The Dynamics of Random Quantum Walks

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The Mathematics of Disorder
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Outline

Quantum Walks: An Overview

Spectral Properties of Quantum Walks on Trees
Quantum Walk on $\mathbb{Z}$

Quantum particle with spin $1/2$ on 1–dim lattice, i.e. $\mathcal{K}_2 = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$

Spin evolution: $C$ unitary operator on $\mathbb{C}^2$, "coin" space

Spin dependent shift: $S_{\uparrow(\downarrow)}$ shift to the right(left) on $\ell^2(\mathbb{Z})$

$$S := S_{\uparrow} \otimes | \uparrow \rangle \langle \uparrow | + S_{\downarrow} \otimes | \downarrow \rangle \langle \downarrow | \quad \text{on } \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$$

One Step dynamics: $U := S(\mathbb{I} \otimes C)$

$$U|x \otimes \uparrow \rangle = C_{\downarrow \uparrow} |(x - 1) \otimes \downarrow \rangle + C_{\uparrow \uparrow} |(x + 1) \otimes \uparrow \rangle$$

$$U|x \otimes \downarrow \rangle = C_{\downarrow \downarrow} |(x - 1) \otimes \downarrow \rangle + C_{\uparrow \downarrow} |(x + 1) \otimes \uparrow \rangle$$

Analogy: $H = -\Delta + V$
Quantum Walk versus Random Walk

Random Walker $\Rightarrow$ Coin

Quantum Walker $\Rightarrow$ Coin Matrix $C \in U(2)$
Quantum Walk on $\mathbb{Z}^2$ :

Setup: \[ \mathcal{K} = l^2(\mathbb{Z}^2) \otimes \mathbb{C}^4 \]
\[ \{|\tau\rangle\}_{\tau \in I_\pm}, \quad I_\pm \equiv \{\pm 1, \pm 2\} \text{ for } \mathbb{C}^4, \]
\[ \{|x\rangle\}_{x \in \mathbb{Z}^2} \text{ for } l^2(\mathbb{Z}^2) \]

- Spin dependent shift: Let $P_\tau$ the projection "on" $|\tau\rangle \in \mathbb{C}^4$
\[ S := \sum_{x \in \mathbb{Z}^2} \sum_{\tau \in I_\pm} |x + \text{sign}(\tau)e_{|\tau|}\rangle \langle x| \otimes P_\tau \text{ on } l^2(\mathbb{Z}^2) \otimes \mathbb{C}^4 \]

- Spin evolution: $C$ unitary operator on $\mathbb{C}^4$

Time one dynamics of a QW:
\[ U(C) := S (\mathbb{I} \otimes C) = \sum_{x \in \mathbb{Z}^2} \sum_{\tau \in I_\pm} |x + \text{sign}(\tau)e_{|\tau|}\rangle \langle x| \otimes (P_\tau C) \]
Quantum Walk on $\mathbb{Z}^2$:

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$\{|\tau\rangle\}_{\tau \in I_{\pm}}$, $I_{\pm} \equiv \{\pm 1, \pm 2\}$ for $\mathbb{C}^4$,  

$\{|x\rangle\}_{x \in \mathbb{Z}^2}$ for $l^2(\mathbb{Z}^2)$

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Time one dynamics of a QW:

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Time one dynamics of a random QW:

$U_\omega (C) := \sum_{x \in \mathbb{Z}^2} \sum_{\tau \in I_{\pm}} |x + \text{sign}(\tau)e_{|\tau|}\rangle \langle x| \otimes (P_\tau C_\omega (x))$
Random Quantum Walk

- A random diagonal unitary operator on $\mathcal{K}$ by

$$D_\omega x \otimes \tau = \exp(i\omega^T\tau)x \otimes \tau.$$ 

The random time one dynamics

$$U_\omega(C) = D_\omega U(C)$$

Assumptions:
Random Quantum Walk

- A random diagonal unitary operator on $\mathcal{K}$ by
  \[ \mathbb{D}_x \otimes \tau = \exp(i \omega^T_x)x \otimes \tau. \]

  The random time one dynamics
  \[ U_\omega(C) = \mathbb{D}_x U(C) \]

  Assumptions:

- \( \{\omega^T_x\}_{x \in \mathcal{T}, \tau \in A} \) i.i.d. random variables on the torus $\mathbb{T}$ with common distribution $d\nu(\theta) = l(\theta)d\theta$, where $l \in L^\infty(\mathbb{T})$

Remarks:
Random Quantum Walk

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Remarks:

- Spatial disorder: $C \mapsto \{C_\omega(x)\}_{x \in \mathcal{T}}$
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Remarks:

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- "Expected" spectral transition between "large" and "small" disorder regimes. Abou-Chacra, Anderson, Thouless '73, Kunz, Souillard '83, Klein '94
Remarks

- First result for RQW [Konno ’09]
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- Localization for $d = 1$ [Joye-Merkli ’10, Ahlbrecht et al ’11]

\[ C_\omega(x) = \begin{pmatrix} te^{i\omega^{-1}_x} & re^{i\omega^{-1}_x} \\ re^{i\omega^+_x} & -te^{i\omega^+_x} \end{pmatrix} \]
Remarks

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- Large disorder localization on $\mathbb{Z}^d$ [Joye’ 12]
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\end{pmatrix}
\]

- Large disorder localization on $\mathbb{Z}^d$ [Joye’ 12]
- Generalization to homogeneous trees.
The Homogeneous tree $\mathcal{T}_4$

$A_2 = \{a, b\}$ generators of a free group

**Root:** origin $e$

**Edges:** Any $x \in \mathcal{T}_4$: Finite reduced word $x = x_1x_2...x_n$, $x_j \in A_2$

**Length:** $|x| = n$

**Distance:** For any $x, y \in \mathcal{T}_4$, $d(x, y) = |x^{-1}y|$
Random Quantum Walk on $T_4$

Hilbert space; $\mathcal{K}_4 = \ell^2(T_4) \otimes \mathbb{C}^4$

Shift on $\ell^2(T_4)$: $S_\tau |x\rangle = |x\tau\rangle$

Spin dependent shift on $\mathcal{K}_4$:

$$S = S_a \otimes |a\rangle\langle a| + S_b \otimes |b\rangle\langle b| + S_{a^{-1}} \otimes |a^{-1}\rangle\langle a^{-1}| + S_{b^{-1}} \otimes |b^{-1}\rangle\langle b^{-1}|$$

RQW: $U_\omega(C) = \mathbb{D}(\omega)S(\mathbb{I} \otimes C)$
Landmarks in $U(4)$

- $C_{(a)(b)(a^{-1})(b^{-1})} = \mathbb{I} \Rightarrow \text{a.c.} \quad U_{\omega}(\mathbb{I}) \simeq S$
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- $C_{(aba^{-1}b^{-1})} \Rightarrow \text{a.c.}$

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\[ \text{same for } C_{(ab^{-1}a^{-1}b)} \]
Landmarks in $U(4)$

- $C_{(aba^{-1}b^{-1})} \Rightarrow$ a.c.  
- $C_{(ab^{-1}a^{-1}b)}$  
- $C_{(aa^{-1})(bb^{-1})} \Rightarrow$ p.p  
- $C_{(abb^{-1}a^{-1})}$, ...

$H_x^a = \text{span} \{x \otimes a, xa^{-1} \otimes a^{-1}\}$

$H_x^b = \text{span} \{x \otimes b, xb^{-1} \otimes b^{-1}\}$

$C = C_a \oplus C_b$ on $\text{span} \{\langle a \rangle, \langle a^{-1} \rangle\} \oplus \text{span} \{\langle b \rangle, \langle b^{-1} \rangle\}$
Landmarks in $U(4)$

- $C_{(aba^{-1}b^{-1})} \Rightarrow \text{a.c.}$

- $C_{(abb^{-1}a^{-1})} \Rightarrow \text{p.p}$

\[ \mathcal{H}_x = \{ x \otimes a, xb \otimes b, x \otimes b^{-1}, xa^{-1} \otimes a^{-1} \}, \]
Spectral Phase Diagram
Neighborhood of $C_{(aba^{-1}b^{-1})}$

Proposition

There exists $\epsilon > 0$. Then, for any $C \in U(4)$, $\|C - C_{(aba^{-1}b^{-1})}\| \leq \epsilon$ implies for any $\omega \in \Omega$

$$\sigma(U_\omega(C)) = \sigma_{ac}(U_\omega(C)).$$

Idea of Proof:
Neighborhood of $C_{(aba^{-1}b^{-1})}$

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Idea of Proof:

- Spectral criteria

$$\mathcal{H}^{ac}(U) = \left\{ \phi \mid \sum_{n \in \mathbb{N}} |\langle \phi | U^n \phi \rangle|^2 < \infty \right\}$$
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- Path counting argument to show $|\langle x \otimes \tau | U^2_{\omega}(C)x \otimes \tau \rangle| \leq (\kappa \epsilon)^n$
Neighborhood of $C_{(aba^{-1}b^{-1})}$

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- Spectral criteria

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- Path counting argument to show

$$|\langle x \otimes \tau | U_\omega^{2n}(C)x \otimes \tau \rangle| \leq (\kappa \epsilon)^n$$

Remark: This result is deterministic and can be generalized for all $q$ even
Neighborhoods of $C_{(abb^{-1}a^{-1})}$ and $C_{(aa^{-1}bb^{-1})}$, ...

**Theorem**

Let $\pi \in \{(abb^{-1}a^{-1}), (aa^{-1}bb^{-1}), (aa^{-1}b^{-1}b), (aa^{-1})(bb^{-1}), (ab^{-1}ba^{-1})\}$ there exists $\epsilon > 0$ such that for all $C \in U(4)$ with $\|C - C_\pi\| \leq \epsilon$

$$\sigma(U_\omega(C)) = \sigma_{pp}(U_\omega(C))$$ almost surely.

**Proof:** Fractional Moment Method [Aizenman-Molchanov 93, H.-Joye-Stolz 09]
Fractional Moment Estimate

Theorem
For all $0 < s < 1/3$, and all $\gamma > 0$, there exist $K(s, \gamma) < \infty$ and $\epsilon(s, \gamma) > 0$ such that for all $C \in U(q)$ with $\|C - C^\Phi_\pi\| \leq \epsilon(s, \gamma)$, all $x, y \in T_q$ with $d(x, y) > 2$, all $z \notin U$, and all $\tau, \sigma \in A_q$, 

$$
\mathbb{E}(\langle x \otimes \tau | (U_\omega (C) - z)^{-1} y \otimes \sigma \rangle |^s) \leq K(s, \gamma) e^{-\gamma d(x, y)}.
$$
Thank you