The Dynamics of Random Quantum Walks

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The Mathematics of Disorder
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Outline

Quantum Walk on $\mathbb{Z}$

Quantum particle with spin $1/2$ on $1-$dim lattice, i.e. $\mathcal{K}_2 = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$

Spin evolution: C unitary operator on $\mathbb{C}^2$, "coin" space

Spin dependent shift: $S_{\uparrow(\downarrow)}$ shift to the right(left) on $\ell^2(\mathbb{Z})$

$$S := S_\uparrow \otimes |\uparrow\rangle\langle\uparrow| + S_\downarrow \otimes |\downarrow\rangle\langle\downarrow| \quad \text{on } \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$$

One Step dynamics: $U := S(\mathbb{I} \otimes C)$

$$U|x \otimes \uparrow\rangle = C_{\downarrow\uparrow}|(x-1) \otimes \downarrow\rangle + C_{\uparrow\uparrow}|(x+1) \otimes \uparrow\rangle$$

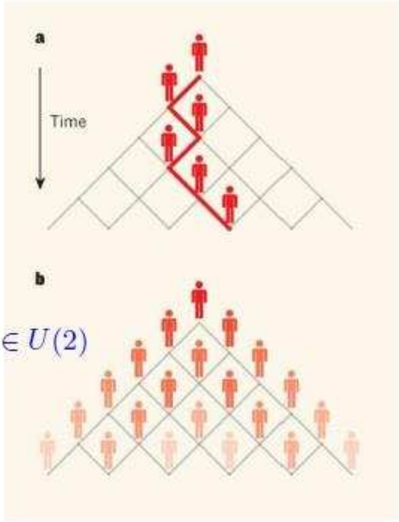
$$U|x \otimes \downarrow\rangle = C_{\downarrow\downarrow}|(x-1) \otimes \downarrow\rangle + C_{\uparrow\downarrow}|(x+1) \otimes \uparrow\rangle$$

Analogy: $H = -\Delta + V$

Quantum Walk versus Random Walk

Random Walker $\Rightarrow$ Coin

Quantum Walker $\Rightarrow$ Coin Matrix $C \in U(2)$

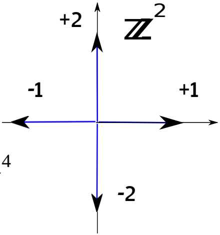


Quantum Walk on $\mathbb{Z}^2$:

Setup: $\mathcal{K} = l^2(\mathbb{Z}^2) \otimes \mathbb{C}^4$

$\{|\tau\rangle\}_{\tau\in I_\pm}$, $I_\pm \equiv \{\pm 1, \pm 2\}$ for $\mathbb{C}^4$,

$\{|x\rangle\}_{x\in\mathbb{Z}^2}$ for $l^2(\mathbb{Z}^2)$



- Spin dependent shift: Let P_τ the projection "on" $|\tau\rangle \in \mathbb{C}^4$

$$S := \sum_{x\in\mathbb{Z}^2} \sum_{\tau\in I_\pm} |x + \text{sign}(\tau) e_{|\tau|}\rangle\langle x| \otimes P_\tau \text{ on } l^2(\mathbb{Z}^2) \otimes \mathbb{C}^4$$

- Spin evolution: C unitary operator on $\mathbb{C}^4$

Time one dynamics of a QW:

$$U(C) := S(\mathbb{I} \otimes C) = \sum_{x\in\mathbb{Z}^2} \sum_{\tau\in I_\pm} |x + \text{sign}(\tau) e_{|\tau|}\rangle\langle x| \otimes (P_\tau C)$$

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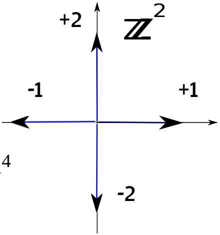

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Time one dynamics of a random QW:

$$U_\omega(C) := \sum_{x \in \mathbb{Z}^2} \sum_{\tau \in I_\pm} |x + \text{sign}(\tau) e_{|\tau|}\rangle\langle x| \otimes (P_\tau C_\omega(x))$$

Random Quantum Walk

- A random diagonal unitary operator on $\mathcal{K}$ by

$$\mathbb{D}_\omega x \otimes \tau = \exp\left(i\omega_x^\tau\right) x \otimes \tau.$$

The random time one dynamics

$$U_\omega(C) = \mathbb{D}_\omega U(C)$$

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Assumptions:

- $\{\omega_x^\tau\}_{x \in \mathcal{T}, \tau \in A}$ i.i.d. random variables on the torus $\mathbb{T}$ with common distribution $d\nu(\theta) = l(\theta)d\theta,$ where $l \in L^\infty(\mathbb{T})$

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- Spatial disorder: $C \mapsto \{C_\omega(x)\}_{x\in\mathcal{T}}$
- "Expected" spectral transition between "large" and "small" disorder regimes. Abou-Chacra, Anderson, Thouless '73, Kunz, Souillard '83, Klein '94

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$$C_\omega(x) = \begin{pmatrix} te^{i\omega_x^{-1}} & re^{i\omega_x^{-1}} \\ re^{i\omega_x^{+1}} & -te^{i\omega_x^{+1}} \end{pmatrix}$$

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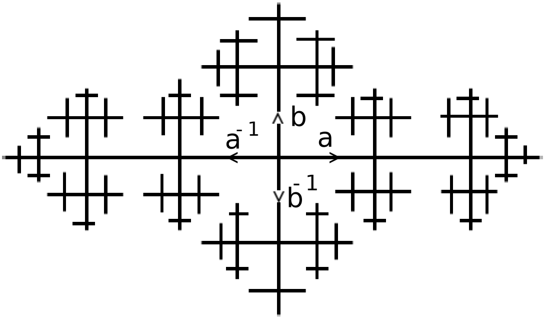
- Generalization to homogeneous trees.

The Homogeneous tree $\mathcal{T}_4$



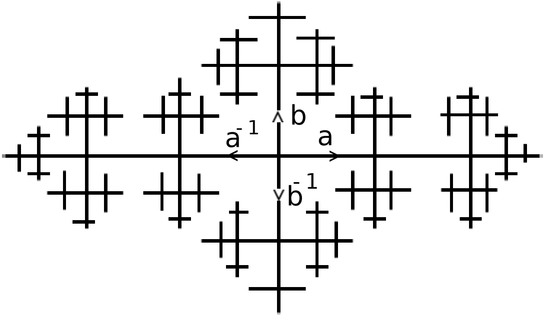
$A_2 = \{a, b\}$ generators of a free group
Root: origin e

Edges: Any $x \in \mathcal{T}_4$: Finite reduced word $x = x_1 x_2 \ldots x_n$, $x_j \in A_2$

Length: $|x| = n$

Distance: For any $x, y \in \mathcal{T}_4$, $d(x, y) = |x^{-1}y|$

Random Quantum Walk on $\mathcal{T}_4$



Hilbert space; $\mathcal{K}_4 = \ell^2(\mathcal{T}_4) \otimes \mathbb{C}^4$

Shift on $\ell^2(\mathcal{T}_4)$: $S_\tau|x\rangle = |x\tau\rangle$

Spin dependent shift on $\mathcal{K}_4$:

$$S = S_a \otimes |a\rangle\langle a| + S_b \otimes |b\rangle\langle b| + S_{a^{-1}} \otimes |a^{-1}\rangle\langle a^{-1}| + S_{b^{-1}} \otimes |b^{-1}\rangle\langle b^{-1}|$$

RQW: $U_\omega(C) = \mathbb{D}(\omega)S(\mathbb{I} \otimes C)$

Landmarks in $U(4)$

- $C_{(a)(b)(a^{-1})(b^{-1})} = \mathbb{I}$ $\Rightarrow$ a.c. $U_\omega(\mathbb{I}) \simeq S$

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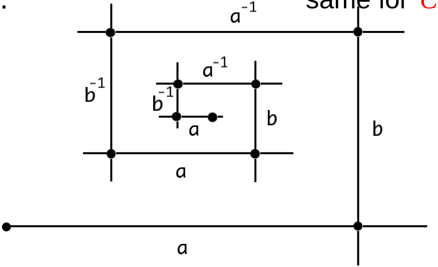
- $C_{(aba^{-1}b^{-1})} \Rightarrow$ a.c. same for $C_{(ab^{-1}a^{-1}b)}$



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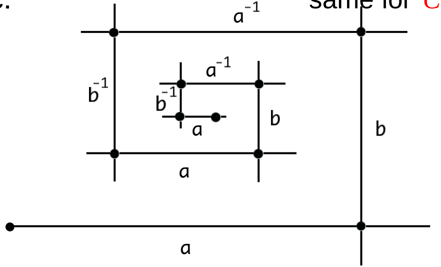

- $C_{(aa^{-1})(bb^{-1})}$ $\Rightarrow$ p.p same for $C_{(abb^{-1}a^{-1})}$, ...

$$\mathcal{H}_x^a = \text{span}\,\{x \otimes a, xa^{-1} \otimes a^{-1}\}$$

$$\mathcal{H}_x^b = \text{span}\,\{x \otimes b, xb^{-1} \otimes b^{-1}\}$$

$C = C_a \oplus C_b$ on span $\{|a\rangle, |a^{-1}\rangle\} \oplus$ span $\{|b\rangle, |b^{-1}\rangle\}$

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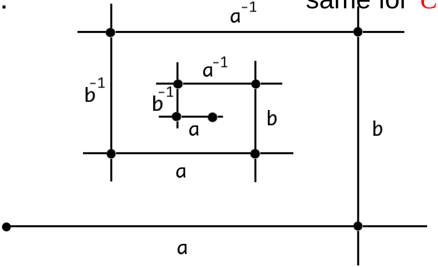

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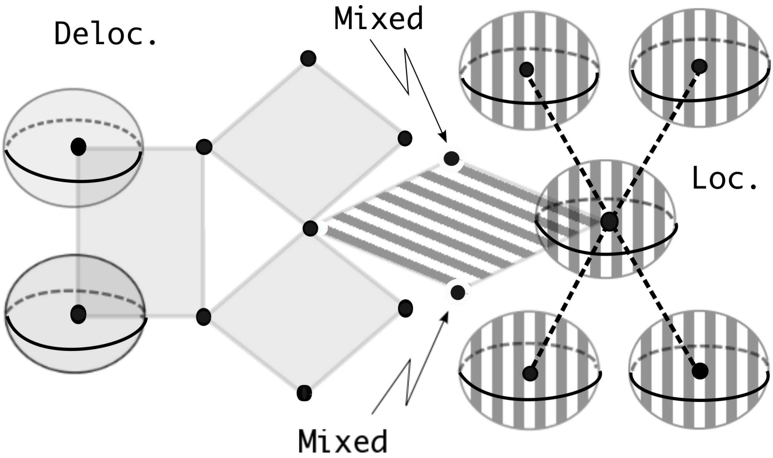
$$\mathcal{H}_x = \{x \otimes a, xb \otimes b, x \otimes b^{-1}, xa^{-1} \otimes a^{-1}\},$$

Spectral Phase Diagram



Neighborhood of $C_{(aba^{-1}b^{-1})}$

Proposition

There exists $\epsilon > 0$. *Then, for any* $C \in U(4)$, $\|C - C_{(aba^{-1}b^{-1})}\| \leq \epsilon$ *implies for any* $\omega \in \Omega$

$$\sigma(U_\omega(C)) = \sigma_{ac}(U_\omega(C)).$$

Idea of Proof:

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- Spectral criteria

$$\mathcal{H}^{ac}(U) = \overline{\left\{\phi \mid \sum_{n \in \mathbf{N}} |\langle \phi | U^n \phi \rangle|^2 < \infty \right\}}$$

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- Path counting argument to show $|\langle x \otimes \tau | U_\omega^{2n}(C) x \otimes \tau \rangle| \leq (\kappa\epsilon)^n$

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Remark: This result is deterministic and can be generalized for all q even

Neighborhoods of $C_{(abb^{-1}a^{-1})}$ and $C_{(aa^{-1}bb^{-1})}$, ...

Theorem

Let $\pi \in \{(abb^{-1}a^{-1}), (aa^{-1}bb^{-1}), (aa^{-1}b^{-1}b), (aa^{-1})(bb^{-1}), (ab^{-1}ba^{-1})\}$ *there exists* $\epsilon > 0$ *such that for all* $C \in U(4)$ *with* $\|C - C_\pi\| \leq \epsilon$

$$\sigma(U_\omega(C)) = \sigma_{pp}(U_\omega(C)) \text{ almost surely.}$$

Proof: Fractional Moment Method [Aizenman-Molchanov 93 , H.-Joye-Stolz 09]

Fractional Moment Estimate

Theorem

For all $0 < s < 1/3$*, and all* $\gamma > 0$*, there exist* $K(s,\gamma) < \infty$ *and* $\epsilon(s,\gamma) > 0$ *such that for all* $C \in U(q)$ *with* $\|C - C_\pi^\Phi\| \leq \epsilon(s,\gamma)$*, all* $x, y \in \mathcal{T}_q$ *with* $d(x,y) > 2$*, all* $z \notin \mathbb{U}$*, and all* $\tau, \sigma \in A_q$*,*

$$\mathbb{E}(|\langle x \otimes \tau| \, (U_\omega(C) - z)^{-1} y \otimes \sigma\rangle|^s) \leq K(s,\gamma) e^{-\gamma d(x,y)}.$$

Thank you