_{xy} \phi_y,$$

and $\mathcal{N}_J$ is a normalization constant

$$\mathcal{N}_J = \int \prod_{x\in\Lambda} d\phi_x e^{-\frac{1}{2}(\phi, J^{-1}\phi)} = \frac{1}{\sqrt{\frac{\det J^{-1}}{2\pi}}}.$$

Performing the change of coordinates $\phi_x = \phi'_x - h_x/\sqrt{\beta}$ we obtain

$$Z_\Lambda(\beta, \mathbf{h}_\Lambda) = e^{-\frac{1}{2\beta}(h_\Lambda, J^{-1}h_\Lambda)} \frac{2^{|\Lambda|}}{\mathcal{N}_J} \int \prod_{x\in\Lambda} d\phi_x e^{-\frac{1}{2}(\phi, J^{-1}\phi)} e^{\frac{1}{\sqrt{\beta}}(\phi, J^{-1}h_\Lambda)} \prod_x \cosh(\sqrt{\beta}\phi_x)$$

In the following (unless stated otherwise) we consider the case $h_x = h \; \forall x$. Then the formula above becomes

$$Z_\Lambda(\beta, \mathbf{h}_\Lambda) = e^{-|\Lambda|\frac{h^2}{2\beta}} \frac{2^{|\Lambda|}}{\mathcal{N}_J} \int \prod_{x\in\Lambda} d\phi_x e^{-\frac{W^2}{2}\sum_{|x-y|=1}(\phi_x-\phi_y)^2} e^{-\sum_x f(\phi_x)}$$

where

$$f(\phi_x) = \frac{\phi_x^2}{2} - \frac{h\phi_x}{\sqrt{\beta}} - \ln\cosh(\sqrt{\beta}\phi_x).$$

and we used $-\Delta 1 = 0$.

3.1.2 Heuristics

Since $W \gg 1$ we expect the integral to be dominated by configurations near $\phi_x = \phi \; \forall x$ (otherwise the corresponding weight is exponentially small). For constant ϕ_x the integrand becomes

$$e^{-|\Lambda| f(\phi)}$$

hence a sadddle point approximation seems justified. The saddle equation for ϕ is

$$f'(\phi) = \phi_x - \frac{h}{\sqrt{\beta}} - \sqrt{\beta}\tanh(\sqrt{\beta}\phi_x)$$

Studying this equation we find there are two minima ϕ_{m1}, ϕ_{m2} with

$$\phi_{m1}(h) = \sqrt{\beta} + \frac{h}{\sqrt{\beta}} + O(e^{-\beta}), \qquad \phi_{m2}(h) = -\sqrt{\beta} + \frac{h}{\sqrt{\beta}} + O(e^{-\beta}).$$

For $h = 0$ we have $\phi_{m1}(0) = -\phi_{m2}(0) = \sqrt{\beta} + O(e^{-\beta})$ and the two minima are at the same height $f(\phi_{m2}(0)) = f(\phi_{m1}(0)) = -\beta/2 + O(1/\beta)$. For $h > 0$, the function f has only one global minimum at ϕ_{m1}. Expanding around $h = 0$ we find

$$f(\phi_{m2}(h)) - f(\phi_{m1}(h)) = -h\frac{\phi_{2m}(0) - \phi_{1m}(0)}{\sqrt{\beta}} + O(h^2) = 2h + O(h^2) + O(e^{-\beta}) > 0$$

since $f'(\phi_{m2}(h)) = f'(\phi_{m1}(h)) = 0$. Moreover the hessian at the two minuma are

$$f''(\phi_{m1}(h)) = 1 - \frac{\beta}{\cosh\sqrt{\beta}\phi_{m1}(h)} = 1 - O(e^{-2\beta}) = 1 - \frac{\beta}{\cosh\sqrt{\beta}\phi_{m2}(h)} = f''(\phi_{m2}(h))$$

since $\cosh\sqrt{\beta}\phi_{m1}(h) = \cosh(\beta + O(e^{-\beta})$. Finally for $h = 0$ there is one local maximum at $\phi_3 = 0$ with $f(0) = 0$. For $0 < h \ll 1$ we have $\phi_3 = -\frac{h}{\beta^{3/2}}$ to the leading order, hence $f(\phi_3) = -\frac{h^2}{2\beta^2}$ to the leading order.

Since we expect the integral to be concentrated near constant configurations $\phi_x = \phi_{m1}(h)\ \forall x$ we perform a global translation

$$\phi_x = \phi'_x + \phi_{m1}$$

so that the global minimum occurs ar $\phi' = 0$. The saddle approximation then says that the potential can be approximated by the quadratic terms

$$f(\phi_x + \phi_{m1}) \simeq f(\phi_{m1}) + \frac{m^2}{2}\phi_x^2.$$

Then the partition function can be written as

$$Z_\Lambda(\beta, h) = e^{-|\Lambda|\left(\frac{h^2}{2\beta} + f(\phi_{m1})\right)} \frac{2^{|\Lambda|}\mathcal{N}_C}{\mathcal{N}_J} \tilde{Z}_\Lambda(\beta, h) = e^{|\Lambda|\left[F_0 + \frac{1}{|\Lambda|}\ln\tilde{Z}_\Lambda(\beta,h)\right]} \tag{3.1.1}$$

where

$$F_0 = -f(\phi_{m1}) - \frac{h^2}{2\beta} + \ln 2 + \frac{1}{|\Lambda|}\ln\frac{\mathcal{N}_C}{\mathcal{N}_J} \tag{3.1.2}$$

is the main contribution to the partition function and the correction is

$$\tilde{Z}_\Lambda(\beta, h) = \int d\mu_C(\phi) e^{-\sum_x V(\phi_x)}, \tag{3.1.3}$$

where $d\mu_C(\phi)$ is the real normalized gaussian measure

$$d\mu_C(\phi) = \frac{d\phi_\Lambda}{\mathcal{N}_C} e^{-\frac{1}{2}(\phi, C^{-1}\phi)} = \frac{d\phi_\Lambda}{\mathcal{N}_C} e^{-\frac{1}{2}\left[W^2 \sum_{|x-y|=1}(\phi_x - \phi_y)^2 + m^2 \sum_x \phi_x^2\right]} \tag{3.1.4}$$

with covariance $C = (-W^2\Delta + m^2)^{-1}$, $d\phi_\Lambda = \prod_{x\in\Lambda} d\phi_x$ and

$$V(\phi_x) = f(\phi_x + \phi_{m1}) - f(\phi_{m1}) - \frac{m^2}{2}\phi_x^2 \tag{3.1.5}$$

is what remains in the interaction after we have extracted the value at the mininum and the quadratic contribution. The normalization constant is

$$\mathcal{N}_C = \left(\sqrt{\det \frac{C^{-1}}{2\pi}}\right)^{-1}.$$

In the following we will study (3.1.1). If our heuristic arguments are true we expect $\tilde{Z}_\Lambda(\beta, h) \sim 1$. More precisely we expect $\frac{1}{|\Lambda|} \ln \tilde{Z}_\Lambda(\beta, h)$ to be small compared to F_0 for any volume Λ.

3.2 Preliminary results.

3.2.1 Eigenvalues and eigenvectors for the discrete Laplacian

Discrete Laplacian on $\mathbb{Z}^d$ For any function $f : \mathbb{Z}^d \to \mathbb{R}$ we define the discrete Laplace operator by

$$(-\Delta f)(x) = \sum_{j=1}^{d} \sum_{\sigma=\pm 1} [f(x) - f(x + \sigma e_j)] \qquad x \in \mathbb{Z}^d,$$

where e_j is the unit vector in direction j. This formula is well defined also when f has no finite l_2 norm.

In the following we will consider the discrete Laplacian restricted to the set $\Lambda_L = [-L, \ldots, L]^2$ a finite cube in $\mathbb{Z}^d$. When dealing with a finite volume one must specify the boundary conditions. We will consider here only two types of b.c.

Discrete Laplacian on Λ_L with periodic b.c. For any function $f : \Lambda \to \mathbb{R}$ we define discrete Laplacian $-\Delta_P$ restricted to Λ with periodic boundary conditions by

$$(-\Delta_P f)(x) = (-\Delta f_P)(x) \qquad x \in \Lambda,$$

where f_P is the periodic extention of f to $\mathbb{Z}^d$, i.e.

$$f_P(x + n(2L+1)e_j) = f_P(x) \qquad \forall x \in \mathbb{Z}^d, \quad \forall j = 1, \ldots, d, \quad \forall n \in \mathbb{Z}. \tag{3.2.6}$$

We consider the family $\{v_k\}$ of functions on $\mathbb{Z}^d$ defined by

$$v_{\mathbf{k}}(x) = \frac{1}{\sqrt{|\Lambda|}} e^{i\mathbf{k}\cdot x} \qquad \mathbf{k} \in \mathbb{R}^d.$$

where $(\mathbf{k} \cdot x) = \sum_{j=1}^{d} x_j k_j$. The prefactor ensures these functions, when restricted to Λ, have unit norm: $\sum_{x\in\Lambda} |v_{\mathbf{k}}(x)|^2 = 1$. Note that $\forall \mathbf{k} \in \mathbb{R}^d$ we have

$$(-\Delta v_k)(x) = \lambda_k v_k(x),$$

where $\lambda_k = 2\sum_{j=1}^{d}(1-\cos k_j)$. The functions u_k, satisfy in addition the periodic condition (3.2.6) if $k_j = \frac{\pi}{2L+1} 2n_j$, with $n_j \in \mathbb{Z}$. Then

$$\mathcal{B}_P = \{v_{k_n}\}_n, \qquad k_n = \tfrac{\pi}{2L+1} 2n, \; n \in \Lambda_L = [-L, \dots, L]^d$$

is a normalised basis of **eigenvectors** for $-\Delta_P$. The corresponding **eigenvalues** are

$$\lambda_{k_n} = 2\sum_{j=1}^{d}(1 - \cos(k_n)_j) \geq 0.$$

Note that for each $n \neq 0$, the corresponding eigenvalue has multiplicity 2. When $n = 0$ then $\lambda_0 = 0$ and the eigenvalue has multiplicity one (with eigenvector $v_0(x) = 1/\sqrt{\Lambda}, \forall x$).

Discrete Laplacian on Λ_L with Neuman b.c. For any function $f : \Lambda \to \mathbb{R}$ we define discrete Laplacian $-\Delta_N$ restricted to Λ with Neuman boundary conditions by

$$(-\Delta_N f)(x) = (-\Delta f_N)(x) \qquad x \in \Lambda,$$

where f_N is the an extension of f to $\mathbb{Z}^d$, satisfying

$$f_N(x + \sigma e_j) - f_N(x) = 0 \qquad \text{whenever } x \in \Lambda \text{ and } x + \sigma e_j \notin \Lambda. \tag{3.2.7}$$

Note that this is equivalent to require $\nabla_N f_{|\partial\Lambda} = 0$, where ∇_N is the (discrete) derivative in the direction orthogonal to the boundary. We consider the family of functions $\{u_k^\sigma\}$ on $\mathbb{Z}$, with $\sigma = \pm 1$ and $k \in \mathbb{R}$ defined by

$$u_k^+(x) = \cos(kx), \qquad u_k^-(x) = \sin(kx).$$

With these functions we construct a set of functions on $\mathbb{Z}^d$,

$$U_{\mathbf{k}}^\alpha(x) = \tfrac{1}{\mathcal{N}_{\mathbf{k}}^\alpha} \prod_{j=1}^{d} u_{k_j}^{\alpha_j}(x_j), \mathbf{k} \in \mathbb{R}^d, \alpha \in \{-1, 1\}^d,$$

where $\mathcal{N}_{\mathbf{k}}^\alpha$ ensures that $\sum_{x\in\Lambda} |U_{\mathbf{k}}^\alpha(x)|^2 = 1$. Note that we have

$$(-\Delta U_{\mathbf{k}}^\alpha)(x) = \lambda_{\mathbf{k}} U_{\mathbf{k}}^\alpha(x)$$

where $\lambda_{\mathbf{k}} = 2\sum_{j=1}^{d}(1 - \cos k_j)$. The function $U_{\mathbf{k}}^\alpha$ satisfy (3.2.7) if

$$\begin{aligned} &\cos(Lk_j) - \cos((L+1)k_j) = 0 \;\text{ when } \alpha_j = +1 \quad \text{ and} \\ &\sin(Lk_j) - \sin((L+1)k_j) = 0 \;\text{ when } \alpha_j = -1. \end{aligned}$$

This is satisfied if $k_j = \frac{\pi}{2L+1} n_j$ with n_j even when $\alpha_j = 1$ and n_j odd when $\alpha_j = -1$. Then

$$\mathcal{B}_N = \{U^{\alpha}_{\mathbf{k}(\alpha)}\}, \qquad \text{with } \alpha \in \{-1,+1\}^d,\ k_n(\alpha) = \tfrac{\pi}{2L+1} n,$$
$$\text{where} \qquad n(\alpha) \in [0,\dots,L]^d, \text{and } n_j(\alpha) \text{ is } \begin{cases} \text{even if} & \alpha_j = +1 \\ \text{odd if} & \alpha_j = -1 \end{cases}$$

is a basis of **eigenvectors** for $-\Delta_N$. The corresponding **eigenvalues** are

$$\lambda_{k_n(\alpha)} = 2\sum_{j=1}^{d} (1 - \cos(k_n(\alpha))_j) \geq 0.$$

Note that all eigenvalues have multiplicity one and that they differ from the corresponding eigenvalues of $-\Delta_P$ only by a factor of order $1/L^2$. The lowest eigenvalue is

$$\alpha = \mathbf{1} \equiv \alpha_j = 1\ \forall j, \quad n_j = 0\ \forall j.$$

The only eigenvector corresponding to $\lambda_{k(\alpha)} = 0$ is $U^{\mathbf{1}}_{k_0}(x) = 1/\sqrt{|\Lambda|}\ \forall x$, where $\mathbf{1} \equiv \alpha_j = 1 \forall j$ and $k_0 = 0$. Note that for any $k(\alpha) \neq 0$ the normalization constant satisfies

$$\mathcal{N}^{\alpha}_{\mathbf{k}} = c_\alpha \sqrt{|\Lambda|}, \qquad 2^{-d/2} \leq c_a \leq 1.$$

To prove it we compute $\sum_{x\in\Lambda} |U^{\alpha}_{\mathbf{k}(\alpha)}(x)|^2$ using $(\cos\alpha)^2 = (1+\cos 2\alpha)/2$ and $(\sin\alpha)^2 = (1-\cos 2\alpha)/2$.

3.2.2 Properties of the covariance

Remember that we defined $C = (-W^2\Delta + m^2)^{-1}$. For the initial model we take Δ to be the discrete Laplacian with periodic boundary conditions. Later we will need to consider also the case of Neuman boundary conditions.

Estimate on the covariance. We will need the following lemma

Lemma 2 *There exists a constant K independent of W, m and $|\Lambda|$ such that*

$$|C_{xx} - C_{xy}| \leq \tfrac{K}{W^2}, \quad \textit{and} \quad |C_{xy}| \leq \tfrac{1}{m^2|\Lambda|} + \tfrac{K}{W^2} \qquad \forall x, y \in \Lambda.$$

This bound holds both for periodic and for Neumann boundary conditions.

Proof in the case of periodic b.c. Since $-\Delta$ is diagonal in the basis $\mathcal{B}_P$ we can write

$$\begin{aligned} C_{xy} &= \sum_{n\in\Lambda_L} \frac{v_{k_n}(x)\bar{v}_{k_n}(y)}{W^2\lambda_{k_n} + m^2} = \frac{1}{|\Lambda|} \sum_{n\in\Lambda_L} \frac{e^{ik_n\cdot(x-y)}}{W^2\lambda_{k_n} + m^2} \\ &= \frac{1}{m^2|\Lambda|} + \frac{1}{|\Lambda|W^2} \sum_{n\neq 0} \frac{e^{ik_n\cdot(x-y)}}{\lambda_{k_n} + \frac{m^2}{W^2}} \end{aligned}$$

where $v_{k_n}(x)\bar{v}_{k_n}(y)$ is the projection on the eigenvector v_{k_n}. Then

$$|C_{xx} - C_{xy}| \leq \tfrac{1}{W^2}\tfrac{1}{|\Lambda|}\sum_{n\neq 0}\frac{\left|e^{ik\cdot(x-y)}-1\right|}{\lambda_{k_n}+\frac{m^2}{W^2}} \leq \tfrac{1}{W^2}\tfrac{1}{|\Lambda|}\sum_{n\neq 0}\frac{1}{\lambda_{k_n}+\frac{m^2}{W^2}}$$

Now set $M^2 = \frac{m^2}{W^2}$. We have

$$\tfrac{1}{|\Lambda|}\sum_{k_n\neq 0}\frac{1}{\lambda_{k_n}+M^2} \leq \tfrac{1}{|\Lambda|}\sum_{|\mathbf{k_n}|\leq 1/10}\frac{1}{\lambda_{k_n}+M^2} + \frac{1}{c_0} \tag{3.2.8}$$

where $c_0 > 0$ is some constant such that $\lambda_{k_n} \geq c_0 > 0\ \forall |k_n| > 1/10$, and we used

$$\tfrac{1}{|\Lambda|}\sum_{|k_n|\geq 1/10}\frac{1}{\lambda_{k_n}+M^2} \leq \frac{1}{c_0+M^2}\frac{|\Lambda|}{|\Lambda|} \leq \frac{1}{c_0}.$$

For small k we can approximate $\lambda_k \geq |k|^2/c_1$ for some $c_1 > 0$ then

$$\tfrac{1}{|\Lambda|}\sum_{|k_n|\leq 1/10}\frac{1}{\lambda_{k_n}+M^2} \leq c_1\int_{|k|\leq 1}\frac{d^dk}{k^2+M^2c_1} \leq K_d \qquad d\geq 3$$

where K_d is some constant that depends only on the dimension. To prove the last inequality one can pass to spherical coordinates in the integral. This completes the proof.

Proof in the case of Neuman b.c. Since $-\Delta_N$ is diagonal in the basis $\mathcal{B}_N$ we can write

$$C_{xy} = \sum_{\alpha,n}\frac{U^\alpha_{k_n(\alpha)}(x)\bar{U}^\alpha_{k_n(\alpha)}(y)}{W^2\lambda_{k_n(\alpha)}+m^2} = \frac{1}{m^2|\Lambda|} + \frac{1}{W^2}\sum_{\alpha,n\neq 0}\frac{U^\alpha_{k_n(\alpha)}(x)\bar{U}^\alpha_{k_n(\alpha)}(y)}{\lambda_{k_n(\alpha)}+\frac{m^2}{W^2}}$$

Then

$$\begin{aligned}|C_{xx} - C_{xy}| &\leq \tfrac{1}{W^2}\sum_{\alpha,n\neq 0}\frac{\left|U^\alpha_{k_n(\alpha)}(x)\bar{U}^\alpha_{k_n(\alpha)}(y) - U^\alpha_{k_n(\alpha)}(x)\bar{U}^\alpha_{k_n(\alpha)}(x)\right|}{\lambda_{k_n(\alpha)}+\frac{m^2}{W^2}}\\ &\leq \tfrac{2^{d+1}}{W^2}\tfrac{1}{|\Lambda|}\sum_{\alpha,n\neq 0}\frac{1}{\lambda_{k_n(\alpha)}+\frac{m^2}{W^2}},\end{aligned}$$

where we used

$$|U^\alpha_{k_n(\alpha)}(x)|^2 \leq \frac{2^d}{|\Lambda|} \quad \forall x, k_n(\alpha).$$

The proof then works exactly as in the case of periodic boundary conditions.

Finally the estimate on $|C_{xy}|$ works in the same way (for periodic and Neuman b.c.) except that now the contribution from the zero eigenvalue $1/(m^2|\Lambda|)$ must be added. □

Ratios of normalization constants for gaussian measures. We will need the following lemma

Lemma 3 *Let*

$$C^{-1} = (-W^2\Delta_\Lambda + m^2), \quad C_q^{-1} = C^{-1} - (1-q)m^2 = (-W^2\Delta_\Lambda + m^2 q),$$

where $1 > q > 0$ *and* Δ_Λ *is the discrete Laplacian on* Λ *with periodic or Neumann boundary conditions. Then there exists some constants* $K_1, K_2 > 0$ *(independent of* W *and* q*) such that*

$$\det \frac{C^{-1}}{C_q^{-1}} \leq \tfrac{K_2}{q} e^{K_1 f_d \frac{(1-q)|\Lambda|}{W^2}} \tag{3.2.9}$$

where

$$f_d = 1 \ \forall d \geq 3, \qquad f_2 = -\ln \tfrac{qm^2}{W^2}, \qquad f_1 = \tfrac{W}{m\sqrt{q}}.$$

Proof We have seen above that the discrete Laplacian wih periodic boundary conditions has eigenvalues indexed by the vectors $\mathbf{k} = \frac{2\pi}{2L+1}\mathbf{n}$ with $\mathbf{n} \in [-L, L]^d$

$$\lambda_{\mathbf{k}} = 2\sum_{i=1}^{d}(1 - \cos k_i).$$

The determinant is then

$$\det \tfrac{C^{-1}}{C_q^{-1}} = \prod_{\mathbf{k}} \tfrac{W^2\lambda_{\mathbf{k}} + m^2}{W^2\lambda_{\mathbf{k}} + qm^2} = \tfrac{1}{q}\prod_{\mathbf{k}\neq 0}\left[1 + \tfrac{m^2(1-q)}{W^2\lambda_{\mathbf{k}} + m^2 q}\right] \leq \tfrac{1}{q} e^{\frac{m^2(1-q)}{W^2}\sum_{\mathbf{k}\neq 0}(\lambda_{\mathbf{k}} + \frac{m^2 q}{W^2})^{-1}}$$

where we used $(1+u) \leq e^u \ \forall u \geq 0$. Note that

$$\begin{aligned}\int_{|\vec{k}|\leq \frac{1}{10}} d\vec{k}\ \frac{1}{\lambda_{\vec{k}} + \frac{m^2 q}{W^2}} &\simeq \int_{|\vec{k}|\leq \frac{1}{10}} d\vec{k}\ \frac{1}{|\vec{k}|^2 + \frac{m^2 q}{W^2}} \\ &= c_d \int_0^{\frac{1}{10}} dk \frac{k^{d-1}}{k^2 + \frac{m^2 q}{W^2}} \begin{cases} \leq \text{const} & d \geq 3 \\ \propto -\ln \frac{m^2 q}{W^2} & d = 2 \\ \propto \frac{W}{m\sqrt{q}} & d = 1 \end{cases}\end{aligned}$$

where c_d is a constant dependent on the dimension d and we used $\frac{m^2 q}{W^2} \ll 1$. Using then the same arguments as in the previous lemma the proof follows. The same arguments apply in the case of Neumann boundary conditions. □

3.3 Finite volume estimate

If the volume is not too large (depending on W and β) we can prove the conjecture above by a generalization of rigorous saddle analysis we performed in Chapter 1.

Theorem 1 *If Λ satisfies $|\Lambda| = W^3\beta$ and $\beta \gg \ln W$ we have*

$$|\tilde{Z}_\Lambda(\beta, h) - 1| \leq e^{-c\beta} + e^{-h|\Lambda|}$$

for some constant $c > 0$ independent of β, h, W.

Proof. The rest of this section is devoted to the proof of this result.

We partition the configuration space $\mathbb{R}^\Lambda$ in four regions

$$\cup_{j=1}^4 I_j = \mathbb{R}^\Lambda, \quad I_i \cap I_j = \emptyset \forall i \neq j.$$

Let $\chi_j(\phi_\Lambda)$ be the corresponding characteristic functions. Then

$$\tilde{Z}_\Lambda = \sum_{j=1}^4 \tilde{Z}_j, \quad \tilde{Z}_j = \int d\mu_C(\phi) e^{-\sum_x V(\phi_x)} \chi_{I_j}(\phi).$$

Region $\mathbf{I_1}$ [all fields near the global minimum]. We say $\phi_\Lambda \in I_1$ if

$$|\phi_x - \phi_y| \leq \tfrac{1}{4}\beta^{\frac{1}{2}} \ \forall x, y \in \Lambda, \quad \text{and} \quad |\phi_x| \leq \tfrac{1}{2}\beta^{\frac{1}{2}} \ \forall x \in \Lambda.$$

We will prove the main contribution comes from this region. Precisely

$$\left| \tilde{Z}_1 - 1 \right| \leq |\Lambda| e^{-c_1\beta}. \tag{3.3.10}$$

Setting $|\Lambda| = W^3\beta$ we can find β_0 such that $\forall b \geq \beta_0$

$$\left| \tilde{Z}_1 - 1 \right| \leq \beta W^3 e^{-c_1\beta} \leq e^{-\frac{c_1}{2}\beta}.$$

Region $\mathbf{I_2}$ [all fields near the second minimum]. We say $\phi_\Lambda \in I_2$ if

$$|\phi_x - \phi_y| \leq \tfrac{1}{4}\beta^{\frac{1}{2}} \ \forall x, y \in \Lambda, \quad \text{and} \quad |\phi_x + \phi_{m1} - \phi_{m2}| \leq \tfrac{1}{2}\beta^{\frac{1}{2}} \forall x \in \Lambda.$$

Using $f(\phi_{m2}) - f(\phi_{m1}) > h$ we will prove that if $h \gg \frac{1}{W^2}$ we have

$$|\tilde{Z}_2| \leq e^{-c_2 h|\Lambda|}. \tag{3.3.11}$$

for some constant $c_2 > 0$ independent of β and W.

Region $\mathbf{I_3}$ [all fields far from both minimums]. We say $\phi_\Lambda \in I_3$ if

$$|\phi_x - \phi_y| \leq \tfrac{1}{4}\beta^{\frac{1}{2}} \ \forall x, y \in \Lambda, \quad \text{and} \quad \exists x_0 \in \Lambda \text{ with } |\phi_{x_0}| > \tfrac{1}{2}\beta^{\frac{1}{2}} \text{ and} |\phi_{x_0} + \phi_{m1} - \phi_{m2}| > \tfrac{1}{2}\beta^{\frac{1}{2}}.$$

This means $|\phi_y| > \frac{1}{4}\beta^{\frac{1}{2}}$ and $|\phi_y + \phi_{m1} - \phi_{m2}| > \frac{1}{4}\beta^{\frac{1}{2}}$ $\forall y \in \Lambda$. We will prove that

$$|\tilde{Z}_3| \leq e^{-c_3\beta|\Lambda|}. \tag{3.3.12}$$

for some constant $c_3 > 0$ independent of β and W.

Region $\mathbf{I_4}$ [large gradients]. We say $\phi_\Lambda \in I_4$ if

$$\exists x_0, y_0 \in \Lambda \quad \text{such that } |\phi_{x_0} - \phi_{y_0}| > \tfrac{1}{4}\beta^{\frac{1}{2}}.$$

We will prove that, when $W^2 \ll |\Lambda| \ll \beta W^4$ we have

$$|\tilde{Z}_4| \leq e^{-c_4 \beta W^2} \tag{3.3.13}$$

for some constant $c_4 > 0$ independent of β, h and W.

Remark. With these definitions $\{I_j\}_{j=1}^4$ defines indeed a partition of the configuration space. It is easy to see that the sets are all disjoint. We need to check that

$$\cup_{j=1}^3 I_j = I_4^c.$$

To ensure this we should define I_3 as

$$I_3 = \{\phi_\Lambda |\ |\phi_x - \phi_y| \leq \tfrac{1}{4}\beta^{\frac{1}{2}}\ \forall x, y \in \Lambda, \quad \text{and} \quad \phi_\Lambda \notin I_1 \cup I_2\}$$

Proof of the bounds in the regions I_2 and I_3. These correspond to the regions where the field is almost constant but far from the global minimum. Using the bounds on V proved in Lemma 4 below we obtain

$$|\tilde{Z}_j| \leq\ e^{a_j} b_j \int d\mu_C(\phi)\ e^{(1-q_j)\frac{m^2}{2}\sum_x \phi_x^2}\ e^{(v_j,\phi)}$$

where for $j = 2$

$$\begin{aligned} b_2 =&1, \quad a_2 = -\left[2h + \frac{m^2}{2}(\phi_{m1} - \phi_{m2})^2\right]|\Lambda| \\ q_2 =&1, \qquad (v_2)_x = -m^2(\phi_{m1} - \phi_{m2})\ \forall x, \end{aligned}$$

and for $j = 3$

$$b_3 = 1, \quad a_3 = -[q\beta]\,|\Lambda|, \qquad q_3 = q, \qquad (v_3)_x = 0\ \forall x,$$

for some $0 < q \ll 1$ independent of β, h, W.

Replacing the values for $j = 2$ we get

$$\begin{aligned} |\tilde{Z}_2| \leq&\ e^{a_2}\int d\mu_C(\phi)\ e^{(v_2,\phi)} = e^{a_2} e^{\frac{1}{2}(v_2, C v_2)} \\ &= e^{a_2} e^{\frac{m^4}{2}(\phi_{m1}-\phi_{m2})^2 (1, Cv1)} = e^{a_2} e^{\frac{m^2}{2}(\phi_{m1}-\phi_{m2})^2|\Lambda|} = e^{-2h|\Lambda|} \end{aligned}$$

where we used $C1 = \frac{1}{m^2}1$ and $(1,1) = |\Lambda|$ since 1 is the eigenvector for the Laplacian with eigenvalue 0.

When $0 < q_j < 1$ we have

$$d\mu_C(\phi)\ e^{q_j \frac{m^2}{2}\sum_x \phi_x^2} = \tfrac{d\phi}{\mathcal{N}_C} e^{-\frac{1}{2}(\phi, C^{-1}\phi)} e^{\frac{1}{2}(1-q_j)m^2(\phi,\phi)} = \frac{\mathcal{N}_{C_{q_j}}}{\mathcal{N}_C} d\mu_{C_{q_j}}(\phi)$$

where

$$C_{q_j}^{-1} = C^{-1} - q_j m^2 = -W^2\Delta + m^2 q_j > 0$$

since $0 < q_j \leq 1$. Then using Lemma 3

$$|\tilde{Z}_j| \leq \; e^{a_j} b_j \frac{\mathcal{N}_{C_{q_j}}}{\mathcal{N}_C} \int d\mu_{C_{q_j}}(\phi)\; e^{(v_j,\phi)} = e^{a_j} b_j \frac{\mathcal{N}_{C_{q_j}}}{\mathcal{N}_C} e^{\frac{1}{2}(v_j, C_{q_j} v_j)}$$
$$\leq e^{a_j} b_j \frac{K_2^{\frac{1}{2}}}{q_j^{\frac{1}{2}}} e^{K_1 \frac{(1-q_j)|\Lambda|}{2W^2}} e^{\frac{1}{2}(v_j, C_{q_j} v_j)}$$

Replacing the values for $j = 3$ we get

$$|\tilde{Z}_3| \leq e^{-q\beta|\Lambda|} \sqrt{\frac{K_2}{q}} e^{K_1 \frac{(1-q)|\Lambda|}{2W^2}} \leq e^{-c_3\beta|\Lambda|}$$

for some constant $c_3 > 0$.

Proof of the bound in the region I_4. This corresponds to the region where some large gradient appears. Using the bounds on V proved in Lemma 4 below and Lemma 3 we obtain

$$|\tilde{Z}_4| \leq \; e^{\eta|\Lambda|} \int d\mu_C(\phi)\; e^{(1-q_4)\frac{m^2}{2}(\phi,\phi)} \chi_4(\phi) \leq \; e^{\eta|\Lambda|} \frac{\mathcal{N}_{C_{q_4}}}{\mathcal{N}_C} \int d\mu_{C_{q_4}}(\phi) \chi_4(\phi)$$

where $q_4 = c\eta/\beta$. The function $\chi_4(\phi)$ is bounded by

$$\chi_4(\phi) \leq \sum_{x_0,y_0} \chi_{[|\phi_{x_0} - \phi_{y_0}| > \delta]}(\phi) \leq \sum_{x_0,y_0} \frac{\cosh(u(\phi_{x_0} - \phi_{y_0}))}{\cosh(u\delta)}$$

where $\delta = \frac{1}{4}\beta^{\frac{1}{2}}$ and u is any positive constant. We will optimize the choice of u later. Inserting this bound in the integral we obtain

$$\tilde{Z}_4 \leq e^{\eta|\Lambda|} \frac{e^{-\delta u}}{1+e^{-2u\delta}} \frac{\mathcal{N}_{C_{q_4}}}{\mathcal{N}_C} \sum_{x_0,y_0 \in \Lambda} \sum_{\sigma = \pm 1} \int d\mu_{C_{q_4}} e^{\sigma u(\phi_{x_0} - \phi_{y_0})}$$
$$= e^{a_4} b_4 \frac{\mathcal{N}_{C_{q_4}}}{\mathcal{N}_C} \sum_{x_0,y_0 \in \Lambda} \int d\mu_{C_{q_h}} e^{(v_{x_0,y_0},\phi)}$$
$$\leq e^{a_4} b_4 \frac{\mathcal{N}_{C_{q_4}}}{\mathcal{N}_C} |\Lambda|^2 \sup_{x_0,y_0} e^{\frac{1}{2}(v_{x_0,y_0}, C_{q_4} v_{x_0,y_0})}$$

where we used the symmetry of the Gaussian measure under $\phi \to -\phi$ and

$$a_4 = \eta|\Lambda| - \delta u, \qquad b_4 = \frac{2}{1+e^{-2u\delta}}, \qquad v_{x_0,y_0}(x) = u(\delta_{x,x_0} - \delta_{x,y_0})$$

Now by Lemma 2 we have

$$(v_{x_0,y_0}, Cv_{x_0,y_0}) = u^2[C_{x_0x_0} + C_{y_0y_0} - C_{x_0y_0} - C_{y_0x_0}] \leq u^2 2\frac{K_1}{W^2}$$

independently of the mass $m^2 > 0$ and the pair x_0, y_0. Then this holds also for C_{q_4} (where the mass is replaced by $m^2 q_4$). Inserting these estimates and using Lemma 3

$$\begin{aligned}\tilde{Z}_4 \leq\ & e^{a_4} b_4 \frac{\mathcal{N}_{C_{q_4}}}{\mathcal{N}_C} |\Lambda|^2 e^{2\frac{K_1}{W^2}} \leq\ e^{a_4} b_4 |\Lambda|^2 \frac{K_2^{\frac{1}{2}}}{q_4^{\frac{1}{2}}} e^{K_1 \frac{(1-q_4)|\Lambda|}{2W^2}} e^{u^2 \frac{K_1}{W^2}} \\ \leq\ & \frac{|\Lambda|^2 \sqrt{\beta K_2}}{\sqrt{c\eta}}\ e^{|\Lambda|\left(\frac{K_1}{2W^2}+\eta\right)}\ e^{u^2 \frac{K_1}{W^2} - u\delta} \\ \leq\ & \tilde{K} |\Lambda|^2 \sqrt{\beta W} e^{|\Lambda| \frac{K_1}{W^2}} e^{-\frac{1}{2} u\delta} = \tilde{K} |\Lambda|^2 \sqrt{\beta W} e^{|\Lambda| \frac{K_1}{W^2}} e^{-\frac{\beta W^2}{4K_1}} \\ \leq\ & e^{-c_4 \beta W^2}\end{aligned}$$

where we set

$$u = \frac{\delta W^2}{2K_1}, \qquad \eta = \frac{K_1}{2W}, \qquad \tilde{K} = \sqrt{\frac{2K_2}{cK_1}}.$$

and we used $|\Lambda| \ll \beta W^4$.

Proof of the bound in the region I_1. We can now prove the estimate in the region where all fields are near the global minimum. Inserting (3.3.14) we have

$$\begin{aligned}|\tilde{Z}_1 - 1| &\leq \int d\mu_C(\phi) \left| e^{V(\phi)} - 1 \right| \chi_{I_1}(\phi) + \int d\mu_C(\phi) [1 - \chi_{I_1}(\phi)] \\ &\leq |\Lambda| e^{-\beta} + \int d\mu_C(\phi) \chi_{I_1^c}(\phi)\end{aligned}$$

Now

$$I_{I_1}^c \subset \{\phi|\ \exists x \in \Lambda,\ |\phi_x| > \frac{1}{8} \beta^{\frac{1}{2}}\}$$

hence

$$\int d\mu_C(\phi) \chi_{I_1^c}(\phi) \leq \sum_{x \in \Lambda} \int d\mu_C(\phi) \chi_{\{|\phi_x| > \frac{1}{8}\beta^{\frac{1}{2}}\}}(\phi)$$

Using the same arguments as in the bound on $\tilde{Z}_4$ we have

$$\begin{aligned}\int d\mu_C(\phi) \chi_{\{|\phi_x| > \delta\}}(\phi) &\leq e^{-\delta u} \int d\mu_C(\phi) \sum_{\sigma_x = \pm 1} e^{\sigma_x u \phi_x} \\ &\leq 2 e^{-\delta u} e^{\frac{1}{2} u^2 C_{xx}} \leq 2 e^{-\delta u} e^{\frac{1}{2} u^2 [\frac{1}{m^2 |\Lambda|} + \frac{K_1}{W^2}]} \\ &\leq e^{-\delta u} e^{u^2 \frac{K_1}{W^2}} \leq e^{-\tilde{c}_1 \beta W^2},\end{aligned}$$

where we used $|\Lambda| \gg W^2$ and in the last line we chose

$$u = \frac{\delta W^2}{2K_1}.$$

Finally

$$\int d\mu_C(\phi) \chi_{I_1^c}(\phi) \leq |\Lambda| e^{-\tilde{c}_1 \beta W^2}.$$

Therefore

$$|\tilde{Z}_1 - 1| \leq |\Lambda| e^{-\beta} + |\Lambda| e^{-\tilde{c}_1 \beta W^2} \leq |\Lambda| e^{-c_1 \beta}.$$

Putting all the bounds together we get

$$|\tilde{Z}_\Lambda - 1| \leq |\Lambda| e^{-c_1\beta} + e^{-c_2 h|\Lambda|} + e^{-c_3\beta|\Lambda|} + e^{-c_4\beta W^2} \leq \beta W^3 e^{-c_1\beta} + e^{-c_2 h|\Lambda|} \leq e^{-\frac{c_1}{2}\beta} + e^{-c_2 h\beta W^3}$$

using $\beta \gg \ln W$. This completes the proof of the theorem. □

It remains now only to prove the estimates on V we used above. This is done in the following lemma.

Lemma 4 *The interaction $V(\phi)$ satisfies the following relations*

1. *When $|\phi_x| \leq \frac{3}{4}\beta^{\frac{1}{2}}$ we have*

$$V(\phi_x) = o(e^{-\beta}) \tag{3.3.14}$$

2. *When $|\phi_x + \phi_{m1} - \phi_{m2}| \leq \frac{1}{4}\beta^{\frac{1}{2}}$ we have*

$$V(\phi_x) \geq 2h + o(e^{-\beta}) + m^2 \tfrac{1}{2}\left[(\phi_{m1} - \phi_{m2})^2 + 2\phi_x(\phi_{m1} - \phi_{m2})\right] \tag{3.3.15}$$

3. *When $|\phi_x| > \frac{1}{4}\beta^{\frac{1}{2}}$ and $|\phi_x + \phi_{m1} - \phi_{m2}| > \frac{1}{4}\beta^{\frac{1}{2}}$ we have*

$$V(\phi_x) \geq q\beta - (1-q)m^2 \tfrac{1}{2}\phi_x^2 \tag{3.3.16}$$

for some $0 < q \ll 1$ independent of β, h, W.

4. *For any value of $\phi_x \in \mathbb{R}$ we have the apriori bound*

$$V(\phi_x) \geq -(1 - q_{\beta,\eta}) m^2 \tfrac{1}{2}\phi_x^2 - \eta \tag{3.3.17}$$

where $0 < \eta < \beta$ is any positive constant (to be fixed later) and $q_{\beta,\eta} = \frac{c\eta}{\beta}$ for some $0 < c \ll 1$ independent of W, β and h).

Proof. To prove (3.3.14) it is enough to use third order Taylor formula with integral remainder

$$V(\phi_x) = \frac{\phi_x^3}{2}\int_0^1 dt(1-t)^2 f'''(\phi_{m1} + t\phi_x) = (\phi_x \beta^{\frac{1}{2}})^3 \int_0^1 dt(1-t)^2 \frac{\tanh(\phi_{m1}+t\phi_x)}{[\cosh(\phi_{m1}+t\phi_x)]^2}.$$

Using $|\tanh(u)| \leq 1$ $\forall u$ and $|\phi_{m1} + t\phi_x| \geq \frac{1}{4}\beta^{\frac{1}{2}}$ for all $t \in [0,1]$ as long as $|\phi_x| \leq \frac{3}{4}\beta^{\frac{1}{2}}$ we have

$$|V(\phi_x)| \leq \beta^3 e^{-2\beta} c = o(e^{-\beta}).$$

To prove (3.3.15) we expand around the secon minimum ϕ_{m2}

$$V(\phi_x) = f(\phi_x + \phi_{m1}) - f(\phi_{m1}) - \tfrac{m^2}{2}\phi_x^2 \tag{3.3.18}$$
$$= [f(\phi_{m2}) - f(\phi_{m1})] + [f(\phi_x + \phi_{m1}) - f(\phi_{m2}) - \tfrac{f''(\phi_{m2})}{2}(\phi_x + \phi_{m1} - \phi_{m2})^2]$$
$$+ [\tfrac{1}{2}(\phi_x + \phi_{m1} - \phi_{m2})^2(f''(\phi_{m2}) - m^2)] + [\tfrac{m^2}{2}\left((\phi_x + \phi_{m1} - \phi_{m2})^2 - \phi_x^2\right)]$$
$$\geq h + o(e^{-\beta}) + \tfrac{m^2}{2}((\phi_{m1} - \phi_{m2})^2 + 2\phi_x(\phi_{m1} - \phi_{m2})) \tag{3.3.19}$$

where we also used $m^2 = f''(\phi_{m1}) = f''(\phi_{m2}) + o(e^{-\beta})$. To prove (3.3.16) we show that the function

$$g(\phi) = f(\phi + \phi_{m1}) - f(\phi_{m1}) - q\tfrac{m^2}{2}\phi^2.$$

is positive definite in the region $|\phi_x| > \frac{1}{4}\beta^{\frac{1}{2}}$ and $|\phi_x + \phi_{m1} - \phi_{m2}| > \frac{1}{4}\beta^{\frac{1}{2}}$, for all $0 < q \leq c$ with c small enough (independent of β). The result then follows taking $q = c/2$. Finally a similar argument proves (3.3.17). In this case we choose $q_{\beta,\eta} = c\eta/\beta$ with $0 < c < 1$ such that

$$\eta - q_{\beta,\eta}\frac{m^2}{2}(\phi_{m1} + \phi_{m2})^2 = 0.$$

Then using the convexity properties of the function f we can show that

$$f(\phi + \phi_{m1}) - f(\phi_{m1}) - q_{\beta,\eta}\frac{m^2}{2}(\phi + \phi_{m1})^2 + \eta \geq 0.$$

This bound holds also when $h = 0$. □

3.4 Infinite volume

All estimates in the previous section work as long as the volume is bounded by $|\Lambda| \leq W^4\beta$. Since the covariance C decays exponentially fast $C_{xy} \leq K_{W,d}e^{-|x|/W}$ we expect the integral to factorize over regions of finite size. Guided by the finite volume estimates let us partition the volume $\Lambda = [-L, \ldots L]^d$ in cubes of side $l = W\beta^{1/d}$. We will denote each cube by Δ and the corresponding center by x_Δ. Note that x_Δ may not be on $\mathbb{Z}^d$. This construction can be extended to the whole lattice $\mathbb{Z}^d$. Without loss of generality we can assume $L/l = n \in \mathbb{N}$. Let us call $\tilde{\mathbb{Z}}^d$ the lattice (of step l) made of the centers of the small cubes and $\tilde{\Lambda}$ the restriction to the set of cubes inside Λ. We say that a cube Δ belongs to Λ, $\Delta \in \Lambda$, if the corresponding center belongs to $\tilde{\Lambda}$. With these definitions

$$\Lambda = \cup_{\Delta | x_\Delta \in \tilde{\Lambda}}\Delta = \cup_{\Delta \in \Lambda}\Delta,$$

and this is a disjoint union. To avoid confusion in the following we will denote the Laplacian operator by $\hat{\Delta}$.

We introduce the factorized covariance

$$C_d = \sum_{\Delta \in \Lambda} \mathbf{1}_\Delta(x) C_{xy} \mathbf{1}_\Delta(y) = \sum_{\Delta \in \Lambda} (C_\Delta)_{xy}.$$

By construction this is a block diagonal matrix. If we replace C by C_d inside $\tilde{Z}_\Lambda$ we obtain

$$\tilde{Z}_\Lambda(C_d) = \prod_{\Delta\in\Lambda} Z_\Delta, \qquad Z_\Delta = \int d\mu_{C_\Delta}(\phi_\Delta) e^{-\sum_{x\in\Delta} V(\phi_x)}.$$

We can then write the partition function as

$$\tilde{Z}_\Lambda = \frac{\tilde{Z}_\Lambda}{\prod_{\Delta\in\Lambda} Z_\Delta} \prod_{\Delta\in\Lambda} Z_\Delta$$

If our intuition is correct we will see that the ratio

$$\frac{\tilde{Z}_\Lambda}{\prod_{\Delta\in\Lambda} Z_\Delta} \simeq 1.$$

We will prove this in the next section via cluster expansions.

Finite volume estimate on Z_Δ. For a single cube we want to apply the finite volume estimates we derived in the previous section. These estimates used heavily the properties of the discrete Laplacian with periodic or Neuman conditions on the boundary of Λ. Unfortunately, the covariance C_Δ has no longer the structure $(-W^2\hat{\Delta} + m^2)^{-1}$. A similar problem will arise when estimating the integrals obtained from the cluster expansion. The solution is explained in the next subsection.

3.4.1 Estimates on the factorized covariance

Let $\mathcal{P}[\tilde{\Lambda}]$ the set of partitions of $\tilde{\Lambda}$ into (not necessarily connected) sets. Let $\Pi \in \mathcal{P}[\tilde{\Lambda}]$ be such a partition. Each cluster $\tilde{X} \in \Pi$ induces a set of cubes

$$X = \{\Delta | x_\Delta \in \tilde{X}\} \subset \Lambda. \tag{3.4.20}$$

With this definition $\{X\}_{\tilde{X}\in\Pi}$ is a partition of Λ. We define the covariance **factorized over the partition** Π

$$(C_\Pi)_{xy} = \sum_{\tilde{X}\in\Pi} \mathbf{1}_X(x) C_{xy} \mathbf{1}_X(y).$$

In the special case $\Pi = \Pi_0 = \{\{j\}_{j\in\tilde{\Lambda}}\}$ each component X is reduced to a single cube. When $\Pi = \Pi_1 = \tilde{\Lambda}$ we have a single component $X = \Lambda$. Then we recover the covariance C_d and C

$$C_d = C_{\Pi_0}, \qquad C = C_{\Pi_1}$$

Note that C_Π is a block diagonal matrix on $\Lambda \times \Lambda$ and it has no longer the structure $(-W^2\hat{\Delta} + m^2)^{-1}$. Now let $\partial\Pi$ be the boundary of **each** component $\tilde{X} \in \Pi$. More precisely

$$\partial\Pi = \{(x,y) \in \Lambda\times\Lambda |\ |x-y| = 1, \text{ and } \exists \tilde{X} \neq \tilde{X}' \in \Pi, \text{with } x \in X, y \in X'\},$$

where X is defined in (3.4.20). Then $\partial\Pi$ is the boundary of the sets in Λ associated to each $\tilde{X} \in \Pi$. With these definitions we can introduce $-\hat{\Delta}_N^{\partial\Pi}$ the discrete Laplacian on Λ with Neuman boundary conditions on the boundary of Π

$$
\begin{aligned}
(f, -\hat{\Delta}_N^{\partial\Pi} f) &= \sum_{x,y\in\Lambda\setminus\partial\Pi} (f_x - f_y)^2 = \sum_{\tilde{X}\in\Pi} \left[\sum_{xy\in X} (f_x - f_y)^2 \right] \\
&= \sum_{\tilde{X}\in\Pi} (f_X, -\hat{\Delta}_N^{\partial X} f_X),
\end{aligned} \tag{3.4.21}
$$

where f_X is the function restricted to X (the set associated to $\tilde{X}$) and $-\hat{\Delta}_N^{\partial X}$ is the discrete Laplacian on X with Neuman conditions on the boundary of X. Let

$$
C_N^{\partial\Pi} = (-W^2 \hat{\Delta}_N^{\partial\Pi} + m^2)^{-1} = \sum_{\tilde{X}\in\Pi} C_X^N \tag{3.4.22}
$$

where $C_X^N = (-W^2 \hat{\Delta}_N^{\partial X} + m^2 1_X)^{-1}$. Note that

$$
C_N^{\partial\Pi} \leq C_N^{\partial\Pi_0} = \sum_{\Delta\in\Lambda} C_\Delta^N \quad \forall \Pi \in \mathcal{P}[\tilde{\Lambda}].
$$

The Laplacian with periodic boundary conditions on Λ satisfies

$$
-\hat{\Delta} \geq -\hat{\Delta}_N^{\partial\Pi} \quad \Rightarrow \quad C \leq C_N^{\partial\Pi} \leq \sum_{\Delta\in\Lambda} C_\Delta^N
$$

for any partition Π. With all these definitions we can prove the following result.

Lemma 5 *The following quadratic form bound holds*

$$
0 < C_\Pi \leq C_N^{\partial\Pi}.
$$

Proof Since both C_Π and $C_N^{\partial\Pi}$ are block diagonal matrices on the partition of Λ induced by Π it is enough to prove the bound for one set X such that $\tilde{X} \in \Pi$. Let $f_X : X \to \mathbb{R}$ a function on X and let $f : \Lambda \to \mathbb{R}$ its extension to Λ defined by

$$
f(x) = \begin{cases} f_X(x) & x \in X \\ 0 & x \notin X \end{cases}
$$

Then

$$
0 < (f, Cf) = (f_X, C_\Pi f_X) = (f, Cf) \leq (f, C_N^{\partial\Pi} f) = (f_X, C_N^{\partial\Pi} f_X).
$$

This completes the proof. □

3.4.2 Finite volume estimate for Z_Δ.

Lemma 6 *If $d = 3$, $|\Delta| = \beta W^3$ and $h \geq W^{\epsilon-1}$ for some fixed $0 < \epsilon < 1$, the partition function restricted to a single cube satisfies*

$$
|Z_\Delta - 1| \leq e^{-\beta^{\frac{1}{2}}}.
$$

Proof. In order to apply the same arguments as in the previous section we need to check that

$$C_\Delta^{-1} - (1-q)m^2 1_\Delta > 0 \qquad \forall 0 < q < 1$$

and that Lemma 2 and 3 can be generalized from C to C_Δ. Using the results from the last subsection we have

$$\begin{aligned} C_{\Delta,q}^{-1} &= C_\Delta^{-1} - (1-q)m^2 1_\Delta \geq (C_\Delta^N)^{-1} - (1-q)m^2 1_\Delta = (C_{\Delta,q}^N)^{-1} \\ &= -W^2 \hat{\Delta}_N^{\partial\Delta} + m^2 q 1_\Delta > 0. \end{aligned}$$

Therefore

$$\begin{aligned} &(v, C_{\Delta,q}, v) \leq (v, C_{\Delta,q}^N, v), \text{ and} \\ &\frac{\mathcal{N}^2_{C_{\Delta,q}}}{\mathcal{N}^2_{C_\Delta}} = \det\big(1 + (1-q)m^2 C_{\Delta,q}\big) \leq \det\big(1 + (1-q)m^2 C_{\Delta,q}^N\big) \end{aligned}$$

hence we can generalize the proofs of Lemma 2 and 3 to C_Δ. □

3.4.3 Cluster expansion around a gaussian meaure

Preliminary definitions

We need to generalize a bit the definitions introduced in Section 2.7.1. For any partition $P \in \mathcal{P}[\Lambda]$ We introduce the function

$$\begin{aligned} \chi_P : \quad \Lambda \times \Lambda \quad &\to \{0,1\} \\ (x,y) \quad &\to \begin{cases} 1 & \exists X \in P,\ x,y \in X \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

With this definition $\chi_P(x,x) = 1$ for any partition P. Note that in Section 2.7.1. we considered only pairs $x \neq y$. Now for each $\Pi \in \mathcal{P}[\tilde{\Lambda}]$ let P_Π the corresponding partition of Λ. Let

$$P_d = \{\Delta\}_{\Delta \in \Lambda}$$

the partition corresponding to $\Pi_0 = \{\{j\}_{j \in \Lambda}\}$. Then

$$\chi_{P_d}(x,y) = \begin{cases} 1 & \exists \Delta,\ x,y \in \Delta \\ 0 & \text{otherwise.} \end{cases}$$

On the other extreme, when $\Pi = \Pi_1 = \{\tilde{\Lambda}\}$ is reduced to only one set then $P = \{\Lambda\}$. The corresponding function is

$$\chi_\Lambda(x,y) = 1 \quad \forall x,y \in \Lambda.$$

We will consider $\mathcal{F}[\tilde{\Lambda}]$ the set of ordered forests on $\tilde{\Lambda}$. Each edge in the forest corresponds now to a pair of cubes $\Delta \neq \Delta'$ (instead of a pair of points in Λ). To each forest with n lines $F_n = (l_1, \ldots, l_n)$ we associate the vector

$$s_{F_n} = (s_1, \ldots, s_n), \qquad 1 \geq s_1 \geq s_2 \geq \cdots \geq s_n \geq 1,$$

and the set of forests

$$F_j = (l_1, \dots, l_j), \quad j = 1, \dots n$$

obtained keeping only the first j edges in F_n. For each forest F_n we denote by Π_{F_n} the partition of $\tilde{\Lambda}$ generated by the connected components of the forest. Let P_{F_n} the corresponding partition of Λ. With these definitions we can introduce the function

$$\begin{array}{lll} u(s_{F_n}): & \Lambda \times \Lambda & \to [0,1] \\ & (x,y) & \to u(s_{F_n})(x,y) \end{array}$$

$$u(s_{F_n}) = s_n(\chi_{P_{F_n}} - \chi_{P_{F_{n-1}}}) + s_{n-1}(\chi_{P_{F_{n-1}}} - \chi_{P_{F_{n-2}}}) + \cdots + s_1(\chi_{P_{F_1}} - \chi_{P_d}) + \chi_{P_d}$$

Note that by construction $u(s_{F_n})(x,y) = 1 \; \forall x, y \in \Delta$, for any cube Δ. Moreover we have

$$u(s_F) = \begin{cases} 1 & \exists \Delta, \; x, y \in \Delta \\ 0 & \exists \Delta \neq \Delta', \;\; x \in \Delta, \; y \in \Delta', \; \Delta, \Delta' \text{ not connected by } F \\ \inf_{l \in \gamma^F_{\Delta\Delta'}} s_l & \exists \Delta \neq \Delta', \;\; x \in \Delta, \; y \in \Delta', \; \Delta, \Delta' \text{ connected by } F \end{cases}$$

This last definition can be generalized to unordered forests.

Theorem 2 (BKAR) *In the case of a real gaussian measure with covariance C the Brydges-Kennedy-Abdesselam-Rivasseau Forest formula becomes*

$$\int d\mu_C(\phi) e^{-\sum_{x\in\Lambda} V(\phi_x)} = \sum_{F \in \mathcal{F}[\tilde{\Lambda}]} \sum_{(x_e, y_e)_{e\in F}} [\prod_{e\in F} C_{x_e, y_e}] \int_{[0,1]^{|F|}} [\prod_{e \in F} ds_e] \tag{3.4.23}$$

$$\cdot \int d\mu_{C(u(s_F))}(\phi) \left[\prod_{e\in F} \frac{\delta}{\delta \phi_{x_e}} \frac{\delta}{\delta \phi_{y_e}} \right] e^{-\sum_{x\in\Lambda} V(\phi_x)}$$

where for each edge $e = (\Delta, \Delta')$ in the forest (x_e, y_e) is any pair of points in $\Lambda \times \Lambda$ with $x_e \in \Delta$, $y_e \in \Delta'$.

Proof The proof works as in Theorem 5 (Chapter 2). For each $n \geq 1$ we define the function

$$\tilde{u}(s_1, .., s_{n+1}) = s_{n+1}(\chi_\Lambda - \chi_{P_{F_n}}) + u(s_{F_n}), \qquad \tilde{u}(s_1) = s_1(\chi_\Lambda - \chi_{P_d}) + \chi_{P_d}.$$

Note that

$$\begin{aligned} C(\tilde{u}(s_1, .., s_{n+1}))(x,y) &= s_{n+1} \; [\chi_\Lambda(x,y) C_{x,y}] + (1 - s_1) \; [\chi_{P_d}(x,y) C_{x,y}] \\ &\quad + \sum_{j=1}^{n} (s_j - s_{j+1}) \; [\chi_{P_{F_j}}(x,y) C_{x,y}] \\ &= s_{n+1} \; C_{xy} + (1 - s_1) \; [C_d]_{xy} + \sum_{j=1}^{n} (s_j - s_{j+1}) \; [C_{\Pi_{F_j}}]_{xy} \end{aligned}$$

Since $C_\Pi > 0$ for any partition Π, $(s_j - s_{j+1}) \geq 0$, $(1-s_1) \geq 0$ and

$$s_{n+1} + (1-s_1) + \sum_{j=1}^{n} (s_j - s_{j+1}) = 1,$$

$C(\tilde{u}(s_1,..,s_{n+1}))$ is a convex combination of positive matrices, hence is positive

$$C(\tilde{u}(s_1,\ldots,s_{n+1})) > 0 \qquad \forall (s_1,\ldots,s_{n+1}).$$

Similarly for

$$\begin{aligned} C(u(s_1,..,s_n))(x,y) &= s_n\ [\chi_{F_n}(x,y)C_{x,y}] + (1-s_1)\ [\chi_{P_d}(x,y)C_{x,y}] \\ &\quad + \sum_{j=1}^{n-1}(s_j - s_{j+1})\ [\chi_{P_{F_j}}(x,y)C_{x,y}] \\ &= s_n\ C_{\Pi_{F_n}} + (1-s_1)\ [C_d]_{xy} + \sum_{j=1}^{n-1}(s_j - s_{j+1})\ [C_{\Pi_{F_j}}]_{xy}. \end{aligned} \tag{3.4.24}$$

Hence

$$C(u(s_1,\ldots,s_n)) > 0 \qquad \forall (s_1,\ldots,s_n).$$

Moreover

$$\begin{aligned} \partial_{s_{n+1}} \int d\mu_{C(\tilde{u})}(\phi) e^{-V(\phi)} &= \partial_{s_{n+1}} \left[\det{}^{\frac{1}{2}} \tfrac{C^{-1}}{2\pi} \int d\phi e^{-\frac{1}{2}(\phi, C^{-1}\phi)} e^{-V(\phi)} \right] \\ &= \frac{1}{2} \int d\mu_{C(\tilde{u})}(\phi) \left[((C^{-1}\phi),\ (\partial_{s_{n+1}}C)\ (C^{-1}\phi)) - \mathrm{tr}[C^{-1}\partial_{s_{n+1}}C] \right] e^{-V(\phi)} \\ &= \sum_{xy} [\partial_{s_{n+1}}C]_{xy} \int d\mu_{C(\tilde{u})}(\phi) \left[\tfrac{1}{2} \tfrac{\delta}{\delta\phi(x)} \tfrac{\delta}{\delta\phi(y)} \right] e^{-V(\phi)}. \end{aligned}$$

where we used

$$\begin{aligned} \partial_s C^{-1}_{xy} &= -\left[C^{-1}(\partial_s C) C^{-1} \right]_{xy}, \\ \partial_s (\det C)^{-1} &= \partial_s e^{-tr \ln C} = -(\det C)^{-1} \mathrm{tr} \left[C^{-1}(\partial_s C) \right], \\ (C^{-1}\phi)(x) e^{-\frac{1}{2}(\phi, C^{-1}\phi)} &= -\frac{\delta}{\delta\phi(x)} e^{-\frac{1}{2}(\phi, C^{-1}\phi)}. \end{aligned}$$

Finally

$$(\partial_{s_{n+1}}C)_{xy} = \sum_{(\Delta \neq \Delta') \notin P_{F_n}} \mathbf{1}_\Delta(x) C_{xy} \mathbf{1}_{\Delta'}(y).$$

The proof then works as for Theorem 5 (Chapter 2). $\square$

Since the integral inside (3.4.23) factors over the connected components of the forest we can write

$$\frac{\tilde{Z}_\Lambda}{\prod_\Delta Z_\Delta} = \sum_{\Pi \in \mathcal{P}[\tilde{\Lambda}]} \left[\prod_{\tilde{X} \in \Pi} \frac{A(\tilde{X})}{\prod_{\Delta \in X} Z_\Delta} \right] = \sum_{\Pi \in \mathcal{P}[\tilde{\Lambda}]} \left[\prod_{\tilde{X} \in \Pi} \mathcal{A}(\tilde{X}) \right]$$

where

$$A(\tilde{X}) = \sum_{T\in\mathcal{T}[\tilde{X}]} \sum_{\{(x_e,y_e)\in\Delta_e\times\Delta'_e\}_{e\in T}} [\prod_{e\in T} C_{x_e,y_e}] \int_{[0,1]^{|T|}} [\prod_{e\in T} ds_e] \tag{3.4.25}$$
$$\cdot \int d\mu_{C(u(s_T))}(\phi_X) \left[\prod_{e\in T} \frac{\delta}{\delta\phi_{x_e}} \frac{\delta}{\delta\phi_{y_e}}\right] e^{-\sum_{x\in X} V(\phi_x)}$$

and X is the connected component of P_Π corresponding to $\tilde{X}$. With these definitions

$$\mathcal{A}(\tilde{X}) = 1 \qquad \text{when } |\tilde{X}| = 1 \;\equiv\; \exists\Delta \text{ s.t } X = \Delta.$$

Using the same arguments as in Chapter 2 (Lemma 4) we can write

$$\frac{1}{|\Lambda|} \ln \frac{\tilde{Z}_\Lambda}{\prod_\Delta Z_\Delta} = \frac{1}{\beta W^3 |\tilde{\Lambda}|} \ln \frac{\tilde{Z}_\Lambda}{\prod_\Delta Z_\Delta} \tag{3.4.26}$$
$$= \frac{1}{\beta W^3|\tilde{\Lambda}|} \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{\tilde{X}_1,\ldots\tilde{X}_n\subset\tilde{\Lambda} \\ |\tilde{X}_i|\geq 2}} \left[\prod_{i=1}^n \mathcal{A}(\tilde{X}_i)\right] V_c(\tilde{X}_1,\ldots,\tilde{X}_n)$$

where we used $|\Lambda|/|\tilde{\Lambda}| = |\Delta| = \beta W^3$. Note that by construction X_i is a set of cubes. By Theorem 4 (Chapter 2) if we have

$$\sup_{\tilde{x}\in\tilde{\mathbb{Z}}^3} \sum_{\substack{2\leq|\tilde{X}|<\infty \\ \tilde{x}\in\tilde{X}}} |\mathcal{A}(\tilde{X})| e^{|\tilde{X}|} < 1 \tag{3.4.27}$$

then the series (3.4.26) above converges in absolute value uniformly in the volume $\tilde{\Lambda}$ (hence also in Λ). Note that requiring that the point $\tilde{x} \in \tilde{X}$ is equivalent to require that the corresponding cube Δ, such that $x_\Delta = \tilde{x}$, satisfies $\Delta \in X$.

3.5 Convergence of the log expansion

By the arguments in the previous section we only need to prove that (3.4.27) holds. We will prove this is true assuming $h \geq \frac{1}{|\Delta|^{1-\epsilon}}$ for some fixed $0 < \epsilon < 1$.

3.5.1 Reorganization of the sum

Note that, using Lemma 6 above we have

$$|\mathcal{A}(\tilde{X})| = \frac{|A(\tilde{X})|}{\prod_{\Delta\in X} Z_\Delta} \leq |A(\tilde{X})| e^{|\tilde{X}| e^{-\beta^{\frac{1}{2}}}} \tag{3.5.28}$$

where we used

$$\frac{1}{1+(Z_\Delta - 1)} \leq \frac{1}{1-e^{-\beta^{\frac{1}{2}}}},$$

and $(1+x) \leq e^x \ \forall x \in \mathbb{R}$. The problem is then to get a reasonable bound on $|A(\tilde{X})|$. Since we will try to apply the finite volume estimates inside each cube we partition each cube configuration ϕ_Δ according to the regions I_j introduced in Theorem 1 (Section 3.3).

$$1 = \prod_\Delta [\sum_{j=1}^{4} \chi_{I_j}(\phi_\Delta)] = \sum_{\{j_\Delta\}} \prod_\Delta \chi_{I_{j_\Delta}}(\phi_\Delta).$$

The amplitude $A(\tilde{X})$ is then written as

$$A(\tilde{X}) = \sum_{T \in \mathcal{T}[\tilde{X}]} \sum_{\{(x_e,y_e) \in \Delta_e \times \Delta'_e\}_{e \in T}} [\prod_{e \in T} C_{x_e,y_e}] \int_{[0,1]^{|T|}} [\prod_{e \in T} ds_e] \tag{3.5.29}$$
$$\cdot \sum_{\{j_\Delta\}} \int d\mu_{C(u(s_T))}(\phi_X) \left[\prod_\Delta \chi_{I_{j_\Delta}}(\phi_\Delta) \right] \left[\prod_{e \in T} \frac{\delta}{\delta \phi_{x_e}} \frac{\delta}{\delta \phi_{y_e}} \right] e^{-\sum_{x \in X} V(\phi_x)}$$

We need to reorganize the expression for $A(\tilde{X})$. For a fixed tree on $\tilde{\Lambda}$ a tree line e corresponds to a pair of cubes Δ_e, Δ'_e. The corresponding covariance acts on Λ so we have to choose in addition two points $x_e \in \Delta_e$ and $y_e \in \Delta'_e$. The same point $x \in \Delta$ may be used as endpoint for several tree lines. To make this precise we

1. fix $d_\Delta \geq 1$ the number of tree lines attached to the cube Δ. This number is determined by the tree;
2. inside each cube Δ we fix m_Δ the number of points in the lattice inside Δ attached to some tree line. By construction $1 \leq m_\Delta \leq d_\Delta$. When $m_\Delta = 1$ all tree lines connected to Δ hook to the same point, when $m_\Delta = d_\Delta$ all tree lines hook to different points;
3. given m_Δ, we choose a subset $V_\Delta \subset \Delta$ made of m_Δ points $|V_\Delta| = m_\Delta$. These are the lattice points where the tree lines actually hook;
4. for each $x \in V_\Delta$ we choose d_x the number of tree lines hooking to the point x. By construction $d_x \geq 1$ and $\sum_{x \in V_\Delta} d_x = d_\Delta$ (since there are d_Δ tree lines hooking inside Δ);
5. finally we have to choose which tree lines (among the d_Δ) hooks to each point $x \in V_\Delta$, respecting the constraint given by d_x.

The sum over (x_e, y_e) can then be written as

$$\sum_{\{(x_e,y_e) \in \Delta_e \times \Delta'_e\}_{e \in T}} = \prod_\Delta \left[\sum_{m_\Delta=1}^{d_\Delta} \sum_{V_\Delta \subset \Delta} \sum_{\{d_x, x \in V_\Delta\}} \right] \sum_{\{(x_e,y_e)\}_{e \in T}}^{*}$$

where $\sum^*$ means the sum must be compatible with the constraints created by d_Δ, V_Δ and the coordination numbers $d_x \ \forall x \in V_\Delta$. Then the expression for

$A(\tilde{X})$ becomes

$$A(\tilde{X}) = \sum_{T\in\mathcal{T}[\tilde{X}]} \sum_{\substack{m_\Delta \\ \Delta\in X}} \sum_{\{V_\Delta\subset\Delta\}} \sum_{\{d_x, x\in V_\Delta\}} \sum^{*}_{\{(x_e,y_e)\}_{e\in T}} [\prod_{e\in T} C_{x_e,y_e}] \int_{[0,1]^{|T|}} [\prod_{e\in T} ds_e]$$
$$\cdot \sum_{\{j_\Delta\}} \int d\mu_{C(u(s_T))}(\phi_X) \prod_{\Delta\in X} \left[\chi_{I_{j_\Delta}}(\phi_\Delta) \left[\prod_{x\in V_\Delta} \left(\frac{\delta}{\delta\phi_x}\right)^{d_x} \right] e^{-\sum_{x\in\Delta} V(\phi_x)} \right] \tag{3.5.30}$$

3.5.2 Estimate on the derivatives inside each cube

The derivatives in the second line of (3.5.30) can be written as

$$\left[\prod_{x\in V_\Delta} \left(\frac{\delta}{\delta\phi_x}\right)^{d_x} \right] e^{-\sum_{x\in\Delta} V(\phi_x)} = P_\Delta(\phi_\Delta) e^{-V(\phi_\Delta)}$$

where $P_\Delta(\phi_\Delta) = \prod_{x\in V_\Delta} P_{d_x}(\phi_x)$,

$$P_d(\phi_x) = e^{V(\phi_x)} \left(\frac{\delta}{\delta\phi_x}\right)^{d} e^{-V(\phi_x)} = \sum_{q=1}^{d} (-1)^q \sum_{\substack{n_1,..n_q\geq 1 \\ \sum_{j=1}^{q} n_j = d}} V^{(n_1)}(\phi_x)\cdots V^{(n_q)}(\phi_x)$$

and

$$V^{(n)}(\phi_x) = \left(\frac{\delta}{\delta\phi_x}\right)^{n} V(\phi_x).$$

We have the following estimate .

Lemma 7 *The contribution from all derivatives in the cube Δ is bounded by*

$$|P_\Delta(\phi_\Delta)| = \prod_{x\in V_\Delta} |P_{d_x}(\phi_x)| \leq \begin{cases} (c_\beta d_\Delta^2)^{d_\Delta} e^{-\frac{1}{4}\beta m_\Delta} & \phi_\Delta \in I_1 \\ (c_\beta d_\Delta^3)^{d_\Delta} & \phi_\Delta \in I_2 \\ (2c_\beta d_\Delta^4)^{d_\Delta} e^{f_\beta \frac{m^2}{2}(\phi_\Delta,\phi_\Delta)} & \phi_\Delta \in I_3\cup I_4 \end{cases}$$

where we set

$$c_\beta = 4\sqrt{\beta}, \quad f_\beta = \frac{1}{m^2} e^{-\beta}.$$

Proof . We study separately the regions I_1, I_2 and $I_3\cup I_4$.

Region I_1. If $\phi_\Delta \in I_1$ then using Lemma 8 below we have

$$|V^{(n)}(\phi_x)| \leq (\sqrt{\beta})^n\ 4^n n!\ e^{-\frac{\beta}{2}}\ \forall n\geq 1.$$

Then

$$|P_d(\phi_x)| \le \sum_{q=1}^{d} \sum_{\substack{n_1,..n_q\ge 1\\ \sum_{j=1}^q n_j=d}} \prod_{j=1}^{q} \left[(\sqrt{\beta})^{n_j}\ 4^{n_j} n_j!\ e^{-\frac{\beta}{2}}\right] = (4\sqrt{\beta})^d \sum_{q=1}^{d} e^{-q\frac{\beta}{2}} \sum_{\substack{n_1,..n_q\ge 1\\ \sum_{j=1}^q n_j=d}} \prod_{j=1}^{q} n_j!$$

$$\le (4\sqrt{\beta})^d d^d \sum_{q=1}^{d} e^{-q\frac{\beta}{2}} d^q \le (4\sqrt{\beta})^d d^{2d} \sum_{q=1}^{d} e^{-q\frac{\beta}{2}} \le (4\sqrt{\beta})^d d^{2d} e^{-\frac{1}{4}\beta}$$

where we used $n_j! \le n_j^{n_j} \le d^{n_j}$ and $\beta \gg 1$. The contribution from the cube Δ is then

$$\prod_{x\in V_\Delta} |P_{d_x}(\phi_x)| \le \prod_{x\in V_\Delta} \left[(4\sqrt{\beta})^{d_x} d_x^{2d_x} e^{-\frac{\beta}{4}}\right] \tag{3.5.31}$$

$$\le (4\sqrt{\beta})^{d_\Delta} d_\Delta^{2d_\Delta} e^{-\frac{1}{4}\beta m_\Delta} \tag{3.5.32}$$

Region I_2. If $\phi_\Delta \in I_2$ then using Lemma 8 below we have

$$|V^{(n)}(\phi_x)| \le (\sqrt{\beta})^n\ 4^n n!\ \ \forall n \ge 1.$$

$$|P_d(\phi_x)| \le \sum_{q=1}^{d} \sum_{\substack{n_1,..n_q\ge 1\\ \sum_{j=1}^q n_j=d}} \prod_{j=1}^{q} \left[(\sqrt{\beta})^{n_j}\ 4^{n_j} n_j!\right] = (4\sqrt{\beta})^d \sum_{q=1}^{d} \sum_{\substack{n_1,..n_q\ge 1\\ \sum_{j=1}^q n_j=d}} \prod_{j=1}^{q} n_j!$$

$$\le (4\sqrt{\beta})^d d^d \sum_{q=1}^{d} d^q \le (4\sqrt{\beta})^d d^{2d+1} \le (4\sqrt{\beta})^d d^{2d+1}$$

The contribution from the cube Δ is then

$$\prod_{x\in V_\Delta} |P_{d_x}(\phi_x)| \le \prod_{x\in V_\Delta} \left[(4\sqrt{\beta})^{d_x} d_x^{2d_x+1}\right] \le (4\sqrt{\beta})^{d_\Delta} d_\Delta^{2d_\Delta+m_\Delta} \le (4\sqrt{\beta} d_\Delta^3)^{d_\Delta}, \tag{3.5.33}$$

where we used $m_\Delta \le d_\Delta$.

Region $I_3 \cup I_4$. If $\phi_\Delta \in I_3 \cup I_4$ then using Lemma 8 below we have

$$|V^{(n)}(\phi_x)| \le (\sqrt{\beta})^n\ 4^n n![1 + |\phi_x| e^{-\beta}] \le (\sqrt{\beta})^n\ 4^n n! 2\sqrt{d_x}\ e^{\frac{1}{2d_x}\phi_x^2 e^{-\beta}}$$

where we used

$$(1 + c\alpha|x|) \le c(1 + \alpha|x|) \le 2ce^{\frac{x^2}{2}}$$

for $0 < \alpha \ll 1$ and $c \geq 1$, we set $c = \sqrt{d_x}$, $\alpha = e^{-\beta/2}$ and $x = \phi_x e^{-\beta}/\sqrt{d_x}$.

$$
\begin{aligned}
|P_{d_x}(\phi_x)| &\leq \sum_{q=1}^{d_x} (2\sqrt{d_x})^q e^{\frac{q}{2d_x}\phi_x^2 e^{-\beta}} \sum_{\substack{n_1,..n_q\geq 1\\ \sum_{j=1}^q n_j = d_x}} \prod_{j=1}^{q} \left[(\sqrt{\beta})^{n_j}\ 4^{n_j} n_j!\right] \\
&\leq (2\sqrt{d})^{d_x} e^{\frac{1}{2}\phi_x^2 e^{-\beta}} (4\sqrt{\beta})^{d_x} \sum_{q=1}^{d_x} \sum_{\substack{n_1,..n_q\geq 1\\ \sum_{j=1}^q n_j = d_x}} \prod_{j=1}^{q} n_j! \\
&\leq\ e^{\frac{\phi_x^2}{2e^{\beta}}} (8\sqrt{\beta})^{d_x} d_x^{\frac{3}{2}d_x} \sum_{q=1}^{d_x} d_x^q \leq (8\sqrt{\beta})^{d_x} d_x^{\frac{5}{2}d_x+1} e^{\frac{\phi_x^2}{2e^{\beta}}} \leq (8\sqrt{\beta})^{d_x} d_x^{3d_x+1} e^{\frac{\phi_x^2}{2e^{\beta}}},
\end{aligned}
$$

where we used $q \leq d_x$. The contribution from the cube Δ is then

$$
\begin{aligned}
\prod_{x\in V_\Delta} |P_{d_x}(\phi_x)| &\leq \prod_{x\in V_\Delta} \left[(8\sqrt{\beta})^{d_x} d_x^{3d_x+1} e^{\frac{\phi_x^2}{2e^{\beta}}},\right] &(3.5.34)\\
&\leq\ e^{\frac{1}{2e^{\beta}}(\phi_\Delta,\phi_\Delta)} (8\sqrt{\beta})^{d_\Delta} d_\Delta^{3d_\Delta+m_\Delta}. &(3.5.35)
\end{aligned}
$$

This concludes the proof. □

Lemma 8 *The first and second derivatives of V satisfy the bounds*

$$
|V'(\phi_x)| \leq \begin{cases} 2\sqrt{\beta} e^{-\frac{\beta}{2}} & \phi_x \in I_1 \\ 3\sqrt{\beta} & \phi_x \in I_2 \\ |\phi_x| e^{-\beta} + 2\sqrt{\beta} & \phi_x \in (I_1 \cup I_2)^c \end{cases}
$$

$$
|V^{(2)}(\phi_x)| \leq \begin{cases} 2(\sqrt{\beta})^2\ e^{-\beta} & \phi_x \in I_1 \cup I_2 \\ 2(\sqrt{\beta})^2 & \phi_x \in (I_1 \cup I_2)^c \end{cases}
$$

All other derivatives satisfy

$$
|V^{(p+2)}(\phi_x)| \leq \begin{cases} (\sqrt{\beta})^{p+2}\ 4^{p-1}(p+1)!\ e^{-2\beta} & \phi_x \in I_1 \cup I_2 \\ (\sqrt{\beta})^{p+2} 4^{p-1}(p+1)! & \phi_x \in (I_1 \cup I_2)^c \end{cases}
$$

where $p \geq 1$.

Proof Remember that

$$
V(\phi) = f(\phi + \phi_{m1}) - f(\phi_{m1}) - \frac{m^2}{2}\phi^2.
$$

The expression for the first derivative is

$$
\begin{aligned}
V'(\phi_x) &= f'(\phi_x + \phi_{m1}) - m^2\phi_x = f'(\phi_x + \phi_{m1}) - f'(\phi_{m1}) - m^2\phi_x \\
&= \phi_x(1-m^2) + (\sqrt{\beta})\left[\tanh(\sqrt{\beta}\phi_{m1}) - \tanh(\sqrt{\beta}(\phi_{m1}+\phi_x)\right] \\
&= \phi_x(1-m^2) - (\sqrt{\beta})\frac{\sinh(\sqrt{\beta}\phi_x)}{\cosh(\sqrt{\beta}\phi_{m1})\cosh(\sqrt{\beta}(\phi_{m1}+\phi_x)},
\end{aligned}
$$

where we used $f'(\phi_{m1}) = 0$. Now $|1-m^2| \leq e^{-\beta}$ and $|\phi_x| \simeq |\phi_x + \phi_{m1}| \simeq |\phi_{m1}| \simeq \sqrt{\beta}$ when $\phi_x \in I_1$. In all other regions the second term on the derivative is bounded by $2\sqrt{\beta}$ since $|\tanh x| \leq 1$.

The second derivative is

$$V''(\phi_x) = (1-m^2) - (\sqrt{\beta})^2 \left((1 - [\tanh(\sqrt{\beta}(\phi_{m1}+\phi_x)]^2\right)$$
$$= (1-m^2) - \frac{(\sqrt{\beta})^2}{[\cosh(\sqrt{\beta}(\phi_{m1}+\phi_x))]^2}$$

The bound follows directly. Finally using $\tanh x' = 1 - (\tanh x)^2$ higher order derivatives can be written as

$$V^{(n+2)}(\phi_x) = (\sqrt{\beta})^{n+2} P_n(\tanh(\sqrt{\beta}(\phi_{m1}+\phi_x))$$

where $P_n : [-1,1] \to \mathbb{R}$ is a polynome of order $n+2$ defined by induction:

$$P_n(x) = Q_n(x)(1-x^2), \quad Q_n(x) = P'_{n-1}(x),\ \forall n \geq 2, \forall x \in [-1,1],$$

and

$$Q_1(x) = x.$$

By construction $Q_n(x)$ is a polynome of order n. Let

$$Q_n(x) = a_0 + a_1 x + \cdots + a_n x^n, \qquad \|Q_n\| = (n+1) \sup_j |a_j|.$$

Then $\|Q_2\| = 2$ and

$$|Q_n(x)| \leq \|Q_n\| \qquad \forall |x| \leq 1, \forall n \geq 1.$$

Using

$$Q_{n+1}(x) = Q'_n(x)(1-x^2) - 2xQ_n(x)$$

one chan check that

$$\|Q_{n+1}\| \leq 4(n+2)\|Q_n\|.$$

Then

$$\|Q_n\| \leq 4^{n-1}(n+1)!\ \forall n \geq 1.$$

Inserting all this we have

$$|V^{(n+2)}(\phi_x)| \leq (\sqrt{\beta})^{n+2} \sup_{x\in[-1,1]} |P_n(x)| \leq (\sqrt{\beta})^{n+2} \|Q_n\|.$$

This completes the proof. $\square$

3.5.3 Estimate on the gaussian integral

Inserting the estimates we proved in Section 3, the ϕ integral in (3.5.30) is bounded by

$$\int d\mu_{C(u(s_T))}(\phi_X)\left[\prod_{\Delta\in X}\chi_{I_{j_\Delta}}(\phi_\Delta)|P_\Delta(\phi_\Delta)|e^{-V(\phi_\Delta)}\right] \tag{3.5.36}$$
$$\leq \sum_{\substack{x'_\Delta,y'_\Delta\\ j_\Delta=4}}\sum_{\substack{\sigma_\Delta=\pm\\ j_\Delta=4}}\left[\prod_{\Delta\in X}(2c_\beta d_\Delta^4)^{d_\Delta}e^{a_\Delta}\right]\int d\mu_{C(u(s_T))}(\phi_X)e^{(v,\phi_X)}\ e^{\frac{m^2}{2}(\phi_X,D\phi_X)}$$

where for each Δ with $j_\Delta=4$ we sum over two points x'_Δ, y'_Δ inside Δ and over a sign $\sigma_\Delta=\pm$,

$$(v,\phi_X)=\sum_{\Delta\in X}(v_\Delta,\phi_\Delta),\quad D=\sum_{\Delta\in X}(1-q_\Delta)Id_\Delta$$

and $a_\Delta, v_\Delta, q_\Delta$ depend on the index j_Δ. Precisely when $j_\Delta=1$ we have

$$a_\Delta=-\frac{1}{4}\beta+c|\Delta|e^{-\beta},\quad v_\Delta(x)=0,\quad q_\Delta=1,$$

and $c>0$ is some constant. When $j_\Delta=2$ we have

$$a_\Delta=-h|\Delta|-\frac{m^2}{2}(\phi_{m1}-\phi_{m2})^2|\Delta|,\quad v_\Delta(x)=-m^2(\phi_{m1}-\phi_{m2}),\quad q_\Delta=1.$$

When $j_\Delta=3$ we have

$$a_\Delta=-q\beta|\Delta|,\quad v_\Delta(x)=0,\quad q_\Delta=q,\qquad 0<q\ll 1.$$

Finally when $j_\Delta=4$ we have

$$a_\Delta=+\eta|\Delta|-\delta u,\quad v_\Delta(x)=\sigma_\Delta u(\delta_{x'_\Delta x}-\delta_{y'_\Delta x}),\quad q_\Delta=\frac{c\eta}{\beta},$$

where $c>0$ is some constant and we set

$$\delta=\frac{\sqrt{\beta}}{4},\qquad u=\delta W^2 c_1,\qquad \eta=\frac{1}{4Wc_4}$$

with some fixed constant c_4. Note that when $j_\Delta=3,4$ there is a correction of order $e^{-\beta}$ (from P_Δ) to $1-q_\Delta$ that is negligeable (even in the region I_4). Finally, the $j_\Delta=1$, the factor $-\beta$ in a_Δ comes from the field derivative. Since the set X has at least two cubes there is always at least one tree line hooking to each cube therefore the factor $-\beta$ appears in *each* cube with $j_\Delta=1$.

The gaussian integral in the last term is finite. This is a consequence of the next lemma.

Lemma 9 *Let $C(u(s_T))$ and D the matrices introduced above. Then*

$$C(u(s_T))^{-1} - m^2 D > 0, \qquad \text{when } 0 < q_\Delta < 1 \ \forall \Delta.$$

Moreover let $C(u(s_T))_D = (C(u(s_T))^{-1} - m^2 D)^{-1}$ and

$$C_D^N = (-W^2 \hat{\Delta}_N^{\partial \Pi_0} + m^2 \hat{q}_\Delta)^{-1} = \sum_\Delta C^N_{\Delta, q_\Delta}$$

where $-\hat{\Delta}_N^{\partial\Pi_0}$ is the Laplacian with Neuman b.c. on each cube and $\hat{q}_\Delta = \sum_\Delta q_\Delta I_\Delta$ is a diagonal matrix. Then we have

$$0 < C(u(s_T))_D \le C_D^N \quad \forall T \text{ and } s_T.$$

Proof. When the forest is reduced to a tree, formula (3.4.24) becomes

$$C(u(s_1, .., s_n)) = s_n \ C_{\Pi_{T_n}} + (1 - s_1) \ [C_d] + \sum_{j=1}^{n-1} (s_j - s_{j+1}) \ [C_{\Pi_{T_j}}]$$

where $n = |T|$ and we ordered the tree lines so that $1 \ge s_1 \ge s_2, \ldots \ge s_n$. By Lemma 5 $C_{\Pi_{T_j}} \le C_N^{\partial\Pi_0}$ for all T_j then

$$0 < C(u(s_1, .., s_n)) \le [s_n + \sum_{j=1}^{n-1} (s_j - s_{j+1}) + (1 - s_1)] C_N^{\partial\Pi_0} = C_N^{\partial\Pi_0}.$$

Then

$$C(u(s_T))^{-1} - m^2 D \ge (C_N^{\partial\Pi_0})^{-1} - m^2 D = \sum_\Delta (C^N_{\Delta, q_\Delta})^{-1} > 0$$

since $q_\Delta > 0 \ \forall \Delta$. Therefore $C(u(s_T))^{-1} - m^2 D$ is invertible and

$$\left[C(u(s_T))^{-1} - m^2 D\right]^{-1} \le C_D^N.$$

This ends the proof. $\square$

Using this result we can compute the gaussian integral in (3.5.36). The result is

$$\int d\mu_{C(u(s_T))} \ e^{\frac{m^2}{2}(\phi_X, D\phi_X)} e^{(v, \phi_X)} = \frac{\mathcal{N}_{C(u(s_T))_D}}{\mathcal{N}_{C(u(s_T))}} \int d\mu_{C(u(s_T))_D} \ e^{(v, \phi_X)} \tag{3.5.37}$$

$$= \frac{\mathcal{N}_{C(u(s_T))_D}}{\mathcal{N}_{C(u(s_T))}} e^{\frac{1}{2}(v, C(u(s_T))_D v)} = \det\left[1 + m^2 D C(u(s_T))_D\right]^{\frac{1}{2}} e^{\frac{1}{2}(v, C(u(s_T))_D v)}$$

Using Lemma 9 above

$$(v, C(u(s_T))_D v) \le (v, C_D^N v) = \sum_{\Delta \in X} (v_\Delta, C^N_{\Delta, q_\Delta} v_\Delta).$$

Moreover since $D > 0$ we can define $D^{\frac{1}{2}}$ and

$$\begin{aligned}\det\left[1 + m^2 DC(u(s_T))_D\right] &= \det\left[1 + m^2 D^{\frac{1}{2}} C(u(s_T))_D D^{\frac{1}{2}}\right]\\ &\le \det\left[1 + m^2 D^{\frac{1}{2}} C_D^N D^{\frac{1}{2}}\right] = \det\left[1 + m^2 DC_D^N\right]\\ &= \prod_{\Delta} \det\left[1 + m^2(1 - q_\Delta) C_{\Delta,q_\Delta}^N\right],\end{aligned}$$

where we used

$$A \ge B \quad \Rightarrow \quad M^T AM \ge M^T BM \quad \forall M \text{ invertible},$$

and

$$A \ge B > 0 \quad \Rightarrow \quad \det A \ge \det B.$$

The first equation is proved by quadratic forms

$$(\phi, M^T AM\phi) = (M\phi, AM\phi) \ge (M\phi, BM\phi),$$

where we use $M\phi \neq 0$ when $\phi \neq 0$. The second can be proved using gaussian integrals

$$\frac{1}{[\det A]^{\frac{1}{2}}} = \int \frac{d\phi}{2\pi}\; e^{-\frac{1}{2}(\phi,A\phi)} \le \int \frac{d\phi}{2\pi}\; e^{-\frac{1}{2}(\phi,B\phi)} = \frac{1}{[\det B]^{\frac{1}{2}}}.$$

Inserting these bounds and using the same estimates we did in Section ?? we obtain

$$\begin{aligned}&\sum_{\substack{x'_\Delta, y'_\Delta\\ j_\Delta=4}} \sum_{\substack{\sigma_\Delta=\pm\\ j_\Delta=4}} \left[\prod_{\Delta\in X} (2c_\beta d_\Delta^4)^{d_\Delta} e^{a_\Delta}\right] \int d\mu_{C(u(s_T))}(\phi_X) e^{(v,\phi_X)}\; e^{\frac{m^2}{2}(\phi_X, D\phi_X)}\\ &\quad\le \prod_{\Delta\in X} (4c_\beta d_\Delta^4)^{d_\Delta} e^{-c_\Delta} \le \prod_{\Delta\in X} (4c_\beta d_\Delta^4)^{d_\Delta} e^{-\beta^\epsilon}\end{aligned}$$

where

$$c_\Delta = \begin{cases} c_1\beta & j_\Delta = 1\\ c_2 h|\Delta| & j_\Delta = 2\\ c_3\beta|\Delta| & j_\Delta = 3\\ c_4 W^2\beta & j_\Delta = 4, \end{cases} \qquad \text{where } h \ge \frac{1}{|\Delta|^{1-3\epsilon}},\ \beta \gg \ln W, \tag{3.5.38}$$

and $c_j > 0$ are some fixed constants. The amplitude $A(\tilde{X})$ is then estimated by

$$\begin{aligned}|A(\tilde{X})| \le &\sum_{T\in\mathcal{T}[\tilde{X}]} \sum_{\substack{m_\Delta\\ \Delta\in X}} \sum_{\{V_\Delta\subset\Delta\}} \sum_{\{d_x, x\in V_\Delta\}} \sum^{*}_{\{(x_e,y_e)\}_{e\in T}} \sum_{\{j_\Delta\}}\\ &\cdot \left[\prod_{e\in T} C_{x_e,y_e}\right] \left[\prod_{\Delta\in X} (4c_\beta d_\Delta^4)^{d_\Delta} e^{-\beta^\epsilon}\right]\end{aligned} \tag{3.5.39}$$

The factor $d_\Delta^{d_\Delta}$ may destroy the convergence. Note that, when d_Δ tree lines hook to the cube Δ, there must be d_Δ *different* cubes in Λ connected to Δ by the tree. Since each cube has volume $|\Delta|$ at least half of these cubes have their center at distance $R \propto d^{\frac{1}{3}}|\Delta|^{\frac{1}{3}} = d^{\frac{1}{3}} W \beta^{\frac{1}{3}}$ from the center of Δ. Using the properties of the discrete Laplacian one can show that

$$C_{xy} \leq \frac{1}{W^2} \frac{1}{(1+|x-y|)} e^{-\frac{|x-y|}{W}} \quad \text{in } d=3.$$

Then extracting a fraction of the exponential decay from each tree line, and noting that each tree line contributes only to two different cubes we have

$$[\prod_{e\in T} C_{x_e,y_e}] \leq \left[\prod_{e\in T} \frac{1}{W^2}\frac{1}{(1+|x-y|)} e^{-\frac{|x-y|}{10W}}\right]\left[\prod_{\Delta} e^{-cd_\Delta d_\Delta^{\frac{1}{3}}\beta^{\frac{1}{3}}}\right].$$

Moreover if we let $\tilde{x}_\Delta$ denote the point in $\tilde{\Lambda}$ corresponding to the cube Δ we have

$$|x_e - y_e| \geq W\beta^{\frac{1}{3}}(|\tilde{x}_e - \tilde{x}'_e| - 1)$$

where $\tilde{x}_e$ is the point corresponding to the cube Δ_e and $|\tilde{x}-\tilde{x}'|$ is the distance in the dual lattice $\tilde{\mathbb{Z}}^3$. Inserting this in the sum above we obtain

$$\begin{aligned}
|A(\tilde{X})| \leq & \sum_{T\in\mathcal{T}[\tilde{X}]} \sum_{\substack{m_\Delta \\ \Delta\in X}} \sum_{\{V_\Delta\subset\Delta\}} \sum_{\{d_x, x\in V_\Delta\}} \sum^{*}_{\{(x_e,y_e)\}_{e\in T}} \sum_{\{j_\Delta\}} \\
& \cdot [\prod_{e\in T} J_{\tilde{x}_e\tilde{x}'_e}]\left[\prod_{\Delta}(4c_\beta d_\Delta^4)^{d_\Delta} e^{-\beta^\epsilon} e^{-cd_\Delta d_\Delta^{\frac{1}{3}}\beta^{\frac{1}{3}}}\right] \\
& \leq \sum_{T\in\mathcal{T}[\tilde{X}]} [\prod_{e\in T} J_{\tilde{x}_e\tilde{x}'_e}]\left[\prod_{\Delta} 4(4\ |\Delta| c_\beta d_\Delta^4)^{d_\Delta} e^{-\beta^\epsilon} e^{-cd_\Delta d_\Delta^{\frac{1}{3}}\beta^{\frac{1}{3}}}\right]
\end{aligned}$$

where we used

$$\sum_{\{j_\Delta\}} 1 = \prod_{\Delta} 4$$

$$\sum_{\substack{m_\Delta \\ \Delta\in X}} \sum_{\{V_\Delta\subset\Delta\}} \sum_{\{d_x,x\in V_\Delta\}} \sum^{*}_{\{(x_e,y_e)\}_{e\in T}} 1 = \sum_{\{(x_e,y_e)\}_{e\in T}} 1 = \prod_{e\in T}|\Delta|^2 = \prod_{\Delta}|\Delta|^{d_\Delta},$$

and we defined

$$J_{\tilde{x}_e\tilde{x}'_e} = \frac{1}{W^2} e^{-(|\tilde{x}_e - \tilde{x}'_e|-1)\beta^{\frac{1}{3}}}.$$

Now

$$4(4\ |\Delta| c_\beta d_\Delta^4)^{d_\Delta} e^{-\beta^\epsilon} e^{-cd_\Delta d_\Delta^{\frac{1}{3}}\beta^{\frac{1}{3}}} = 4e^{-d_\Delta(cd_\Delta^{\frac{1}{3}}\beta^{\frac{1}{3}} - 4\ln d_\Delta - \ln c_\beta - \ln|\Delta|)} e^{-\beta^\epsilon} \leq e^{-d_\Delta \beta^{\frac{1}{4}}}$$

for any value of d_Δ since β is large. Then

$$|A(\tilde{X})| \leq \sum_{T \in \mathcal{T}[\tilde{X}]} [\prod_{e \in T} \lambda J_e]$$

where $\lambda = e^{-\beta^{\frac{1}{4}}} \ll 1$ and $\sum_{\tilde{x}' \in \tilde{\mathbb{Z}}^3} J_{\tilde{x}\tilde{x}'} < 1$. Then we can repeat the same argument we used in Section 2.8 to prove that

$$\sup_{\tilde{x}_0} \sum_{\substack{2 \leq |\tilde{X}| < \infty \\ \tilde{x}_0 \in \tilde{X}}} |\mathcal{A}(\tilde{X})| e^{|\tilde{X}|} \leq \sum_{T \in \mathcal{T}[\tilde{X}]} [\prod_{e \in T} \lambda J_e] e^{c|\tilde{X}|} \leq \sum_{T \in \mathcal{T}[\tilde{X}]} [\prod_{e \in T} \lambda^{\frac{1}{2}} J_e] e^{|\tilde{X}|} < 1,$$

where $c = 1 + e^{-\beta^{\frac{1}{2}}}$ (see (3.5.28)). This concludes the proof of (3.4.27). $\square$