Notes for the lecture: PDE and Modelling
Preliminary version: lectures 1-24

Summer semester 2016, University of Bonn
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This is a short summary of the topics discussed in the lectures and cannot replace a careful study of the literature. Relevant references include

- H. Brezis, Functional Analysis, Sobolev spaces and partial differential equations, Springer 2010
- L.C. Evans, Partial Differential Equations, American Math. Soc. 1998

These notes are based on lecture notes from courses taught by B. Niethammer (summer term 2014), S. Müller (summer term 2013) and S. Conti (summer term 2012). They are only intended for the students of the course V3B2/F4B1 "PDE and modelling" at the university of Bonn in the summer term 2016. Please send typos and corrections to disertori(at)iam.uni-bonn.de

Plan:

1. Chapter 1. Basic notions of continuum mechanics
2. Chapter 2. PDEs and techniques in fluid mechanics
3. Chapter 3. solid mechanics, calculus of variations

Contents

1. Basic notions of continuum mechanics

[Lecture 1: 13.04]

1.1 Introduction: modelling a physical system

goal: translate a physical problem in fluid or solid mechanics into a system of PDEs

Examples: a) flow of a fluid around an obstacle or inside a pipe, b) deformations in a solid (ex. a sponge)

Physical description:

- microscopic level: atoms/molecules. There are $\sim 10^{26}$ equations to study, very hard!
- macroscopic level: see the material as a continuous object $\rightarrow$ continuous mass distribution

Notation: For partial derivatives we will use alternatively one of the following three notations: $\frac{\partial}{\partial x_j} = \partial_{x_i} = \partial_i$.

1.1.1 From physics to PDE

Physical description:

1. identify the mathematical object to study
2. use physical laws+assumptions on the material to extract a system of PDEs
3. boundary conditions (how the sample interacts with the external world)
4. intial conditions

Example: fluid flowing around an obstacle (a stone in water, an aiplane in air)
1. The obstacle is described by a compact region $K \subset \mathbb{R}^3$.
The fluid motion is described by $\vec{u}(x,t)$ = velocity of a small portion of fluid near the position $\vec{x} \in \mathbb{R}^3 \backslash K$ at time $t > 0$.
Goal: find u.
2. physical laws+incompressible fluid+other assumptions $\rightarrow$ Navier-Stokes equation

$$\begin{cases} \rho_0 \left[\partial_t \vec{u} + \vec{u} \cdot \vec{\nabla} \vec{u} \right] = -\vec{\nabla} p + \eta \Delta \vec{u} & \\ \operatorname{div}[\vec{u}] = 0 & x \in \mathbb{R}^3 \backslash K, \ t \geq 0 \end{cases} \tag{1.1.1}$$

where $\rho_0 > 0$ (mass density) and $\eta > 0$ (dynamical viscosity) are constants, $\vec{u} : \mathbb{R}^3 \backslash K \to \mathbb{R}^3$ is the velocity field, $p : \mathbb{R}^3 \backslash K \to \mathbb{R}$ is the pressure field,

$$(\Delta \vec{u})_i = \Delta \vec{u}_i := \sum_{j=1}^{3} \left(\frac{\partial}{\partial x_j} \right)^2 u_i \tag{1.1.2}$$

$$[\vec{u} \cdot \vec{\nabla} \vec{u}]_i := \sum_{j=1}^{3} \left[u_j \frac{\partial}{\partial x_j} \right] u_i \tag{1.1.3}$$

$$\mathrm{div}[\vec{u}] := \vec{\nabla} \cdot \vec{u} = \sum_{j=1}^{3} \frac{\partial}{\partial x_j} u_j \tag{1.1.4}$$

$$[\vec{\nabla} p]_i := \frac{\partial}{\partial x_j} p \tag{1.1.5}$$

3. There are two boundaries: ∂K and infinity. The two most standard boundary conditions are

- Dirichlet: fix the value of u on ∂K. Example: $u_{|\partial K} = 0$ (no slip)
- Neumann: fix the normal deivative $\nabla_N u$ on ∂K (i.e. derivative in the direction perpendicular to the boundary).

To fix the fluid behavior at infinity, we assume the fluid moves at constant speed $\vec{u}_\infty$, when no obstacle is present. The obstacle creates a perturbation only in the vicinity of K, then

$$\lim_{|x| \to \infty} \vec{u}(x,t) = \vec{u}_\infty \quad \forall t \geq 0. \tag{1.1.6}$$

4. initial condition: $\vec{u}(x,0) = \vec{u}_0(x)$ (initial velocity field).

1.1.2 From PDE to physics

Dimensional analysis.

To check if a PDE is 'physically meaningful' we evaluate the dimension of each term and check if they coincide.
The basic dimensions are: length L, mass M and time T. We denote the dimension of a quantity by $[quantity]$.
Some important dimensions ($d = 3$):

$$\begin{aligned} &[area] = L^2, \quad [velocity] = \frac{L}{T}, \quad [force] = [mass \cdot acceleration] = \frac{ML}{T^2} \\ &[pressure] = \frac{[force]}{[area]} = \frac{M}{LT^2}, \quad [mass\ density] = \frac{[mass]}{[volume]} = \frac{M}{L^3} \end{aligned} \tag{1.1.7}$$

Application to NS, eq. (1.1.1)

$$\begin{aligned} &[\vec{\nabla} p] = \frac{[p]}{L} = \frac{M}{L^2 T^2}, \quad [\partial_t \vec{u}] = \frac{[u]}{[T]} = \frac{L}{T^2} = \frac{[u]^2}{L} = [\vec{u} \cdot \vec{\nabla} \vec{u}] \\ &\Rightarrow [\rho_0 \partial_t \vec{u}] = \frac{M}{L^3} \frac{L}{T^2} = \frac{M}{L^2 T^2} = [\vec{\nabla} p] \end{aligned}$$

Finally $[\Delta u] = \frac{[u]}{L^2} = \frac{1}{LT}$. Then $[\eta] = \frac{M}{LT}$.

Similarity principle

Typical problem: we check the validity of a PDE in some experimental lab on a small scale sample. How do we find the PDE describing a real size (generally larger) sample?

Geometry. Two triangles are similar if all the corresponding adimensional parameters (i.e. the angles) coincide.

Physical process. Two physical processes are similar if the corresponding adimensional parameters coincide.

Compare the PDEs for two physical processes. We need to write the PDEs in adimensional form and then compare the adimensional parameters appearing in the equations.

PDE in adimensional form

Let us see how to write (1.1.1) in adimensional form. The variables in the functions are $\vec{x}$ (space dimension) and t (time dimension). We introduce the adimensional variables

$$\vec{y} := \frac{1}{x_0}\vec{x}, \qquad \text{with } x_0 > 0, [x_0] = L,$$
$$\tau := \frac{1}{t_0}t, \qquad \text{with } t_{0>0}, [t_0] = T,$$

where x_0, t_0 are fixed reference parameters. A natural choice for x_0 is the size of the obstacle: $x_0 := \text{diam}K$. Then $|y| \simeq 1$ when we are near K. At the moment t_0 is still free.
The functions appearing in the PDE become

$$\tilde{u}(y,\tau) := u(x(y), t(\tau)), \qquad u(x,t) = \tilde{u}(y(x), \tau(t))$$
$$\tilde{p}(y,\tau) := p(x(y), t(\tau)), \qquad p(x,t) = \tilde{p}(y(x), \tau(t))$$

where $\tilde{u}, \tilde{p}$ satisfy

$$\begin{cases} \rho_0 \left[\frac{1}{t_0}\partial_\tau \tilde{u} + \frac{1}{x_0}\tilde{u}\cdot\vec{\nabla}_y\tilde{u}\right] = -\frac{1}{x_0}\vec{\nabla}_y\tilde{p} + \eta\frac{1}{x_0^2}\Delta_y\tilde{u} \\ \text{div}_y[\tilde{u}] = 0 & \vec{y} \in \mathbb{R}^3 \backslash K/x_0, \tau \geq 0 \\ \lim_{|y|\to\infty} \tilde{u}(y,t) = u_\infty \end{cases} \tag{1.1.8}$$

Now y, τ are adimensional, but $\tilde{p}, \tilde{u}$ still have a dimension. We introduce the adimensional functions

$$\vec{v}(y,\tau) := \frac{1}{u_0}\vec{\tilde{u}}(y,\tau), \qquad \text{with } u_0 > 0, [u_0] = \frac{L}{T},$$
$$q(y,\tau) := \frac{1}{p_0}\tilde{p}(y,\tau), \qquad \text{with } p_0 > 0, [p_0] = \frac{M}{LT^2},$$

where u_0, p_0 are fixed reference parameters. A natural choice for u_0 is the velocity at infinity: $u_0 := |u_\infty|$ Then $|v(y,\tau)| \to 1$ when $|y| \to \infty$. At the moment p_0 is still free.
Replacing v, q in the PDE we get

$$\partial_\tau \vec{v} + \frac{u_0 t_0}{x_0}\vec{v}\cdot\vec{\nabla}_y\vec{v} = -\frac{p_0 t_0}{x_0\rho_0 u_0}\vec{\nabla}_y q + \frac{\eta t_0}{\rho_0 x_0^2}\Delta_y\vec{v}$$

where all functions and prefactors are adimensional. Note that

- the parameters x_0, u_0 are already fixed
- the parameters ρ_0, η are given (by the physical problem)
- the parameters t_0, p_0 are still free.

We can choose t_0 such that $u_0 = \frac{x_0}{t_0}$ and p_0 such that $\frac{p_0 t_0}{x_0 \rho_0 u_0} = \frac{p_0}{\rho_0 u_0^2} = 1$. Finally we obtain

$$\begin{cases} \partial_\tau \vec{v} + \vec{v} \cdot \vec{\nabla}_y \vec{v} = -\vec{\nabla}_y q + \frac{1}{Re} \Delta_y \vec{v} \\ \mathrm{div}_y[\vec{v}] = 0 \\ \lim_{|y| \to \infty} \vec{v}(y,t) = u_\infty \end{cases} \qquad \vec{y} \in \mathbb{R}^3 \backslash K/x_0, \tau \geq 0 \tag{1.1.9}$$

where $Re := \frac{\rho_0 u_0 x_0}{\eta}$ is the Reynolds number. Note that the only information on the physical system is now *only* in the Reynolds number. Therefore any system with the same Re is described by the same PDE system.

[Lecture 2: 15.04]

1.2 Point mechanics

True particles occupy a finite volume, but it is often convenient to describe them as point-particles located at the center of mass. This holds also for large objects (ex a planet).
We consider a system of N particles with masses $m_i > 0$, $i = 1, \dots, N$ moving in $\mathbb{R}^d$ ($d = 1, 2, 3$).

1.2.1 Kinematics

The movement is described by the positions $\vec{x_i}(t) \in \mathbb{R}^d$ = position of the $i-$the particle at time $t \geq 0$, $i = 1, \dots n$. We also define $\vec{V_i}(t) = \vec{x_i}'(t) \in \mathbb{R}^d$ = velocity of the $i-$the particle at time $t \geq 0$, $i = 1, \dots n$.
If no force acts on the particles, they will move forever at constant speed

$$\vec{V_i}(t) = \vec{V_i}(0) \quad \forall t$$
$$\vec{x_i}(t) = \vec{x_i}(0) + t\vec{V_i}(0) \quad \forall t.$$

The trajectoris are lines.

1.2.2 Dynamics

To modify the trajectories we apply forces. we consider two types of forces acting on the particle i:

1. external force: $\vec{f_i}(t)$
2. (internal) force generated on i due to the particle $j = \vec{f_{ij}}(t)$.

Newton 2nd law. $force = mass \cdot accelaration$. The **equation of motion** for the $i-$th particle is then given by

$$m_i \vec{x_i}''(t) = \vec{p_i}'(t) = \vec{f_i}(t) + \sum_{j, j \neq i} \vec{f_{ij}}(t) \tag{1.2.10}$$

where we defined $\vec{p_i}(t) = m_i \vec{x}_i'(t)$ as the momentum of the particle i.

Newton 3rd law. The force on i due to j equals - force on j due to i, i.e.

$$\vec{f_{ij}}(t) = -\vec{f_{ji}}(t) \qquad \forall t, \ \forall i \neq j. \tag{1.2.11}$$

Then f_{ij} must be of the form

$$\vec{f_{ij}} = \frac{\vec{x}_i - \vec{x}_j}{|\vec{x}_i - \vec{x}_j|} g_{ij}(|\vec{x}_i - \vec{x}_j|) \tag{1.2.12}$$

with $g_{ij}(r) = g_{ji}(r)$.
Two important examples are

1. gravitational force: $g_{ij}(r) := -G\frac{m_i m_j}{r^2}$, G = gravitational constant

2. electrical force: $g_{ij}(r) := +K\frac{Q_i Q_j}{r^2}$, Q_i = charge of i-th particle

1.2.3 Conservation laws

In the following we consider relations on the whole system of N particles, rather than on each single particle.

Conservation of (linear) momentum

Let

$$\vec{p}(t) := \sum_{i=1}^{N} \vec{p}_i(t), \qquad \vec{f}(t) := \sum_{i=1}^{N} \vec{f}_i(t)$$

the total momentum and external force respectively. Then

$$(\mathbf{c1}) \qquad \vec{p}\,'(t) = \vec{f}(t) \tag{1.2.13}$$

Remark. Only the external forces play a role here!

Proof.

$$\vec{p}\,'(t) = \sum_{i=1}^{n} \vec{p}_i'(t) = \sum_{i=1}^{n} \vec{f}_i(t) + \sum_{i=1}^{n} \sum_{j, j\neq i} \vec{f}_{ij}(t) = \vec{f}(t) + \frac{1}{2}\sum_{j\neq i}(\vec{f}_{ij} + \vec{f}_{ji}) = \vec{f}(t),$$

where in the first step we used Newton's 2nd law and in the third Newton's 3rd law. □

Conservation of angular momentum ($d = 3$)

Motivation: the total force does not contain enough information. As an example consider a horizontal rod of length $2l > 0$ with the two extremities at positions $\vec{x}_1 = (-l, 0, 0)$, $\vec{x}_2 = (l, 0, 0) = -\vec{x}_1$. We consider two force configurations.

- Configuration 1: we apply the forces $\vec{F}_1 = (0, 0, -F)$, $\vec{F}_2 = \vec{F}_1$ $\vec{F}_3 = (0, 0, 2F) = -\vec{F}_1$ at positions $\vec{x}_1$, $\vec{x}_2$ and $\vec{x}_3 = 0$ respectively.
- Configuration 2: we apply the forces $\vec{F}_1 = (0, 0, F)$, $\vec{F}_2 = -\vec{F}_1 = (0, 0, -F)$ at positions $\vec{x}_1$, and $\vec{x}_2$ respectively.

Though in both configurations the total force is zero, in the second the system rotates.

Definitions and conservation law Let $\vec{x}_0$ be a fixed reference point, $\vec{L}_i(t) := (\vec{x}_i - \vec{x}_0) \times \vec{p}_i$ be the angular momentum of the $i-$th particle and $\vec{M}_i(t) := (\vec{x}_i - \vec{x}_0) \times \vec{f}_i$ the angular momentum of the external force (torque) acting on i. Let

$$\vec{L}(t) := \sum_{i=1}^{N} \vec{L}_i(t), \qquad \vec{M}(t) := \sum_{i=1}^{N} \vec{M}_i(t)$$

the total angular momentum and total torque respectively. Then

$$(\mathbf{c2}) \qquad \vec{L}\,'(t) = \vec{M}(t) \tag{1.2.14}$$

Proof.

$$\begin{aligned}
\vec{L}'(t) =& \sum_{i=1}^{n} \vec{L}_i'(t) = \sum_{i=1}^{n} [(\vec{x}_i - \vec{x}_0) \times \vec{p}_i]' \\
=& \sum_{i=1}^{n} [\vec{x}_i' \times \vec{p}_i + (\vec{x}_i - \vec{x}_0) \times \vec{p}_i'] \\
=& \sum_{i=1}^{n} m_i \vec{x}_i' \times \vec{x}_i' + \sum_{i=1}^{n} (\vec{x}_i - \vec{x}_0) \times \vec{f}_i + \frac{1}{2} \sum_{j \neq i} (\vec{x}_i - \vec{x}_j) \times \vec{f}_{ij} \\
=& \vec{M}(t) + \frac{1}{2} \sum_{j \neq i} \frac{g_{ij}(|\vec{x}_i - \vec{x}_j|)}{|\vec{x}_i - \vec{x}_j|} (\vec{x}_i - \vec{x}_j) \times (\vec{x}_i - \vec{x}_j)
\end{aligned}$$

□

Remark. If we change reference point from $\vec{x}_0$ to $\vec{x}_0'$, the the total torque (resp. the angular momentum) changes to $\vec{M}(\vec{x}_0') = \vec{M}(\vec{x}_0) + (\vec{x}_0 - \vec{x}_0') \times \vec{f}$, while $\vec{L}(\vec{x}_0') = \vec{L}(\vec{x}_0) + (\vec{x}_0 - \vec{x}_0') \times \vec{p}$. Hence (**c2**) holds for any choice of the reference point x_0

Reminder: properties of the cross (vector) product The cross product is a bilinear operation defined by

$$\begin{aligned}
\times : \quad & \mathbb{R}^3 \times \mathbb{R}^3 && \to \mathbb{R}^3 \\
& (\vec{a}, \vec{b}) && \to \vec{a} \times \vec{b} = |a||b| \sin\theta \vec{n}
\end{aligned}$$

where $\theta \in [0, \pi)$ is the angle between $\vec{a}$ and $\vec{b}$, and $\vec{n}$ is a unit vector orthogonal to both vectors, with direction given by the 'right hand rule'. Alternatively one may write

$$(\vec{a} \times \vec{b})_i = \sum_{jk} \epsilon^{ijk} a_j b_k$$

where ϵ^{ijk} is the Levi-Civita symbol. The cross product is antysymmetric $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$. In particular $\vec{a} \times \vec{a} = 0$.

An example. Let us consider the example given above in the two configurations.
Configuration 1. The total force $\vec{f} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0$, hence by (**c1**) $p' = 0$ (the system does not drift). To compute the angular momentun and the torque we set $x_0 = 0$. Indeed, since $\vec{f} = 0$ the torque is independent of the choice oif x_0. Then $\vec{M} = \vec{x}_1 \times \vec{F}_1 + \vec{x}_2 \times \vec{F}_2 + \vec{x}_3 \times \vec{F}_3 = 0$. Hence $\vec{L}' = 0$ (no rotation).
Configuration 2. The total force $\vec{f} = \vec{F}_1 + \vec{F}_2 = 0$, hence by (**c1**) $p' = 0$ (no drift). As before we can set $x_0 = 0$. Then $\vec{M} = \vec{x}_1 \times \vec{F}_1 + \vec{x}_2 \times \vec{F}_2 = 2\vec{x}_1 \times \vec{F}_1 \neq 0$. Hence $\vec{L}' \neq 0$ (we have rotation).

Conservation of energy

Energy is 'stored' in the movement of particles (kinetic energy) and the internal forces (potential energy).

Potential energy. We assume the potential energy can be modified only through internal forces. To quantify this assumption we introduce the notion of work.
We define the *work per unit time* done by the force $\vec{F}_i$ on the particle i by $W_i := \vec{F}_i \cdot \vec{x}_i'$. Its dimension is $[W] = [force] \cdot L/T$. In our case $\vec{F}_i = \vec{f}_i + \sum_{j, j \neq i} \vec{f}_{ij}$. Hence

$$W_i = W_i^{ext} + W_i^{int}, \qquad W_i^{ext} = \vec{f}_i \cdot \vec{x}_i', \qquad W_i^{int} = \vec{x}_i' \cdot [\sum_{j, j \neq i} \vec{f}_{ij}],$$

Let $W_{ext} = \sum_i W_i^{ext}$ be the total work per unit time by external forces and $W_{int} = \sum_i W_i^{int}$ be the total work per unit time by internal forces. The the statement above reads

$$E_p'(t) = -W_{int}(t). \tag{1.2.15}$$

By Newton 3rd law, this identity implies that E_p must be of the form

$$E_p = -\frac{1}{2}\sum_{i\neq j} G_{ij}(|\vec{x}_i - \vec{x}_j|) \tag{1.2.16}$$

where, if $\vec{f}_{ij} = \frac{\vec{x}_i - \vec{x}_j}{|\vec{x}_i - \vec{x}_j|} g_{ij}(|\vec{x}_i - \vec{x}_j|)$, with $g_{ij}(r) = g_{ji}(r)$, then $G_{ij}(r)$ is a primitive of g_{ij}. This follows from

$$\begin{aligned} E_p' = -W_{int} &= -\sum_i \sum_{j, j\neq i} \vec{x}_i' \cdot \vec{f}_{ij} = -\frac{1}{2}\sum_{i\neq j}(\vec{x}_i' \cdot \vec{f}_{ij} + \vec{x}_j' \cdot \vec{f}_{ji}) \\ &= -\frac{1}{2}\sum_{i\neq j}(\vec{x}_i' - \vec{x}_j') \cdot \vec{f}_{ij}) = -\frac{1}{2}\sum_{i\neq j}(\vec{x}_i' - \vec{x}_j') \cdot \frac{\vec{x}_i - \vec{x}_j}{|\vec{x}_i - \vec{x}_j|} g_{ij}(|\vec{x}_i - \vec{x}_j|) \\ &= -\frac{1}{2}\sum_{i\neq j} \frac{d}{dt}|\vec{x}_i - \vec{x}_j| g_{ij}(|\vec{x}_i - \vec{x}_j|) = \left[-\frac{1}{2}\sum_{i\neq j} G_{ij}(|\vec{x}_i - \vec{x}_j|)\right]'. \end{aligned}$$

Kinetic energy.

$$E_k := \sum_{i=1}^N \frac{m_i |\vec{x}_i'(t)|^2}{2} = \sum_{i=1}^N \frac{|\vec{p}_i|^2}{2m_i}.$$

As E_p, also E_k is related to the work. Precisely we have

$$E_k'(t) = W_{ext} + W_{int}. \tag{1.2.17}$$

This follows immediately from

$$E_k' = \frac{d}{dt}\sum_{i=1}^N \frac{m_i \vec{x}_i' \cdot \vec{x}_i'}{2} = \sum_{i=1}^N m_i \vec{x}_i'' \cdot \vec{x}_i' = \sum_{i=1}^N F_i \cdot \vec{x}_i' = W_{ext} + W_{int}.$$

Conservation law. Let $E(t) = E_k(t) + E_p(t)$ the total energy of the particle system. Then (1.2.15)+(1.2.17) imply

$$(\mathbf{c3}) \qquad E'(t) = W_{ext}. \tag{1.2.18}$$

[Lecture 3: 20.04]

1.3 Continuum mechanics: kinematics

1.3.1 From particles to continuum

Let $N \gg 1$ and $C_h(x)$ be a cube of side $h > 0$ centered at $x \in \mathbb{R}^d$. We define the fraction of mass (resp. momentum) inside $C_h(x)$ by

$$\rho_h(t, \vec{x}) := \frac{1}{|C_h|} \sum_{i, \vec{x}_i(t) \in C_h} m_i, \qquad \vec{p}_h(t, \vec{x}) := \frac{1}{|C_h|} \sum_{i, \vec{x}_i(t) \in C_h} \vec{p}_i$$

where $|C_h| = h^d$ is the volume of the cube. As $N \to \infty$ and $h \to 0$ (in the appropriate way) $\rho_h(t, \vec{x}) \to \rho(t, \vec{x})$, $\vec{p}_h(t, \vec{x}) \to \vec{p}(t, \vec{x})$ mass and momentum density.

Remark. The velocity fraction is defined through $\vec{v}_h := \frac{1}{\rho_h}\vec{p}_h \neq \frac{1}{|C_h|}\sum_{i,\vec{x}_i(t)\in C_h}\vec{v}_i$.

Dictionary. *initial position* of the N particles $\vec{X}_i = \vec{x}_i(0)$, $i = 1, \dots N$ $\to$ region occupied by the material at time 0 $\Omega \subseteq \mathbb{R}^d$ open and connected
trajectory of particle i located at $\vec{X}_i$ at time 0 is $\vec{x}_i(\vec{X}_i, t)$ $\to$ trajectory of a small portion of material located at $\vec{X} \in \Omega$ at time 0 $\vec{x}(t, \vec{X})$.

1.3.2 Deformations

For N particles the allowed configurations (at each time t) are the N-tuples $(x_1, \dots ., x_N) \in \mathbb{R}^{Nd}$ such that $x_i \neq x_j$ for all $i \neq j$ (particles do not overlap).
For a continuous body, the possible configurations are all possible deformations (translations, rotations, stretching...) of a reference configuration Ω.

Definition 1 (Deformation.) *Let $\Omega \subseteq \mathbb{R}^d$ be a domain (i.e. open and connected) and let $k \geq 1$. A C^k- deformation (or simply a deformation) is a map $\varphi : \Omega \to \mathbb{R}^d$ such that*

1. $\varphi \in C^k(\Omega; \mathbb{R}^d)$,
2. *φ has a continuous extension to $\bar{\Omega}$ and this extension is invertible*
3. *φ preserves orientations i.e.* $\det D\varphi(x) > 0$ $\forall x \in \Omega$.

where $D\varphi(x) \in \mathbb{R}^{d\times d}$ is defined by $(D\varphi)_{ij}(x) := \frac{\partial \varphi_i}{\partial x_j}$. Ω is called the reference configuration.

Remark. $\varphi(x) \neq \varphi(y)$ $\forall x \neq y$ is the analog of non-overlapping particles
Moreover, since φ is invertible, we have $\det D\varphi(x) \neq 0$ for all x, then it is either always positive or always negative. Taking $\det D\varphi > 0$ allows to include the case $\varphi = Id$ and excludes flipping.

A special class of deformations is given by translations and rotations.

Definition 2 (Rigid deformation) *A deformation $\varphi : \Omega \to \mathbb{R}^d$ is called a rigid deformation if $D\varphi(x) \in SO(d)$ for all $x \in \Omega$.*

Reminder. $SO(d) = \{A \in \mathbb{R}^{d\times d} | A^T A = Id, \text{ and } \det A = 1\}$.
For any $A \in SO(d)$, the linear map $T_A \in \mathcal{L}(\mathbb{R}^d)$ defined by $T_A x = Ax$ is an isometry.

Example. An affine map $\varphi(x) = Ax + b$, with $A \in SO(d)$ and $b \in \mathbb{R}^d$ is a motion iff $\det D\varphi = \det A > 0$. If in addition $A \in SO(d)$ this is an affine rigid motion. The following theorem proves that all rigid motions are affine.

Theorem 1 (Liouville) *Let Ω be a domain and $\varphi : \Omega \to \mathbb{R}^d$ a deformation. The following statements are equivalent.*

(i) *φ is a rigid deformation.*

(ii) *φ is a rigid affine deformation i.e. $\exists b \in \mathbb{R}^d$ and $A \in SO(d)$ s.t. $\varphi(x) = Ax + b$.*

(iii) *$\forall y, z \in \Omega$ it holds $|\varphi(x) - \varphi(y)| = |x - y|$.*

(ii)' *(local version of (ii)) φ is a local rigid deformation i.e. $\forall x \in \Omega$ $\exists r > 0$, $b \in \mathbb{R}^d$ and $A \in SO(d)$ s.t. $\varphi(y) = Ay + b$ $\forall y \in B(x, r)$.*

(iii)' *(local version of (iii)) $\forall x \in \Omega$ $\exists r > 0$, s.t. $|y - z| = |\varphi(y) - \varphi(z)|$ $\forall y, z \in B(x, r)$.*

Proof (exercise) We prove first that $(iii)' \Rightarrow (ii)' \Rightarrow (ii) \Rightarrow (i) \Rightarrow (iii)'$. To conclude we will prove $(ii) \Rightarrow (iii) \Rightarrow (iii)'$.
$(iii)' \Rightarrow (ii)'-$ Applying $\partial_{y_i}\partial_{z_j}$ to $|y-z|^2 = |\varphi(y)-\varphi(z)|^2$ we get

$$I = (D\varphi(y))^T(D\varphi(z)) \quad \forall y, z \in B(x,r)$$

Taking $y = z$ this gives $D\varphi(y) \in SO(d)\ \forall y$, and taking now all $z \in B(x,r)$ we have $D\varphi(z) = ((D\varphi(y))^T)^{-1} =$ constant matrix. Finally from

$$\varphi(z) - \varphi(y) = \int_0^1 (D\varphi)(y + s(z-y))(z-y)ds \tag{1.3.19}$$

we deduce $(ii)'$
$(ii)' \Rightarrow (ii) \Rightarrow (i)$ obvious.
$(i) \Rightarrow (iii)'$. Using (1.3.19) and $D\varphi \in SO(d)$ (hence $T_{D\varphi}$ is an isometry) we obtain the inequality $|\varphi(z) - \varphi(y)| \leq |z-y|$. To obtain the other inequality let $\psi = \varphi^{-1}$ the inverse function. Since $D\psi = (D\varphi)^{-1} \circ \psi$, we have $D\psi \in SO(d)$ too. Applying the inequality above to ψ we obtain the result.
$(ii) \Rightarrow (iii)$ since $\varphi(y) - \varphi(z) = |Ay - Az| = |y-z|$ since T_A is an isometry.
Finally $(iii) \Rightarrow (iii)'$ is obvious. □

Theorem 2 *Let $\Omega \subset \mathbb{R}^d$ be a domain and let $\varphi : \Omega \to \mathbb{R}^d$ a deformation. Then for any measurable set $U \subseteq \Omega$ and for any $g \in L^1(\varphi(U))$ we have*

$$\int_{\varphi(U)} g(x)dx^n = \int_U g(\varphi(X))(\det D\varphi(X))\ dX^n \tag{1.3.20}$$

Proof. change of coordinates. □

1.3.3 Motions

Definition 3 (Motion) *Let $\Omega \subset \mathbb{R}^d$ be a domain. A C^3 map $x : \mathbb{R} \times \Omega \to \mathbb{R}^d$ is a* <u>motion</u> *if for each $t \in \mathbb{R}$ the map $x_t := x(t,\cdot)$ is a deformation.*

Eulerian and Lagrangian coordinates. We can describe the body motion
- as a function of (t, X), where $X \in \Omega$ is the (initial) position in the reference configuration. These are called Lagrangian (or material) coordinates;
- as a function of (t, x), where $x = x(t, X)$ is the position at time t of a small portion of the body near X at $t = 0$. These are called Eulerian (or spatial) coordinates.
To make this precise we need some definitions.

Definition 4 *Let $x : \mathbb{R} \times \Omega \to \mathbb{R}^d$ be a motion. We define*

1. *$\Omega_t := x_t(\Omega)$ the region occupied by the body at time t,*
2. *$\mathcal{T} := \{(t,x) : t \in \mathbb{R},\ x \in \Omega_t\}$ the trajectory of the body in space and time,*
3. *$x_t^{-1} : \Omega_t \to \Omega$ the reference map at time t (also called the back-to-labels map),*
4. *$x^{-1} : \mathcal{T} \to \mathbb{R} \times \Omega$ $(t,x) \to (t, x_t^{-1}(x))$ the reference map.*

Definition 5 (spatial and material fields) *Let $x : \mathbb{R} \times \Omega \to \mathbb{R}^d$ be a motion.*

1. *A map $\Phi : \mathbb{R} \times \Omega \to \mathbb{R}^m$, $m \geq 1$ of Lagrangian coordinates $(t, X) \to \Phi(t, X)$ is called a material field.*
2. *A map $\varphi : \mathcal{T} \to \mathbb{R}^m$, $m \geq 1$ of Eulerian coordinates $(t, x) \to \varphi(t, x)$ is called a spatial field.*

Material fields are often denoted by capital letters while spatial fields by non capital letters. One may also write φ_s, φ_m to indicate if we work in spatial or material coordinates.

Definition 6 *We can relate spatial and material fields as follows.*

1. *Let $\varphi : \mathcal{T} \to \mathbb{R}^m$, be a spatial field. The map $\varphi_m : \mathbb{R} \times \Omega \to \mathbb{R}^m$ defined by $\varphi_m(t, X) := \varphi(t, x(t, X))$ is called the material description of φ.*
2. *Let $\Phi : \mathbb{R} \times \Omega \to \mathbb{R}^m$, be a material field. The map $\Phi_s : \mathcal{T} \to \mathbb{R}^m$ defined by $\Phi_s(t, x) := \Phi(t, \Phi_t(x))$ is called the spatial description of Φ.*

Examples. The motion $x : \mathbb{R} \times \Omega \to \mathbb{R}^d$ is a material field. The reference map $x^{-1} : \mathcal{T} \to \mathbb{R} \times \Omega$ is a spatial field. The function $\partial_X x(t, X) \in \mathbb{R}^{d \times d}$ is a material field and is called the deformation gradient. We will often use the velocity field in spatial coordinates defined by

$$v(t, x) := V_s(t, x) = [\partial_t x(t, X)]_{|X = x^{-1}(t,x)} .$$

Trajectories and streamlines

Lemma 1 *Let $v(t, x)$ be a given spatial field, and $X \in \Omega$ a fixed reference point. Then the motion starting at point X (i.e the function $x_X : \mathbb{R} \to \mathcal{T}$, with $x_X(t) = x(t, X)$) compatible with the velocity field (in spatial coordinates) v is a solution $y : \mathbb{R} \to \mathcal{T}$ of the (nonlinear) ODE*

$$y'(t) = v(t, y(t)).$$

Proof. Indeed

$$y'(t) = x'_X(t) = \partial_t x(t, X) = V(t, X) = v(t, x(t, X)) = v(t, y(t)).$$

□

Definition 7 *Let $v(t, x)$ a given spatial field.*

1. *We call trajectory a solution of the ODE*
$$y'(t) = v(t, y(t)). \tag{1.3.21}$$
2. *We call streamline a solution of the ODE*
$$z'(s) = v(t, z(s)), \tag{1.3.22}$$
where t is a fixed parameter.

Remark. If the velocity field is constant wrt t, the trajectory and stremline coincide. Indeed the streamline obtained from $v(t_0, x)$ is the trajectory of a body with constant velocity field $v(t, x) = v(t_0, x)\ \forall t$.

[Lecture 4: 22.04]

Time derivative and Reynolds transport theorem

Definition 8 (time derivative) *Let $\varphi : \mathcal{T} \to \mathbb{R}^m$, be a spatial field. The time derivative of φ is the usual partial derivative*

$$\frac{\partial \varphi}{\partial t}(t, x).$$

The material time derivative of φ is the partial derivative in time of the material description of φ, evaluated then in spatial coordinates. Precisely:

$$\frac{D\varphi}{Dt}(t, x) := [\partial_t \varphi_m]_s (t, x)$$

Lemma 2 *Let $\varphi : \mathcal{T} \to \mathbb{R}^m$, be a spatial field. The material time derivative can be written as*

$$\frac{D\varphi}{Dt}(t,x) = \partial_t \varphi(t,x) + \vec{v} \cdot \vec{\nabla} \varphi(t,x) \tag{1.3.23}$$

where $\vec{v} \cdot \vec{\nabla}\varphi(t,x) := \sum_{l=1}^d v_l(t,x)\partial_{x_l}\varphi(t,x)$.

Proof. Using $\Phi(t,X) = \varphi(t, x(t,X))$ we have $\partial_t \Phi = (\partial_t \varphi)_m + \sum_l V_l (\partial_{x_l}\varphi)_m$. Going back to spatial coordinates, the result follows. □

Definition 9 *We define a <u>test volume</u> U as a (small) portion of the body in the reference configuration. Precisely $U \subset \Omega$ is open, connected, finite and with C^1 (or piecewise C^1 boundary).*

Let U be a test volume and $\varphi(t,x)$ some given spatial field. Our goal is to study integrals of the form

$$\int_{U(t)} \varphi(t,X)dx,$$

where $dx = \prod_{l=1}^d dx_l$ is the product Lebesgue measure.

Key remark. By performing a coordinate change we have

$$\int_{U(t)} \varphi(t,X)dx = \int_U \varphi_m(t,X)\mathcal{J}(t,X)dX, \tag{1.3.24}$$

where $\varphi_m(t,X) = \varphi(t,x(t,X))$ and

$$\mathcal{J}(t,X) = \det\left[\partial_{X_j} x_i(t,X)\right]_{i,j=1}^d = \det(Dx) \tag{1.3.25}$$

is the Jacobian.
The main consequence of (1.3.24) is the following theorem.

Theorem 3 (Reynolds' transport theorem) *Let $x : \mathbb{R} \times \Omega \to \mathbb{R}^d$ be a motion and $\varphi : \mathcal{T} \to \mathbb{R}^m$ a C^1spatial field. Then for all $i = 1, \ldots, m$*

$$\frac{d}{dt}\left[\int_{U(t)} \varphi_i(t,x)dx\right] = \int_{U(t)} \left[\partial_t \varphi(t,x) + \operatorname{div}(\varphi_i \vec{v})(t,x)\right] dx \tag{1.3.26}$$

$$= \int_{U(t)} \left[D_t \varphi(t,x) + \varphi_i(t,x)\operatorname{div}(\vec{v})(t,x)\right] dx \tag{1.3.27}$$

where

$$\operatorname{div}(\vec{v})(t,x) := \sum_{l=1}^d \partial_{x_l} v_l(t,x) \tag{1.3.28}$$

$$\operatorname{div}(\varphi_i \vec{v})(t,x) := \sum_{l=1}^d \partial_{x_l}[\varphi_i v_l](t,x) \tag{1.3.29}$$

Proof. Using (1.3.24)

$$\frac{d}{dt}\left[\int_{U(t)} \varphi(t,x)dx\right] = \frac{d}{dt}\left[\int_U \varphi_m(t,X)\mathcal{J}(t,X)dX\right]$$

$$= \int_U [\partial_t \varphi_m(t,X)]\mathcal{J}(t,X)dX + \int_U \varphi_m(t,X)[\partial_t \mathcal{J}(t,X)]dX$$

Going back to spatial coordinates the first integral becomes

$$\int_U [\partial_t \varphi_m(t,X)]\mathcal{J}(t,X)dX = \int_{U(t)} [\partial_t \varphi_m]_s(t,x)dx = \int_{U(t)} D_t\varphi(t,x)dx$$

To compute the second integral we use (1.3.30) (see Lemma below) and obtain

$$\int_U \varphi_m(t,X)\mathcal{J}(t,X)[\mathrm{div}(\vec{v})]_m(t,X)dX = \int_{U(t)} \varphi(t,x)[\mathrm{div}(\vec{v})](t,x)dx.$$

Hence the result. □

Lemma 3 *We have*

$$\partial_t \mathcal{J}(t,X) = \mathcal{J}(t,X)[\mathrm{div}(\vec{v})]_m(t,X). \tag{1.3.30}$$

Proof. For any C^1 matrix-valued function $A : \mathbb{R}\times\Omega \to \mathbb{R}^{d\times d}$, with $A(t,X)$ invertible for all (t,X), we have

$$\partial_t(\det A)(t,X) = \sum_{ij} \partial_t A_{ij}\ \partial_{A_{ij}}(\det A) = \sum_{ij} \partial_t A_{ij}\ (A^{-1})_{ji}(\det A) = (\det A)\mathrm{tr}\left[A^{-1}\partial_t A\right]$$

where we used $\partial_{A_{ij}}(\det A) = (\mathrm{cof}A)_{ij} = (\det A)(A^{-1})_{ji}$, since A is invertible. Replacing now $A = Dx$ we have $\partial_t\mathcal{J}(t,X) = \mathcal{J}(t,X)\mathrm{tr}(A^{-1}\partial_t A)$, where

$$\begin{aligned}\partial_t A_{ij} &= \partial_t\partial_{X_j}x_i = \partial_{X_j}\partial_t x_i = \partial_{X_j}V_i(t,X)\\ &= \partial_{X_j}v_i(t,x(t,X)) = \sum_{l=1}^d \partial_{x_l}v_i(t,x(t,X))\ \partial_{X_j}x_l(t,X) = \sum_{l=1}^d (\partial_{x_l}v_i)_m\ A_{lj}.\end{aligned}$$

Hence

$$\mathrm{tr}(A^{-1}\partial_t A) = \sum_{ij}(A^{-1})_{ji}\partial_t A_{ij} = \sum_{ijl} A_{lj}(A^{-1})_{ji}(\partial_{x_l}v_i)_m = \sum_{il}\delta_{li}(\partial_{x_l}v_i)_m = \sum_l(\partial_{x_l}v_l)_m = (\mathrm{div}(v))_m.$$

□

Example. If $m = 1$ and φ is the constant function $\varphi(t,x) = 1$ for all t,x then $\int_{U(t)}\varphi(t,x)dx = \mathrm{Vol}(U(t))$ is the volume occupied by U at time t . Then by Reynolds transport theorem we have

$$\frac{d}{dt}\mathrm{Vol}(U(t)) = \int_{U(t)} \mathrm{div}(\vec{v})dx = \int_{\partial U(t)} \vec{v}(t,x)\cdot\vec{n}_x\ d\mathcal{H}^{d-1}(x)$$

where for each $x \in \partial U(t)$, $\vec{n}_x$ is the unit vector orthogonal to the boundary.

1.4 Continuum mechanics: conservation laws

1.4.1 Mass and momentum

We need to introduce now the notion of mass, momentum, angular momentum and force.

Mass Instead of having the mass concentrated on points, we assume the mass is now distributed uniformly on the volume occupied by the body, i.e. we replace the Dirac measure $\sum_{i=1}^N \delta_{x_i(t)}$ by a measure $dm(t,x) = \rho(t,x)dx$ absolutely continuous with respect to Lebesgue.

Definition 10 *A reference mass density (i.e. in the reference configuration Ω) is a function $\rho_0 \in L^1(\Omega)$ such that $\rho_0(X) \geq 0$ for all $X \in \Omega$. For each $U \subset \Omega$ open, $\int_U \rho_0(X)dX$ is the total mass inside U.*

Definition 11 *Let* $x : \mathbb{R} \times \Omega \to \mathbb{R}^d$ *be a motion. A* <u>*mass density*</u> *is a spatial field* $\rho : \mathcal{T} \to \mathbb{R}^+$ *such that* $\rho(t, \cdot) \in L^1(\Omega_t)$ *for all* t *and*

$$\int_{U(t)} \rho(t,x)dx = \int_U \rho_0(X)dX \tag{1.4.31}$$

for all $U \subset \Omega$ *open.*

The last identity means the mass of a piece of material (that can be seen as a 'particle') does not change in time.

Definition 12 *Let* $x : \mathbb{R} \times \Omega \to \mathbb{R}^d$ *be a motion and* $\rho : \mathcal{T} \to \mathbb{R}^+$ *a mass density.*
The (linear) <u>*momentum*</u> *of a test voume* $U \in \Omega$ *is a function* $l : \mathbb{R} \to \mathbb{R}^d$ *defined as*

$$l(U,t) = \int_{U(t)} \rho(t,x)v(t,x)dx. \tag{1.4.32}$$

The <u>*angular momentum*</u> *of a test voume* $U \in \Omega$ *is a function* $L : \mathbb{R} \to \mathbb{R}^{d\times d}$ *defined as*

$$L(U,t) = \int_{U(t)} \rho(t,x)[x \wedge v(t,x)]dx. \tag{1.4.33}$$

Reminder. The wedge (vector) product is a map

$$\begin{aligned} \wedge : \quad \mathbb{R} \times \mathbb{R} \to \quad & \mathbb{R}^{d\times d} \\ (a,b) \to \quad & (a \wedge b)_{ij} = a_i b_j - a_j b_i. \end{aligned}$$

By construction $a \wedge b$ is a skew-symmetric matrix i.e. $(a \wedge b)^t = -(a \wedge b)$.
Special cases:
$d = 1 : (a \wedge b) = 0$ for all $a, b \in \mathbb{R}$.
$d = 2 : (a \wedge b) = \begin{pmatrix} 0 & (a\wedge b)_{12} \\ -(a\wedge b)_{12} & 0 \end{pmatrix}$
$d = 3 : (a \wedge b) = \begin{pmatrix} 0 & (a\wedge b)_{12} & (a\wedge b)_{13} \\ -(a\wedge b)_{12} & 0 & (a\wedge b)_{23} \\ -(a\wedge b)_{13} & -(a\wedge b)_{23} & 0 \end{pmatrix} = \begin{pmatrix} 0 & (a\times b)_3 & -(a\times b)_2 \\ -(a\times b)_3 & 0 & (a\times b)_1 \\ (a\times b)_2 & -(a\times b)_1 & 0 \end{pmatrix}$
In $d = 3$ the wedge and cross products can be identified (through the independent matrix elements).

1.4.2 Mass conservation

Integral version. By (1.4.31) we have

$$\frac{d}{dt}\int_{U(t)} \rho(t,x)dx = \frac{d}{dt}\int_U \rho_0(X)dX = 0 \tag{1.4.34}$$

for all test volumes U.

PDE version. If $\rho \in C^1$, then by Reynolds transport theorem

$$\frac{d}{dt}\int_{U(t)} \rho(t,x)dx = \int_{U(t)} [D_t\rho + \rho \mathrm{div}(v)]dx = 0$$

Since U, hence $U(t)$ is arbitrary we obtain

$$D_t\rho(t,x) + \rho\mathrm{div}(v)(t,x) = 0 \quad \forall (t,x) \in \mathcal{T}. \tag{1.4.35}$$

This is called the <u>continuity equation</u>.
From now on we will always assume that ρ is at least C^1.

The following lemmas are consequences of mass conservation.

Lemma 4 *Let $\psi : \mathcal{T} \to \mathbb{R}^m$ be a C^1spatial field. Then*

$$\frac{d}{dt}\int_{U(t)} \rho(t,x)\psi(t,x)dx = \int_{U(t)} \rho(t,x)D_t\psi(t,x)dx. \tag{1.4.36}$$

Proof By Reynolds transport theorem

$$\frac{d}{dt}\int_{U(t)} \rho(t,x)\psi(t,x)dx = \int_{U(t)} [D_t(\rho\psi) + \rho\psi\mathrm{div}(v)]dx$$
$$= \int_{U(t)} [D_t(\rho)\psi + \rho D_t(\psi) + \rho\psi\mathrm{div}(v)]dx = \int_{U(t)} \rho D_t(\psi) + [D_t(\rho) + \rho\mathrm{div}(v)]\psi dx$$

The result now follows from (1.4.35). □

Lemma 5 *Let $V \subset \mathbb{R}^n$ a fixed open connected set (with piecewise C^1boundary) such that $V \subset \Omega_t$ for all $t \in (t_1, t_2)$. Then*

$$\frac{d}{dt}\int_V \rho(t,x)dx = -\int_{\partial V} \rho(t,x)\ \vec{v}(t,x)\cdot\vec{n}_x\ d\mathcal{H}^{d-1}(x) \quad \forall t_1 < t < t_2.$$

This result implies that the variation of the total mass contained in the (fixed) volume V is the mass flux through the boundary.

Proof Reynolds transport theorem+ Gauss. □

Lemma 6 *The material is incompressible i.e. $Vol(U(t)) = Vol(U)$ for all t, if and only if we have* $\mathrm{div}(v)(t,x) = 0$ *for all $(t,x) \in \mathcal{T}$.*

Proof By Reynolds transport theorem with $m = 1$ and $\phi(t,x) = 1$ for all (t,x) we have

$$\frac{d}{dt}Vol(U(t)) = \frac{d}{dt}\int_{U(t)} \phi(t,x)dx = \int_{U(t)} [\partial_t\phi + \mathrm{div}(\phi v)](t,x)dx = \int_{U(t)} \mathrm{div}(v)dx = 0$$

for all test volume U. This holds iff $\mathrm{div}(v)(t,x) = 0$. □

[Lecture 5: 27.04]

1.4.3 Forces

We consider two types of forces.

1. Volume forces: forces acting on every part of the body (external forces such as the gravity)
2. Surface forces: contact forces between two parts of the material (example: friction, pressure difference...)

   External forces acting on the boundary of the body are also a type of surface force, but will be treated as part of the boundary conditions (since they apply only on the external boundary $\partial\Omega$).

We will use the following **assumptions**.

Volume forces. We assume there exists a volume force density $f : \mathcal{T} \to \mathbb{R}^d$ such that the (volume) force acting on a test volume U at time t is given by

$$F_v(U,t) = \int_{U(t)} f(t,x)dx.$$

For example the gravitational force (in $d = 3$) is given by $F(U,t) = -g\rho(t,x)(0,0,1)$.

Surface forces. We assume the Cauchy hypothesis holds, i.e. there exists a surface force density $S : \mathcal{T} \times \mathcal{S}^{d-1} \to \mathbb{R}^d$, where $\mathcal{S}^{d-1}$ is the unit sphere in $\mathbb{R}^d$, such that the (surface) force that the portion $\Omega_t \backslash U(t)$ of the material exerts on $U(t)$ is given by

$$F_s(U,t) = \int_{\partial U(t)} S(t,x,n_x) d\mathcal{H}^{d-1}(x)$$

where $n_x \in \mathcal{S}^{d-1}$ is the direction orthogonal to $\partial U(t)$ at x. S is called the stress vector. For example, the hydrostatic pressure is given by $S(t,x,n) = -p(t,x)n$ with $p : \mathcal{T} \to \mathbb{R}$.

Definition 13 *A system of forces is a pair (f, S) (volume and surface force density) with*

- $f \in C(\mathcal{T}; \mathbb{R}^d)$, *and*
- $S : \mathcal{T} \times \mathcal{S}^{d-1} \to \mathbb{R}^d$ *is Borel measurable, locally bounded in all variables and for every $n \in \mathcal{S}^{d-1}$ $S(\cdot, \cdot, n) \in C(\mathcal{T}; \mathbb{R}^d)$.*

The total force acting on a test volume U is a function $F : \mathbb{R} \to \mathbb{R}^d$ defined by

$$F(U,t) = F_v(U,t) + F_s(U,t) = \int_{U(t)} f(t,x)dx + \int_{\partial U(t)} S(t,x,n_x) d\mathcal{H}^{d-1}(x). \tag{1.4.37}$$

The total torque on a test volume U is a function $M : \mathbb{R} \to \mathbb{R}^{d \times d}$

$$M(U,t) = \int_{U(t)} x \wedge f(t,x)dx + \int_{\partial U(t)} x \wedge S(t,x,n_x) d\mathcal{H}^{d-1}(x). \tag{1.4.38}$$

1.4.4 Conservation of momentum and angular momentum.

Integral version. For any test volume U, the variation of the (linear) momentum is given by

$$\frac{dl}{dt}(U,t) = F(U,t). \tag{1.4.39}$$

By (1.4.36) this can be written as

$$\int_{U(t)} \rho(t,x) \frac{Dv}{Dt}(t,x)dx = \int_{U(t)} f(t,x)dx + \int_{\partial U(t)} S(t,x,n_x) d\mathcal{H}^{d-1}(x). \tag{1.4.40}$$

For any test volume U, the variation of the angular momentum is given by

$$\frac{dL}{dt}(U,t) = M(U,t). \tag{1.4.41}$$

This can be written as

$$\int_{U(t)} \rho(t,x) \left(x \wedge \frac{Dv}{Dt}(t,x) \right) dx = \int_{U(t)} (x \wedge f(t,x)) \, dx + \int_{\partial U(t)} (x \wedge S(t,x,n_x)) \, d\mathcal{H}^{d-1}(x). \tag{1.4.42}$$

Indeed, applying again (1.4.36), the time derivative enters the integral and becomes $D_t(x \wedge v) = (D_t x) \wedge v + x \wedge (D_t v)$. To compute $D_t x$ we consider the spatial field $\phi(t,x) = x$:

$$D_t x_j = D_t \phi_j(t,x) = \partial_t \phi_j(t,x) + (\vec{v} \cdot \vec{\nabla}) \phi_j(t,x) = v_j(t,x).$$

Finally we use $v \wedge v = 0$.

PDE version. To obtain a PDE, we need first some additional information on the stress vector.

1.4.5 Cauchy theorem

Theorem 4 *let x be a motion, (f, S) a system of forces such that S is continuously differentiable in x. Then conservation of linear and angular momentum* (1.4.40) (1.4.42) *hold iff there exists a matrix-valued spatial field $\sigma : \mathcal{T} \to \mathbb{R}^{d\times d}$ such that*

(1.) $S(t,x,n) = \sigma(t,x)n$ $(S_i = \sum_{j=1}^d \sigma_{ij} n_j)$ *i.e. S is linear in the n argument,*

(2.) $\sigma^T = \sigma$,

(3.) σ satisfies the <u>*equation of motion*</u>

$$\rho \frac{Dv}{Dt} = \mathrm{div}(\sigma) + f \tag{1.4.43}$$

where we define

$$[\mathrm{div}(\sigma)]_i := \sum_{j=1}^d \partial_{x_j} \sigma_{ij}.$$

The spatial field σ is called the <u>*stress tensor*</u>.

Proof of $\Rightarrow$. We prove that (1.4.40), (1.4.42) $\Rightarrow$ (1.) + (2.) + (3.) .

First step. Let us assume we have already proved existence, i.e. (1.) holds. We prove that (1) $\Rightarrow$ (3), i.e. existence of σ implies the equation of motion (1.4.43).
Indeed, replacing $S_i(t,x,n) = [\sigma(t,x)n]_i = \sum_{j=1}^d \sigma_{ij}(t,x)n_j$ in the conservation law (1.4.40) we get

$$\int_{U(t)} \rho(t,x)\frac{Dv_i}{Dt}(t,x)dx = \int_{U(t)} f_i(t,x)dx + \int_{\partial U(t)} [\sigma(t,x)n_x]_i d\mathcal{H}^{d-1}(x). \tag{1.4.44}$$

For each i, let $\omega^i \in \mathbb{R}^d$ be the vector defined by $\omega^i_j := \sigma_{ij}$ (i.e. ω^i is the $i-$th row of σ). Then $[\sigma n]_i = \vec{\omega}^i \cdot \vec{n}$ and

$$\int_{\partial U(t)} [\sigma(t,x)n_x]_i d\mathcal{H}^{d-1}(x) = \int_{\partial U(t)} [\vec{\omega}^i(t,x)\cdot \vec{n}_x] d\mathcal{H}^{d-1}(x) = \int_{U(t)} \mathrm{div}(\omega^i)dx = \int_{U(t)} [\mathrm{div}(\sigma)]_i(t,x)\,dx,$$

where we used Gauss theorem. Finally (1.4.40) becomes

$$\int_{U(t)} \rho(t,x)\frac{Dv_i}{Dt}(t,x)dx = \int_{U(t)} \left(f_i(t,x) + [\mathrm{div}(\sigma)]_i(t,x)\right)dx$$

Since U, hence $U(t)$, is arbitrary we obtain (1.4.43).

Second step. Assume (1.) holds. We prove that (1) $\Rightarrow$ (2).
We have already proved (1) $\Rightarrow$ (3), hence $\rho\frac{Dv}{Dt} - f = [\mathrm{div}\sigma]$. Replacing this relation and $S_i = [\sigma n]_i$ in equation (1.4.42) (conservation of angular momentum) we get

$$\begin{aligned} 0 = \int_{U(t)} \left(x \wedge \left[\rho(t,x)\frac{Dv}{dt}(t,x) - f(t,x)\right]\right) dx - \int_{\partial U(t)} \left(x \wedge S(t,x,n_x)\right) d\mathcal{H}^{d-1}(x) \\ = \int_{U(t)} \left(x \wedge [\mathrm{div}\sigma]\right) dx - \int_{\partial U(t)} \left(x \wedge S(t,x,n_x)\right) d\mathcal{H}^{d-1}(x) \end{aligned}$$

Now, for fixed i, j we have

$$(x \wedge S)_{ij} = x_i S_j - x_j S_i = \sum_l (x_i \sigma_{jl} - x_j \sigma_{il}) n_l = \vec{W}^{ij} \cdot \vec{n}$$

where $W^{ij} \in \mathbb{R}^d$, is defined by $W^{ij}_l := (x_i\sigma_{jl} - x_j\sigma_{il})$. Then

$$\int_{\partial U(t)} [x \wedge S(t,x,n_x)]_{ij}\, d\mathcal{H}^{d-1}(x) = \int_{\partial U(t)} \left(\vec{W}^{ij}(t,x)\cdot \vec{n}_x\right) d\mathcal{H}^{d-1}(x) = \int_{U(t)} \mathrm{div}(\vec{W}^{ij})(t,x)\, dx$$

where in the last step we used Gauss theorem and $\mathrm{div}(\vec{W}^{ij}) = \sum_l \partial_{x_l} W^{ij}_l$. Now

$$\mathrm{div}(\vec{W}^{ij}) = \sum_l \partial_{x_l}(x_i\sigma_{jl} - x_j\sigma_{il}) = \sigma_{ji} - \sigma_{ij} + x_i[\mathrm{div}\sigma]_j - x_j[\mathrm{div}\sigma]_i = \sigma_{ji} - \sigma_{ij} + (x \wedge [\mathrm{div}\sigma])_{ij}.$$

Inserting this in the integrals above we have

$$0 = \int_{U(t)} \left[(x\wedge[\mathrm{div}\sigma])_{ij} - \mathrm{div}(\vec{W}^{ij})\right] dx = -\int_{U(t)} (\sigma_{ji} - \sigma_{ij})(t,x)\, dx.$$

Since U, and hence $U(t)$, is arbitratry this implies $\sigma_{ij} = \sigma_{ji}$. for all ij. Then $\sigma^t = \sigma$.

Third step. Finally we prove that (1.4.40)+ (1.4.42) $\Rightarrow$ (1.). We will need the following result (whose proof is given later).

Lemma 7 *Let $\vec{n} \in \mathcal{S}^{d-1}$ a fixed arbitrary direction and let $\vec{f}_1, \dots \vec{f}_d$ an orthonormal basis of $\mathbb{R}^d$ s.t. $n^f_j = (\vec{n}\cdot\vec{f}_j) > 0\ \forall j = 1,..d$ (we can always rotate the axis to garantee this condition). Then we can write S as linear combination of stresses in the elementary directions f_j, i.e.*

$$\vec{S}(t,x,n) = -\sum_{j=1}^{d} n^f_j \vec{S}(t,x,-f_j). \tag{1.4.45}$$

With this lemma we will show that, if $\vec{e}_1, \dots, \vec{e}_d$ is the standard basis and $n_j := (\vec{n}\cdot\vec{e}_j) \in \mathbb{R}$, we have

- (a) $\vec{S}(t,x,n) = \sum_{j=1}^d n_j \vec{S}(t,x,e_j)$, and
- (b) $\vec{S}(t,x,-e_j) = -\vec{S}(t,x,e_j)\ \forall j = 1,\dots d$, and, more generally,
  $\vec{S}(t,x,n) = -\vec{S}(t,x,-n)\ \forall n \in \mathcal{S}^{d-1}$ ('action-reaction' principle).

Proof of (a) and (b). Let $\vec{n}_0 \in \mathcal{S}^{d-1}$ a fixed orientation and let $\vec{f}_1, \dots \vec{f}_d$ and orthornormal basis (depending on n_0) such that $(n_0)^f_j > 0$ for all j. Then we can find a neighborhood $\mathcal{U}(n_0)$ of n_0 in $\mathcal{S}^{d-1}$ such that $n^f_j > 0$ for all $j = 1,..,d$ and for all $n \in \mathcal{U}(n_0)$. By Lemma 7 above we have $\vec{S}(t,x,n) = -\sum_{j=1}^d n^f_j \vec{S}(t,x,-f_j)$ for all $n \in \mathcal{U}(n_0)$ hence S is locally continuous on $\mathcal{S}^{d-1}$.
Then, by continuity, (1.4.45) remains valid also in the limit $n^f_j = (\vec{n}\cdot\vec{f}_k) \downarrow 0$. In particular we can choose $\vec{n} = \vec{e}_j$ for some j. Then $n_l = \delta_{lj} \geq 0$ and from (1.4.45) we get $\vec{S}(t,x,e_j) = -\vec{S}(t,x,-e_j)$. For any $n \in \mathcal{S}^{d-1}$ we can find an orthonormal basis such that $n = f_j$ for some j, hence $\vec{S}(t,x,n) = -\vec{S}(t,x,-n)$. This gives (b).
To prove (a), fix some n, let $\vec{e}_1, \dots, \vec{e}_d$ be the standard basis and let $\vec{f}_j := \mathrm{sign}(n_j)\vec{e}_j$ a new family of vectors. Then $\vec{f}_1, \dots, \vec{f}_d$ is an o.n. basis and $n^f_j = \mathrm{sign}(n_j)n_j = |n_j| \geq 0\ \forall j$. Then $\vec{S}(t,x,n) = -\sum_{j=1}^d n^f_j \vec{S}(t,x,-f_j)$, where

$$\begin{aligned} n^f_j \vec{S}(t,x,-f_j) &= n_j \vec{S}(t,x,-e_j) = -n_j \vec{S}(t,x,e_j) \qquad &\text{if } n_j \geq 0,\\ n^f_j \vec{S}(t,x,-f_j) &= -n_j \vec{S}(t,x,e_j) &\text{if } n_j < 0, \end{aligned}$$

where in the first line we used (b). Then $\vec{S}(t,x,n) = -\sum_{j=1}^d n^f_j \vec{S}(t,x,-f_j) = \sum_{j=1}^d n_j \vec{S}(t,x,e_j)$. This gives (a).

Finally, from (a) we get $S_i(t,x,n) = \sum_j \sigma_{ij}(t,x)n_j$ where we defined

$$\sigma_{ij}(t,x) := S_i(t,x,e_j).$$

With this definition $\sigma \in C(\mathcal{T};\mathbb{R}^{d\times d})$. This concludes the proof of the $\Rightarrow$ part.

[Lecture 6: 29.04]

Proof of $\Leftarrow$. We prove that $(1.)+(2.)+(3) \Rightarrow$ (1.4.40) and (1.4.42). Easy: replace the relations in $(1.), (2.), (3.)$ above in the integrals.

Proof of Lemma 7. (Cauchy's tetrahedron argument) Let

$$\Delta = \{x \in \mathbb{R}^d |\ (\vec{x}\cdot\vec{f}_j) \geq 0\ \forall j \text{ and } (\vec{x}\cdot\vec{n}) \leq 1\}$$

be the simplex created by intersecting the plane $(\vec{x}\cdot\vec{n}) = 1$ with the d planes orthogonal to the d axis. The region Δ has $d+1$ corners: the origin $\vec{0}$ and the d points $\vec{c}_j = \frac{1}{n_j}\vec{f}_j$, $j = 1,\dots,d$. Δ has also $d+1$ faces.

- The first d faces are the intersections of Δ with the planes orthogonal to each direction f_j. Precisely the $j-$th face is given by

$$a_j := \{x \in \Delta |\ \ x_j = 0\}, \qquad j = 1,\dots d$$

- The last face is the intersection of Δ with the plane $(\vec{x}\cdot\vec{n}) = 1$:

$$a = \{x \in \Delta |\ (\vec{x}\cdot\vec{n}) = 1\}.$$

Let $|a_l|$, $|a|$ be the area of the face a_l, a respectively. We have

$$|a_l| = n_l \frac{I_{d-1}}{n_\Delta}, \quad |a| = \frac{I_{d-1}}{n_\Delta}$$

where $I_d := \int_{[0,1]^d} \mathbf{1}_{\{(\vec{x}\cdot\vec{n})\leq 1\}} dy^d$ and $n_\Delta := \prod_{j=1}^d n_j$. Hence

$$|a_l| = n_l|a|, \quad l = 1,\dots,d.$$

For a fixed t, we take now as test volume $U(t) = U_\delta(x_0) = \{x_0\} + \Delta_\delta$, where $x_0 \in \Omega_t$ and Δ_δ is the rescaled simplex $\Delta_\delta = \{x \in \mathbb{R}^d |\ x = \delta y,\ y \in \Delta\}$, with $\delta > 0$. Clearly $\sup_{x\in U_\delta(x_0)} |x - x_0| \to 0$ as $\delta \to 0$, the volume satisfies $Vol(U_\delta(x_0)) = O(\delta^d)$ and the surface satisfies $|\partial U_\delta(x_0)| = O(\delta^{d-1})$. By (1.4.40) we have, for all $\delta > 0$,

$$\frac{1}{\delta^{d-1}}\left[\int_{U_\delta}\left[\rho(t,x)\frac{Dv_i}{dt}(t,x) - f(t,x)\right]dx - \int_{\partial U_\delta} S(t,x,n_x)d\mathcal{H}^{d-1}(x)\right] = 0.$$

Since the first integral scales like δ^d we have

$$\lim_{\delta\to 0}\frac{1}{\delta^{d-1}}\int_{U_\delta}\left[\rho(t,x)\frac{Dv_i}{dt}(t,x) - f(t,x)\right]dx = \lim_{\delta\to 0} O(\delta) = 0.$$

Hence we must have

$$\lim_{\delta\to 0}\frac{1}{\delta^{d-1}}\int_{\partial U_\delta} S(t,x,n_x)d\mathcal{H}^{d-1}(x) = 0.$$

Since $U_\delta \to \{x_0\}$ the integral above is given (up to higher order corrections in δ)

$$S(t,x_0,n)|a| + \sum_{j=1}^d S(t,x_0,-f_j)|a_j| = 0.$$

Replacing $|a_j| = n_j|a|$ we get the result. This concludes the proof of the lemma, hence the proof of the Cauchy Theorem. □

1.4.6 Equations in spatial and material coordinates

Until now we obtained two main PDEs, both in spatial coordinates

$$\frac{D\rho}{Dt} + \rho \mathrm{div}(v) = 0 \qquad \text{continuity eq.}$$
$$\rho \frac{Dv}{Dt} = \mathrm{div}(\sigma) + f \qquad \text{eq. of motion}$$

We now look for similar equations in spatial coordinates.

Continuity eq. in material coordinates.

Lemma 8 *The material description of the mass density* $(\rho)_m$ *satisfies*

$$\rho_m(t,X) = \frac{\rho_0(X)}{\det Dx(t,X)}. \tag{1.4.46}$$

Proof. By (1.4.31) and a coordinate change we have

$$\int_U \rho_0(X)dX = \int_{U(t)} \rho(t,x)dx = \int_U \rho_m(t,X) \det Dx(t,X)dX.$$

Since the test volume U is arbitrary this implies the result. □

Remark. Eq.(1.4.46) is the analog of the continuity equation in material coordinates:

$$\rho_m(t,X) = \frac{\rho_0(X)}{\det Dx(t,X)} \quad \Leftrightarrow \quad \left[\frac{D\rho}{Dt} + \rho \mathrm{div}(v)\right]_m = 0$$

Indeed $(D_t\rho)_m = \partial_t(\rho_m)$ by definition of D_t, and from (1.4.46) $\partial_t \rho_m = -\rho_m \frac{\partial_t \det Dx}{\det Dx}$. Moreover, by Lemma 1.3.30 $(\mathrm{div}(v))_m = \frac{\partial_t \det Dx}{\det Dx}$. Hence the result.

Equation of motion in material coordinates

Our goal is to write $\rho\frac{Dv}{Dt} = \mathrm{div}(\sigma) + f$ in material coordinates.

Lemma 9 *The equation of motion in material coordinates becomes*

$$\rho_0(X)\partial_t^2 x(t,X) = (\det Dx)(t,X) f_m(t,X) + \mathrm{DIV}(\mathcal{S})(t,X) \tag{1.4.47}$$

where $\mathcal{S} : \mathbb{R} \times \Omega \to \mathbb{R}^{d\times d}$ *is a matrix-valued material field defined by*

$$\mathcal{S}_{ij} := [\sigma_m \mathrm{Cof}(Dx)]_{ij} \tag{1.4.48}$$

and DIV denotes the divergence in material coordinates

$$[\mathrm{DIV}(\mathcal{S})]_i := \sum_j \partial_{X_j} \mathcal{S}_{ij}.$$

$\mathcal{S}$ *is called the* *Piola-Kirchhoff* *tensor.*

Proof. By definition of D_t we have $(D_t v)_m = \partial_t V = \partial_t^2 x =$ acceleration. By the continuity equation above we know that $\rho_m = \rho_0/(\det Dx)$. The equation becomes

$$\rho_0(X)\partial_t^2 x(t,X) = (\det Dx)(t,X) f_m(t,X) + (\det Dx)(\mathrm{div}\sigma)_m(t,X)$$

It remains to write $(\det Dx)(\mathrm{div}\sigma)_m$ as a divergence in material coordinates. Using the relation $\sigma(t,x) = \sigma_m(t, X(t,x))$ we get

$$[\mathrm{div}(\sigma)]_i = \sum_l \partial_{x_l}\sigma_{il} = \sum_{ll'} \frac{\partial X_{l'}}{\partial x_l}[\partial_{X_{l'}}(\sigma_m)_{il}]_s$$

Hence

$$(\det Dx)[(\mathrm{div}\sigma)_m]_i = \sum_{ll'}(\det Dx)\left(\frac{\partial X_{l'}}{\partial x_l}\right)_m \partial_{X_{l'}}(\sigma_m)_{il}$$

Note that for any material field Φ we have $\partial_{X_{l'}}\Phi = \sum_j (\partial_{x_j}\Phi_s)_m \frac{\partial x_j}{\partial X_{l'}} = \sum_j (\partial_{x_j}\Phi_s)_m (Dx)_{jl'}$. Applying this to $\Phi = X_l$ we get

$$\left(\frac{\partial X_{l'}}{\partial x_l}\right)_m = [(Dx)^{-1}]_{l'l}.$$

The expression for $(\det Dx)[(\mathrm{div}\sigma)_m]_i$ above becomes

$$\sum_{ll'}(\det Dx)[(Dx)^{-1}]_{l'l}(\partial_{X_{l'}}\sigma_m)_{il} = \sum_{l'}\partial_{X_{l'}}\left[(\det Dx)\sum_l (Dx)^{-1}_{l'l}(\sigma_m)_{il}\right] - \sum_l (\sigma_m)_{il}\sum_{l'}\partial_{X_{l'}}\left[(\det Dx)(Dx)^{-1}_{l'l}\right]$$

We will prove later that $\sum_{l'}\partial_{X_{l'}}\left[(\det Dx)(Dx)^{-1}_{l'l}\right] = 0$. Then

$$(\det Dx)[(\mathrm{div}\sigma)_m]_i = \sum_{l'}\partial_{X_{l'}}\left[(\det Dx)\sum_l (Dx)^{-1}_{l'l}(\sigma_m)_{il}\right] = \sum_{l'}\partial_{X_{l'}}\left[\sigma_m \mathrm{Cof}(Dx)\right]_{il'}.$$

This concludes the proof. □

Lemma 10 *Let x be a motion. Then*

$$\sum_{l'}\partial_{X_{l'}}\left[(\det Dx)(Dx)^{-1}_{l'l}\right] = 0 \qquad \forall l. \tag{1.4.49}$$

Proof For any matrix-valued material field $A : \mathbb{R}\times\Omega \to \mathbb{R}^{d\times d}$ differentiable in X, with $A(t,X)$ invertible for all (t,X), we have

$$\sum_{l'}\partial_{X_{l'}}\left[(\det A)A^{-1}_{l'l}\right] = (\det A)\left[\sum_{l'}\partial_{X_{l'}}(A^{-1}_{l'l}) + \sum_{ijl'}(A^{-1}_{l'l})(A^{-1}_{ji})\partial_{X_{l'}}A_{ij}\right],$$

where we used $\partial_{A_{ij}}\det A = (\mathrm{Cof}A)_{ij} = \det A(A^{-1})_{ji}$. Using $A = Dx$ we have $\partial_{X_{l'}}A_{ij} = \partial_{X_{l'}}\partial_{X_j}x_i = \partial_{X_j}A_{il'}$. Then

$$\sum_{ij}(A^{-1}_{ji})\partial_{X_{l'}}A_{ij} = \sum_{ij}(A^{-1}_{ji})\partial_{X_j}A_{il'} = \sum_j \partial_{X_j}\left[\sum_i (A^{-1}_{ji})A_{il'}\right] - \sum_{ij}A_{il'}\partial_{X_j}(A^{-1}_{ji})$$

The first term gives $\partial_{X_j}\delta_{jl'} = 0$. It remains

$$\sum_{ijl'}(A^{-1}_{l'l})(A^{-1}_{ji})\partial_{X_{l'}}A_{ij} = -\sum_{ijl'}A_{il'}(A^{-1}_{l'l})\partial_{X_j}(A^{-1}_{ji}) = -\sum_j \partial_{X_j}(A^{-1}_{jl}).$$

Hence the result. □

1.4.7 Energy

Remember that for particles we had

$$
\begin{aligned}
E_k &= \frac{1}{2}\sum_{i=1}^{N} m_i |x_i'(t)|^2 \\
E &= E_k + E_p \\
P &= \sum_{i=1}^{N} f_i \cdot x_i' \\
E'(t) &= P
\end{aligned}
$$

where E_k, E_p, E are the kinetic, potential and total energy respectively, and P is the total work per time (power) of the external forces. Our goal is to define the analogous quantities in the continous context. The main difference is that, in the continuous case, the energy is not just modified through forces (power) but also through *heat exchange*. There are two types of heat exchanges.

1. Volume effect: heat supply (we heat the whole system, example by radiation).
2. Surface effect: heat exchange between different parts of the material (one area may be hotter). This is described by a heat flux through the surface of a test volume U.

Definition 14 *Let x be a motion, ρ a C^1 mass density, (f, S) a system of forces and ϵ a C^1 spatial field (the internal energy density).*
The kinetic energy of a test volume U is defined by

$$E_k(U,t) := \frac{1}{2}\int_{U(t)} \rho(t,x)|v(t,x)|^2 dx. \tag{1.4.50}$$

The potential energy of a test volume U is defined by

$$E_k(U,t) := \int_{U(t)} \rho(t,x)\epsilon(t,x) dx. \tag{1.4.51}$$

The total work per unit time (power) done solely by forces on a test volume U is

$$P_f(U,t) := \int_{U(t)} \vec{f}(t,x)\cdot\vec{v}(t,x)\ dx + \int_{\partial U(t)} \vec{S}(t,x,n_n)\cdot\vec{v}(t,x)\ d\mathcal{H}^{d-1}(x) \tag{1.4.52}$$

The total work per time (power) done by heat on a test volume U is defined by

$$P_h(U,t) := \int_{U(t)} r(t,x)\ dx - \int_{\partial U(t)} \vec{q}(t,x)\cdot\vec{n}_x\ d\mathcal{H}^{d-1}(x) \tag{1.4.53}$$

where $r : \mathcal{T} \to \mathbb{R}$ is the heat supply and $q : \mathcal{T} \to \mathbb{R}^d$ is the heat (outward) flux.
Finally the total energy and total power are defined by

$$
\begin{aligned}
E(t,U) &:= E_k(t,U) + E_p(t,U) \\
P(t,U) &:= P_f(t,U) + P_h(t,U).
\end{aligned}
$$

[Lecture 7: 4.05]

Lemma 11 *The total energy and total power can be written as*

$$E(t,U) = \int_{U(u)} \rho \left[\frac{|v|^2}{2} + \epsilon\right] dx \tag{1.4.54}$$

$$P(t,U) = \int_{U(u)} \left[f \cdot v + \mathrm{div}(\sigma^t v) + r - \mathrm{div}(q)\right] dx \tag{1.4.55}$$

where

$$\mathrm{div}(\sigma^t v) = \sum_{ij} \partial_{x_i}(\sigma_{ji} v_i), \qquad \mathrm{div}(q) = \sum_i \partial_{x_i} q_i.$$

Proof. The first relation (1.4.54) follows directly from the definitions. For (1.4.55) we use $S = \sigma n$, $(v, \sigma n) = (\sigma^t v, n)$, where $(a,b) = a \cdot b$ is the standart scalar product in $\mathbb{R}^d$. Finally we use Gauss theorem.

Conservation of Energy

Integral version. For any test volume U and any time t we have

$$\frac{d}{dt} E(t,U) = P(t,U). \tag{1.4.56}$$

PDE version

Lemma 12 *Energy conservation* (1.4.56) *holds for all* U *iff*

$$\rho \frac{D\epsilon}{Dt} = \mathrm{tr}(\sigma^t Dv) + r - \mathrm{div}(q) \qquad \forall (t,x) \in \mathcal{T} \tag{1.4.57}$$

where $(Dv)_{ij} := \frac{\partial v_i}{\partial x_j}$.

Proof. By Reynolds transport theorem and mass conservation we have

$$\frac{d}{dt} E(t,U) = \int_{U(t)} \rho \frac{D}{Dt} \left[\frac{|v|^2}{2} + \epsilon\right] dx.$$

Now, using the equation of motion we get

$$\rho D_t |v|^2 = 2v \cdot (\rho D_t v) = 2v \cdot [f + \mathrm{div}(\sigma)].$$

Hence

$$\int_{U(t)} \rho D_t \epsilon \, dx = \int_{U(u)} \left[\mathrm{div}(\sigma^t v) - v \cdot \mathrm{div}(\sigma)\right] dx + \int_{U(u)} [r - \mathrm{div}(q)] \, dx$$

Note that

$$\mathrm{div}(\sigma^t v) - v \cdot \mathrm{div}(\sigma) = \sum_{ij} \partial_{x_i}(\sigma_{ji} v_j) - \sum_{ij} v_j \partial_{x_i} \sigma_{ji} = \sum_{ij} \sigma_{ji} \partial_{x_i} v_j = \mathrm{tr}(\sigma^t Dv).$$

The result follows. □

1.4.8 Summary

Functions we introduced.

- motion, velocity: x, V, v
- mass: $\rho(t,x), \rho_0(X), \rho_m(t,X)$

- forces: $f, S = [\sigma n]$, $\mathcal{S} := [\sigma_m \mathrm{Cof}(Dx)]$
- energy, power: $E(t,U), P(t,U)$ $\epsilon(t,x)$
- heat: $r(t,x), q(t,x)$

Material fields.

- scalars: ρ_0, ρ_m
- vectors: x, V
- matrices: $Dx, \sigma_m, \mathcal{S}$

Spatial fields.

- scalars: ρ, ϵ, r
- vectors: v, f, S, q
- matrices: Dv, σ

Integral identities.

- mass: $\int_{U(t)} \rho(t,x)dx = \int_U \rho_0(X)dX$
- momentum: $\frac{d}{dt}\int_{U(t)} \rho(t,x)v(t,x)dx = \int_{U(t)} [f(t,x) + \mathrm{div}(\sigma)(t,x)]\, dx$
- angular momentum: $\frac{d}{dt}\int_{U(t)} x \wedge (\rho v)dx = \int_{U(t)} x \wedge f dx + \int_{\partial U(t)} x \wedge S \; d\mathcal{H}^{d-1}(x)$
- energy: $\frac{d}{dt}\int_{U(u)} \rho \left[\frac{|v|^2}{2} + \epsilon\right] dx = \int_{U(u)} [f \cdot v + \mathrm{div}(\sigma^t v) + r - \mathrm{div}(q)]\,.$

PDEs in spatial coordinates.

- continuity eq. $\frac{D\rho}{Dt} + \rho \mathrm{div}(v) = 0$
- eq. of motion $\rho\frac{Dv}{Dt} = f + \mathrm{div}(\sigma)$
- energy cons. $\rho\frac{D\epsilon}{Dt} = \mathrm{tr}(\sigma^t Dv) + r - \mathrm{div}(q)$

PDEs in material coordinates.

- continuity eq. $\rho_m = \frac{\rho_0}{\det Dx}$
- eq. of motion $\rho\partial_t^2 x = (\det Dx)f_m + \mathrm{DIV}(\mathcal{S})$

1.5 Constitutive laws

The explicit form of σ, q, ϵ depends on ther material properties.

Example 1: heat flux Let $\theta : \mathcal{T} \to \mathbb{R}$ the temperature function and assume the heat flux depends only on local temperature differences $(D\theta)_i = \partial_{x_i}\theta$, $i = 1, \ldots d$. Then there exists a function

$$\begin{aligned} \hat{q} : \quad \mathbb{R}^d &\to \mathbb{R}^d \\ w &\to \hat{q}(w) \end{aligned}$$

such that $q(t,x) = \hat{q}(D\theta(t,x))$.

Example 2: elastic material Assume the stress tensor depends only on local deformations $(Dx)_{ij} = \partial_{X_j} x_i$, $i, j = 1, \dots d$. Then there exists a function

$$\begin{array}{lll} \hat{\sigma}: & GL_+(\mathbb{R}^d) & \to \mathbb{R}^{d\times d}_{sym} \\ & F & \to \hat{\sigma}(F) \end{array}$$

such that $\sigma(t, x) = \hat{\sigma}(Dx_s)$.

Example 3: fluid with viscosity Assume the stress tensor depends only on local deformations $(Dx)_{ij} = \partial_{X_j} x_i$, $i, j = 1, \dots d$, and on local velocity changes $(Dv)_{ij} = \partial_{x_j} v_i$, $i, j = 1, \dots d$ (neighbor portions at different velocities creat a friction). Then there exists a function

$$\begin{array}{lll} \hat{\sigma}: & GL_+(d) \times \mathbb{R}^{d\times d} & \to \mathbb{R}^{d\times d}_{sym} \\ & (F, G) & \to \hat{\sigma}(F, G) \end{array}$$

such that $\sigma(t, x) = \hat{\sigma}(Dx_s(t, x), Dv(t, x))$.

$\hat{q}, \hat{\sigma}$ are called constitutive laws.

Notation

$$\begin{array}{ll} GL(d) = \{F \in \mathbb{R}^{d\times d} |\ \det F \neq 0\} & SL(d) = \{F \in \mathbb{R}^{d\times d} |\ \det F = 1\} \\ GL_+(d) = \{F \in \mathbb{R}^{d\times d} |\ \det F > 0\} & SO(d) = \{F \in \mathbb{R}^{d\times d} |\ F^t F = I\ \det F = 1\} \\ \mathbb{R}^{d\times d}_{sym} = \{F \in \mathbb{R}^{d\times d} |\ F^t = F\} & \mathbb{R}^{d\times d}_{skw} = \{F \in \mathbb{R}^{d\times d} |\ F^t = -F\} \end{array}$$

[Lecture 8: 6.05]

1.5.1 Coordinates changes

We have introduced two coordinate systems (t, X) and (t, x), where $X \in \Omega$ describes the reference configuration of the material, while $x \in \Omega_t$ describes the material 'seen' at time t, (point of view of an oberver taking a photo at some given time). Both $\Omega, \Omega_t \subset \mathbb{R}^d$, and until now we used the same orthonormal basis $e_1, \dots, e_d$ to parametrize both coordinates. But we could also use a different basis. This corresponds to perform a change of coordinates either in X or in x separately. As a result we will obtain some contraints on the possible form of the constitutive laws.
We will consider here two types of coordinate changes.

1. Changing the observable. We will consider only time-independent rotations

$$(t, x) \to (t, x^*) = (t, Qx), \qquad Q \in SO(d).$$

2. Deformations of the material. We will consider only deformations that do not change the physical structure i.e.

$$X \in \Omega \to X^* \in g\Omega, \quad X^* = gX \qquad g \in G,$$

where G is a symmetry group for the material.

Examples of material symmetry groups:

material	G
cristalline solid	discrete subgroup of $SO(3)$
istropic solid	$SO(3)$
ideal fluid	$SL(3)$

Indeed, in the case of a cristal, the description does not change if we perform rotations that are compatible with the lattice directions. In a isotropic solid we can do any rotation, while in a liquid we can also change the shape.

Transformation rules under observable change

Let $Q \in SO(d)$ a rotation. We define the transformation

$$Q: \begin{array}{ll} \mathcal{T} & \to Q\mathcal{T} = \{(t,x^*)|\, t \in \mathbb{R}, x^* \in Q\Omega_t\} \\ (t,x) & \to (t,x^*(x)) = (t,Qx) \end{array}$$

The quantitities we introduced transform as follows

1. spatial coordinate : $x \to x^*(x) = Qx$
2. deformation gratient: $(Dx)(t,X) \to (Dx^*)(t,X) = Q(Dx)(t,X)$
3. velocity: $v(t,x) \to v^*(t,x^*(x)) = Qv(t,x)$
4. density: $\rho(t,x) \to \rho^*(t,x^*(x)) = \rho(t,x)$
5. spatial gradient: $\frac{\partial}{\partial x_j} \to \frac{\partial}{\partial x_j^*} = \left(Q\frac{\partial}{\partial x}\right)_j$
6. force density: $f(t,x) \to f^*(t,x^*(x)) = Qf(t,x)$,
   $S(t,x,n) \to S^*(t,x^*(x),n^*(x)) = QS(t,x,n)$, where $n^* = Qn$
7. stress tensor: $\sigma(t,x) \to \sigma^*(t,x^*(x)) = Q\sigma(t,x)Q^t$
   Piola-Kirchhoff tensor $\mathcal{S}(t,X) \to \mathcal{S}^*(t,X) = Q\mathcal{S}(t,X)$
8. energy and heat supply: $\epsilon(t,x) \to \epsilon^*(t,x^*) = \epsilon(t,x(x^*))$, $r(t,x) \to r^*(t,x^*) = r(t,x(x^*))$
9. heat flux: $q(t,x) \to q^*(t,x^*(x)) = Qq(t,x)$
10. temperature gradient: $D\theta(t,x) \to D^*\theta^*(t,x^*) = [QD\theta](t,x(x^*))$
11. velocity gradient: $Dv(t,x) \to D^*v^*(t,x^*) = [QDvQ^t](t,x(x^*))$

Proof **1.** is just a definition.
For **2.** we write $(Dx^*)_{ij} = \frac{\partial x_i^*}{\partial X_j} = \sum_l Q_{il}\frac{\partial x_l}{\partial X_j} = \sum_l Q_{il}(Dx)_{lj}$.
For **3.** remember that $v(t,x) = V(t,X(t,x)) = [\partial_t x(t,X)]_{|X=X(t,x)}$. Then
$v^*(t,x^*) = [\partial_t x^*(t,X)]_{|X=X(t,x^*)} = [Q\partial_t x(t,X)]_{|X=X(t,x^*)} = Qv(t,x(x^*))$.
4. follows from the fact that ρ is a scalar field.
For **5.** take any function $f(x(x^*))$. Then using $Q^{-1} = Q^t$ we have
$\frac{\partial f}{\partial x_j^*} = \sum_l \frac{\partial x_l}{\partial x_j^*}\left(\frac{\partial f}{\partial x_l}\right) = \sum_l Q^{-1}_{lj}\left(\frac{\partial f}{\partial x_l}\right) = \left(\sum_l Q_{jl}\frac{\partial}{\partial x_l}\right) f$.
6. Under the rotation Q the test volume and its boundary rotate $U(t) \to QU(t)$. and $\partial U(t) \to Q\partial U(t)$ as well as the corresponding normals $n_x \to n^*_{x_*} = Qn_{x(x^*)}$. Hence the force density transforms as $f^*(t,x^*) = Qf(t,Q^{-1}x)$ and $S(t,x,n) \to S^*(t,x^*,n^*) = QS(t,Q^{-1}x,Q^{-1}n)$.
7. From $S = \sigma n$ and $S^* = \sigma^* n^* = QSn$ we get $Q^T\sigma^* Qn = \sigma n$ for all n, hence the first result.

For the second identity, since $\mathcal{S} = \sigma_m \mathrm{Cof}(Dx)$ we have $\mathcal{S}^*(t,X) = \sigma_m^*(t,X)\mathrm{Cof}(Dx^*)(t,X)$. Inserting $Dx^* = QDx$ we get

$$\mathrm{Cof}(Dx^*) = (Dx^*)^{-t}\det Dx^* = Q(Dx)^{-t}\det Dx$$

where we used $\det Q = 1$ and $Q^t = Q^{-1}$. Finally $\sigma_m^*\mathrm{Cof}(Dx^*) = Q\sigma_m Q^t Q\mathrm{Cof}(Dx) = Q\sigma_m\mathrm{Cof}(Dx)$.
8. Same argument as for the density
9. As for **6.**
10. We have $(D^*\theta^*)_i = \frac{\partial\theta^*}{\partial x_i^*} = \sum_l Q_{il}\frac{\partial\theta}{\partial x_l} = (QD\theta)_i(x(x^*))$.
11. We have
$(D^*v^*)_{ij}(t,x^*) = \frac{\partial v_i^*}{\partial x_j^*}(t,x^*) = \sum_l Q_{il}\frac{\partial v_l}{\partial x_j^*}(t,x(x^*)) = \sum_{ll'} Q_{il}Q_{jl'}\left[\frac{\partial v_l}{\partial x_{l'}}\right](t,x(x^*)) = (QDvQ^t)_{ij}(t,x(x^*))$.

Definition 15 *A quantity $g(t,x)$ is called a*

- *objective scalar if $g^*(t,x^*(x)) = g(t,x)$,*
- *objective vector if $g^*(t,x^*(x)) = Qg(t,x)$,*
- *objective tensor if $g^*(t,x^*(x)) = Qg(t,x)Q^t$,*

for any $Q \in SO(d)$.

In our case we have

- objective scalars: ρ, ϵ, r
- objective vectors: $x, Dx, v, \partial_x, f, q, D\theta, \mathcal{S}$
- objective tensor: σ, Dv

Remark. Though Dx and $\mathcal{S}$ are matrix-valued functions they act as objective vectors under rotations on x.

Transformation rules under material deformation

We will consider only linear transformations on Ω :

$$X \to X^* = gX,$$

where $g \in G$, G is some subgroup of $GL(d)$ (it will depend on the material).
We will always assume $\det g = 1$.

Lemma 13 *Under the change of coordinates $X \to X^* = gX$ the deformation gradient and the Piola-Kirchhoff tensor transform as*

$$Dx(t,X) \to Dx^*(t,X^*(X)) = Dx(t,X)g^{-1}, \qquad \mathcal{S}^*(t,X^*(X)) = S_P(t,X)g^t. \tag{1.5.58}$$

All the other quantities we introduced above are invariant i.e.

$$\Phi^*(t,X^*(X)) = \Phi(t,X). \tag{1.5.59}$$

In particular $x^*(t,X^*) = x(t,X)$.

Proof $(Dx^*)_{ij} = \frac{\partial x_i^*}{\partial X_j^*} = \sum_l \frac{\partial X_l}{\partial X_j^*}\frac{\partial x_i}{\partial X_l} = \sum_l \frac{\partial x_i}{\partial X_l} g_{lj}^{-1} = (Dx\, g^{-1})_{ij}$.
$\mathcal{S}^*(t,X^*(X)) = \sigma_m^*[(Dx^*)^{-1}]^t\det DX^* = \sigma_m(t,X)[(Dx\,g^{-1})^{-1}]^t\det[Dx\,g^{-1}] = S_P(t,X)g^t$, where we used $\det g = 1$. □

Definition 16 (Frame indifference) *We say that a material satisfies frame indifference if its constitutive laws are independent of the observer, i.e. they are invariant under the transformation $x \to x^* = Qx$.*

Definition 17 (Material symmetry) *The group $G \subset GL_+(d)$ is a material symmetry for material if the constitutive laws are invariant under the transformation $X \to X^* = gX$ $\forall g \in G$.*

1.5.2 Frame indifference in elastic and hyperelastic materials

Elastic material

The constitutive law for an elastic material is a function $\hat{\sigma} : GL_+(d) \to \mathbb{R}^{d\times d}_{sym}$ such that $\sigma(t,x) = \hat{\sigma}(Dx)$. Since σ is an objective tensor, $\sigma^*(t,x^*(x)) = Q\sigma(t,x)Q^t$, and $Q\hat{\sigma}(Dx)Q^t = Q\sigma Q^t = \sigma^* = \hat{\sigma}^*(Dx^*) = \hat{\sigma}^*(QDx)$. Hence $\hat{\sigma}^*$ must satisfy $\hat{\sigma}^*(QF) = Q\hat{\sigma}(F)Q^t$.
Given $\hat{\sigma}$, the constitutive law for the Piola-Kirchhoff tensor $\mathcal{S} = \sigma_m \mathrm{Cof}(Dx)$ is the function $\hat{\mathcal{S}} : GL_+(d) \to \mathbb{R}^{d\times d}$ defined by

$$\hat{\mathcal{S}}(F) = \hat{\sigma}(F)\mathrm{Cof}(F). \tag{1.5.60}$$

Since $\mathcal{S}$ is an objective vector $\hat{\mathcal{S}}^*(QF) = Q\hat{\mathcal{S}}(F)$.
Note that though $\hat{\sigma}$ is a symmetric matrix, $\hat{\mathcal{S}}(F)$ is in general not symmetric, but it must satisfy

$$\hat{\mathcal{S}}(F)F^t = \sigma(F)\det(F) \in \mathbb{R}^{d\times d}_{sym}. \tag{1.5.61}$$

Definition 18 *We say that an elastic material with constitutive functions $\hat{\sigma}$ and $\hat{\mathcal{S}}$ satisfies frame indifference if $\hat{\sigma}^*(Dx^*_s) = \hat{\sigma}(Dx^*_s)$, and $\hat{\mathcal{S}}^*(Dx^*) = \hat{\mathcal{S}}(Dx^*)$.*

By frame indifference we have

$$\hat{\sigma}(QF) = Q\hat{\sigma}(F)Q^t \qquad \forall F \in GL_+(d), \forall Q \in SO(d) \tag{1.5.62}$$

$$\hat{\mathcal{S}}(QF) = Q\hat{\mathcal{S}}(F) \qquad \forall F \in GL_+(d), \forall Q \in SO(d). \tag{1.5.63}$$

Hyperelastic material

Definition 19 *Consider an elastic material with constitutive law for the stress tensor $\sigma(t,x) = \hat{\sigma}((Dx)_s)$, and constitutive law for the Piola-Kirchhoff tensor $\mathcal{S}(t,X) = \hat{\mathcal{S}}(Dx)$, with $\hat{\mathcal{S}}(F) = \sigma(F)\mathrm{Cof}(F)$. $\hat{\mathcal{S}} : GL_+(d) \to \mathbb{R}^{d\times d}$.*
The material is called hyperelastic if there is a function $\hat{W} \in C^1(GL_+(d);\mathbb{R})$ such that

$$\hat{S}_{ij}(F) = \frac{\partial}{\partial F_{ij}}\hat{W}(F).$$

The function $\hat{W}$ is called the stored energy.

Definition 20 *We say that hyperelastic material with constitutive functions $\hat{\sigma}$, $\hat{\mathcal{S}}$, $\hat{W}$ satisfies frame indifference if $\hat{\sigma}^* = \hat{\sigma}$, $\hat{\mathcal{S}}^* = \hat{\mathcal{S}}$ and $\hat{W}^* = \hat{W}$.*

Lemma 14 *For a hyperelastic material the following three assertions are equivalent.*

(i) The constitutive law for the stress tensor $\hat{\sigma} : GL_+(d) \to \mathbb{R}^{d\times d}_{sym}$ satisfies frame indifference i.e.

$$\hat{\sigma}(QF) = Q\hat{\sigma}(F)Q^t \qquad \forall F \in GL_+(d), \forall Q \in SO(d).$$

(ii) The constitutive law for the Piola-Kirchhoff tensor $\hat{\mathcal{S}} : GL_+(d) \to \mathbb{R}^{d\times d}$. satisfies frame indifference i.e.

$$\hat{\mathcal{S}}(QF) = Q\hat{\mathcal{S}}(F) \qquad \forall F \in GL_+(d), \forall Q \in SO(d).$$

and $\hat{\mathcal{S}}(F)F^t \in \mathbb{R}^{d\times d}_{sym}$.

(iii) The stored energy function $\hat{W} : GL_+(d) \to \mathbb{R}$ satisfies $\hat{W}^* = \hat{W}$ and

$$\hat{W}(QF) = \hat{W}(F) \qquad \forall F \in GL_+(d), \forall Q \in SO(d).$$

Proof $(i) \Leftrightarrow (ii)$ is just a consequence of the definitions of σ and $\mathcal{S}$.
$(iii) \Rightarrow (ii)$. Remember that $\hat{\mathcal{S}}_{ij}(F) = \frac{\partial}{\partial F_{ij}}\hat{W}(F)$. Since $\hat{W}(QF) = \hat{W}(F)$ $\forall Q \in SO(d)$ we have

$$\frac{\partial}{\partial F_{ij}}(\hat{W}(QF)) = \frac{\partial}{\partial F_{ij}}(\hat{W}(F) = \hat{\mathcal{S}}_{ij}(F).$$

Performing the derivatives in the first term we get

$$\frac{\partial}{\partial F_{ij}}(\hat{W}(QF)) = \sum_{ll'} \partial_{F_{ij}}(QF)_{ll'} \left[\partial_{F_{ll'}}\hat{W}\right]_{|QF} = (Q^t\hat{\mathcal{S}}(QF))_{ij}$$

Hence $Q^t\hat{\mathcal{S}}(QF) = \hat{\mathcal{S}}(F)$. It remains to prove that $\hat{\mathcal{S}}(F)F^t \in \mathbb{R}^{d\times d}_{sym}$. Since $\hat{W}(QF) = \hat{W}(F)$ $\forall Q \in SO(d)$, this is true also for all Q of the form $Q_\alpha = e^{\alpha X}$ with $\alpha \in \mathbb{R}$ and $X^t = -X$. Then we have

$$0 = \partial_\alpha \hat{W}(Q_\alpha F)_{|\alpha=0} = \sum_{ll'}[\partial_\alpha(Q_a F)_{ll'}]_{|\alpha=0}\partial_{F_{ll'}}\hat{W}(F) = \mathrm{tr}(\hat{\mathcal{S}}(F)(XF)^t) = \mathrm{tr}(\hat{\mathcal{S}}(F)F^tX^t.$$

For any matrix M, the identity $\mathrm{tr}MX = 0$ for all $X^t = -X$ iff $M^t = M$. Indeed for any skew-matrix X we can write

$$\mathrm{Tr}(MX) = \sum_{i<j}(M_{ij} - M_{ji})X_{ij} = 0.$$

Since the matrix elements $(X_{ij})_{i<j}$ are independent it follows $M_{ij} = M_{ji}$. This concludes the proof of symmetry.
$(ii) \Rightarrow (iii)$. We want to prove that $\hat{W}(QF) = \hat{W}(F)$ i.e 'morally' $\partial_Q\hat{W}(QF) = 0$. Since $SO(d)$ is not a linear space, we cannot apply the derivative directly, but we have to pass through the corresponding Lie group (tangent space.) The analog of ∂_{x_j} (derivative in direction j) for a linear space is $\partial_\alpha\hat{W}(e^{\alpha X}QF)_{|\alpha=0}$. For any $X^t = -X$ (corresponds to a direction in the tangent space) we have

$$\partial_\alpha\hat{W}(e^{\alpha X}QF)_{|\alpha=0} = \sum_{ll'}[\partial_\alpha(e^{\alpha X}QF)_{ll'}]_{|\alpha=0}[\partial_{F_{ll'}}\hat{W}](QF) = \mathrm{tr}(\hat{\mathcal{S}}(QF)(XQF)^t) = \mathrm{tr}(Q\hat{\mathcal{S}}(F)F^tQ^tX^t = 0$$

where in the last step we used $(\hat{\mathcal{S}}(QF) = Q(\hat{\mathcal{S}}(F)$ and $Q\hat{\mathcal{S}}(F)F^tQ^t$ is symmetric. Then $\hat{W}(QF) = \hat{W}(F)$. □

1.5.3 Material symmetries in elastic materials

Definition 21 *The group $G \subset GL_+(d)$ is a material symmetry for an elastic material with constitutive law $\hat{\sigma}$ if $\hat{\sigma}$ satisfies*

$$\hat{\sigma}(Fg) = \hat{\sigma}(F) \quad \forall F \in GL_+(d),\ \forall g \in G.$$

We have seen that $G = SO(d)$ for a homogeneous solid and $G = SL(d)$ for a fluid. The following result garantees there is no symmetry between a fluid and a solid. An elastic material can be only a solid or a fluid.

Theorem 5 (W. Noll) . *Let $d \geq 2$ and $SO(d) \subset G \subset SL(d)$ group. Then we must have $G = SO(d)$ or $G = SL(d)$*

Proof. See W. Noll[1]. □

[1] 'Proof of the maximality of the orthogonal group in the unimodular group', Arch. Rat. Mech. Anal. 18 (1965), pp. 100-102

Elastic fluid

Theorem 6 *Let us consider a Cauchy-elastic material satisfying frame indifference, and with material symmetry group $G = SL(d)$, i.e.*

- *there exists* $\hat{\sigma} : GL_+(d) \to \mathbb{R}^{d\times d}_{sym}$ *such that* $\sigma(t,x) = \hat{\sigma}(Dx)$,
- *by frame indifference* $\hat{\sigma}(QF) = Q\hat{\sigma}(F)Q^t \qquad \forall F \in GL_+(d), \forall Q \in SO(d)$,
- *by material symmetry* $\hat{\sigma}(Fg) = \hat{\sigma}(F)\ \forall g \in SL(d)$.

Then there exists a function $\hat{p} : (0,\infty) \to \mathbb{R}$ *such that*

$$\hat{\sigma}(F) = -\hat{p}\left(\frac{1}{\det F}\right) Id \tag{1.5.64}$$

where $\hat{p}$ *is called the pressure.*

Proof. exercise

Hyperelastic materials

The material symmetry implies restrictions on $\hat{\sigma}$ and $\hat{\mathcal{S}}$, but NOT becessarily on the stored energy $\hat{W}$. This is the content of the next lemma.

Lemma 15 $\hat{W} \in C^1(GL_+(d);\mathbb{R})$ *be the stored energy for an hyperelastic material and let* $G \subset SL(d)$ *a group. Let us consider the following three statements.*

1. $\hat{W}(F) = \hat{W}(Fg)\ \forall g \in G,\ \forall F \in GL_+(d)$.
2. $\hat{\mathcal{S}}(Fg)g^t = \hat{\mathcal{S}}(F)\ \forall F \in GL_+(d)$.
3. $\hat{\sigma}(Fg) = \hat{\sigma}(F)\ \forall F \in GL_+(d)$.

Then 2. *holds if and only* 3. *holds,* 1. *implies* 2.*, but* 2. *does not imply* 1..

Proof exercise sheet

In the special case of a hyperleastic solid with material symmetry in the stored energy, the function $\hat{W}$ takes a simple form.

Lemma 16 *The two following statements are equivalent.*

1. *for any* $F \in GL_+(d)$ *we have material symmetry* $\hat{W}(Fg) = \hat{W}(F)\ \forall g \in SO(d)$ *(material group for a isotropic solid), and frame indifference* $\hat{W}(QF) = \hat{W}(F)\ \forall Q \in SO(d)$.
2. *there exists a function* $g : (0,\infty)^d \to \mathbb{R}$ *such that*

   $$\hat{W}(F) = g(\lambda_1(F), \ldots, \lambda_d(F))$$

   where $\lambda_i(F)$, $i = 1, \ldots, d$ *are the singular values of* F.

Reminder. For any $F \in \mathbb{R}^{d\times d}$ there exists three real matrices $R, T \in SO(d)$ and $\hat{\lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_d)$ such that

$$F = R\hat{\lambda}T.$$

The diagonal elements $\lambda_i(F)$, $i = 1, \ldots, d$ are called the singular values of F.

Proof. Let $F = R\hat{\lambda}T$ be the singular value decomposition of F. Then

$$\hat{W}(F) = \hat{W}(R\hat{\lambda}T) = \hat{W}(\hat{\lambda}T) = \hat{W}(\hat{\lambda}) = g(\hat{\lambda})$$

where in the second equality we used frame indifference and in the third material symmetry. □

[Lecture 9: 11.05]

1.5.4 Frame indifference and heat flux

We need the following preliminary definition

Definition 22 *A material is* <u>*isotropic*</u> *if the reference mass density is constant i.e.*

$$\rho_0(X) = \rho_0 \qquad \forall X \in \Omega.$$

Lemma 17 *An isotropic material with constant reference density* ρ_0 *has density*

$$\rho(t,x) = \frac{\rho_0}{\det(Dx)_s(t,x)}. \tag{1.5.65}$$

Proof. The continuity equation in material coordinates gives

$$\rho_m(t,X) = \frac{\rho_0(X)}{\det Dx(t,X)} = \frac{\rho_0}{\det Dx(t,X)}.$$

Hence the result. □

Let $\theta : \mathcal{T} \to \mathbb{R}$ the temperature function and assume the heat flux has constitutive law

$$\begin{aligned} \hat{q} :\ & \mathbb{R}^d && \to \mathbb{R}^d \\ & w && \to \hat{q}(w) \end{aligned}$$

such that $q(t,x) = \hat{q}(D\theta(t,x))$.

Change of observable. Since q is an objective vector then $q^*(t, x^*(x)) = Qq(t,x)$, and $Q\hat{q}(D\theta) = Qq = q^* = \hat{q}^*(D^*\theta^*) = \hat{q}^*(QD\theta)$. Then $\hat{q}^*$ must satisfy

$$\hat{q}^*(Qw) = Q\hat{q}(w) \qquad \forall w \in \mathbb{R}^d, \forall Q \in SO(d). \tag{1.5.66}$$

Definition 23 *We say that the heat flux satisfies frame indifference if* $\hat{q}^*(D^*\theta^*) = \hat{q}(D^*\theta^*)$

By frame indifference we obtain

$$\hat{q}(Qw) = Q\hat{q}(w) \qquad \forall w \in \mathbb{R}^d, \forall Q \in SO(d). \tag{1.5.67}$$

Energy equation

We assume that the material is

- homogeneous: the reference density is constant $\rho_0(X) = \rho_0\ \forall X \in \Omega$,
- static: no motion i.e. $x(t,X) = x(0,X)\ \forall t$, hence $v = 0$.

Then the energy conservation reduces to

$$\partial_t \epsilon(t,x) = r(t,x) - \mathrm{div}(q), \tag{1.5.68}$$

where we used $D_t\epsilon = \partial_t\epsilon + v \cdot \nabla\epsilon = \partial_t\epsilon$, and $\mathrm{div}(\sigma^t Dv) = 0$, since $v = 0$.

Fourier's law. Let $q \in C^1$ and assume the following properties.

1. The material is isotropic i.e. the description does not change with rotations $\hat{q}(w) = \hat{q}^*(w)$ $\forall w$. Hence
$$\hat{q}(Qw) = Q\hat{q}(w) \qquad \forall w \in \mathbb{R}^d, \forall Q \in SO(d). \tag{1.5.69}$$

2. The heat flux is parallel and opposite to the temperature gradient and $q = 0$ when $D\theta = 0$ (if no temperature gradient is present, there is no flux) Then there exists a function $h : \mathbb{R}_+ \to \mathbb{R}_+$, such that $\hat{q}(w) = -h(|w|)w$.

3. The flux depends linearly on $D\theta$, (Fourier's law) i.e. $\exists$ a matrix $H \in \mathbb{R}^{d\times d}$ such that
$$\hat{q}(D\theta) = H\, D\theta$$

Linearity+isotropy (**1.** and (**3.**) imply that $Q^{-1}HQw = Hw$ for all $w \in \mathbb{R}^d$ and all $Q \in SO(d)$. Then $H = \alpha Id$, where $\alpha \in \mathbb{R}$. By (**2.** we have $\alpha = -h < 0$. As a result
$$\text{div} q = -h\Delta\theta.$$

Heat equation. We assume the following constitutive law for the internal energy
$$\epsilon(t,x) = c\,\theta(t,x)$$
where $c > 0$ is a constant called the specific heat. Then (1.5.68) becomes
$$\partial_t\theta = \frac{h}{\rho_0 c}\Delta\theta + \frac{r}{\rho_0 c}$$
where $\frac{h}{\rho_0 c}$ is a positive constant and $\frac{r}{\rho_0 c}$ may depend on (t,x) through r. This is the heat equation in d dimensions with an external source.

[Lecture 10: 13.05]

1.5.5 Frame indifference in fluids

Ideal fluid and Euler equation

Definition 24 *An ideal fluid is an isotropic elastic material with constitutive law for the stress tensor σ satisfying frame indifference and with material symmetry group $G = SL(d)$ i.e.*
$$\rho_0(X) = \rho_0 > 0 \quad \forall X \in \Omega$$
$$\left.\begin{array}{l}\hat{\sigma}(QF) = Q\hat{\sigma}(F)Q^t \\ \hat{\sigma}(Fg) = \hat{\sigma}(F)\end{array}\right\} \quad \forall F \in GL_+(d), Q \in SO(d), g \in SL(d).$$

From Theorem 6 above, there exists a function $\hat{p} : (0,\infty) \to \mathbb{R}$ such that
$$\hat{\sigma}(F) = -\hat{p}\left(\frac{1}{\det F}\right) Id$$
where $\hat{p}$ is called the pressure. From Lemma 17 above we know that $\rho(t,x) = \frac{\rho_0}{\det(Dx)_s(t,x)}$, hence
$$\sigma(t,x) = \hat{\sigma}((Dx)_s) = -\hat{p}\left(\frac{1}{(\det Dx)_s}\right) Id = -\hat{p}\left(\frac{\rho(t,x)}{\rho_0}\right) Id \tag{1.5.70}$$

Lemma 18 *An ideal fluid with no external forces, and $\rho_0 = 1$, satisfies the system of equations*

$$\begin{cases} \frac{D\rho}{Dt} + \rho \operatorname{div} v & = 0 \\ \rho \frac{Dv}{Dt} + \nabla \hat{p}(\rho) & = 0 \end{cases} \tag{1.5.71}$$

These are called compressible Euler equations and can be written also as follows

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) & = 0 \\ \rho \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla \hat{p}(\rho) & = 0 \end{cases} \tag{1.5.72}$$

where

$$\begin{array}{lll} \otimes : & \mathbb{R}^d \times \mathbb{R}^d & \to \mathbb{R}^{d\times d} \\ & (a, b) & \to (a \otimes b)_{ij} = a_i b_j \end{array}$$

and

$$[\operatorname{div}(a \otimes b)]_i = \sum_j \partial_{x_j} (a \otimes b)_{ij}.$$

Proof (1.5.71) is just a consequence of the realations above. To prove the second equation in (1.5.72) we replace in the equation of motion

$$\begin{aligned} \rho D_t v_i = \rho \partial_t v_i + \rho (v \cdot \nabla) v_i &= \partial_t(\rho v_i) - (\partial_t \rho) v_i \rho (v \cdot \nabla) v_i \\ &= \partial_t(\rho v_i) + \sum_l [\partial_{x_l}(\rho v_l) v_i + \rho v_l \partial_{x_l} v_i] = \partial_t(\rho v_i) + \sum_l \partial_{x_l}(\rho v_l v_i) \end{aligned}$$

where in the second line we used the continuity equation. □

Viscous fluid and Navier-Stokes equation

We assume now that σ depends on local deformations Dx and also on local velocity changes Dv. The constitutive law is then a function

$$\begin{array}{lll} \hat{\sigma} : & GL_+(d) \times \mathbb{R}^{d\times d} \to & \mathbb{R}^{d\times d}_{sym} \\ & (F, G) \to & \hat{\sigma}(F, G) \end{array}$$

such that $\sigma(t, x) = \hat{\sigma}((Dx)_s, Dv)$. Remember that Dx is an objective vector, while σ and Dv are objective tensors. Then $Q\hat{\sigma}(Dx, Dv)Q^t = Q\sigma Q^t = \sigma^* = \hat{\sigma}^*(Dx^*, D^*v^*) = \hat{\sigma}^*(QDx, QDvQ^t)$. By frame-indifference $\hat{\sigma} = \hat{\sigma}^*$, hence $\hat{\sigma}$ must satisfy

$$\hat{\sigma}(QF, QGQ^t) = Q\hat{\sigma}(F, G)Q^t \qquad \forall F \in GL_+(d), G \in \mathbb{R}^{d\times d}, \forall Q \in SO(d) \qquad \forall Q \in SO(d). \tag{1.5.73}$$

Under material deformation $\sigma^* = \sigma$, $Dx^* = Dxg^{-1}$ and $Dv^* = Dv$, hence material symmetry $\hat{\sigma} = \hat{\sigma}^*$, implies

$$\hat{\sigma}(Fg^{-1}, G) = \hat{\sigma}(F, G) \quad \forall g \in SL(d).$$

Additional assumption. We assume viscosity is a small perturbation of the idea fluid i.e.

$$\hat{\sigma}(F, G) = \hat{\sigma}_1(F, G) + \mathcal{L}(G)$$

with $\mathcal{L}(0) = 0$. Frame indifference and material symmetry hold for any $F \in GL_+(d), G \in \mathbb{R}^{d\times d}$ hence also for $G = 0$, then $\hat{\sigma}_1$ must satisfy $\hat{\sigma}_1(QF) = Q\hat{\sigma}_1(F)Q^t$ and $\hat{\sigma}_1(Fg^{-1}) = \hat{\sigma}_1(F)$, hence $\hat{\sigma}_1(F) = -\hat{p}\left(\frac{1}{\det F}\right) Id$. As a consequence $\mathcal{L}$ must satisfy

$$Q^t \mathcal{L}(G) Q = \mathcal{L}(QGQ^t).$$

If we assume isotropy, then

$$\sigma(t, x) = -\hat{p}\left(\frac{\rho}{\rho_0}\right) + \mathcal{L}(Dv). \tag{1.5.74}$$

Theorem 7 (fluid with linear viscosity) *Let $d = 3$ and assume $\mathcal{L} : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}_{sym}$ is linear. Assume that $\sigma(t,x) = -\hat{p}\left(\frac{\rho}{\rho_0}\right) + \mathcal{L}(Dv)$ and $\mathcal{L}$ satisfies frame indifference $\mathcal{L}(QGQ^t) = Q\mathcal{L}Q^t$ $\forall G \in \mathbb{R}^{3\times 3}$, $\forall Q \in SO(3)$. Then there exists $\lambda, \mu \in \mathbb{R}$ such that*

$$\mathcal{L}(G) = 2\mu \frac{G + G^t}{2} + \lambda(\mathrm{tr}G)Id. \tag{1.5.75}$$

Remark Note that since $\mathrm{tr}G = \mathrm{tr}\frac{G+G^t}{2}$ the function $\mathcal{L}$ depends only on $\mathrm{Re}G = \frac{G+G^t}{2}$.

Proof exercise sheet.

Lemma 19 *A fluid with linear viscosity, no external forces, and $\rho_0 = 1$, satisfies the system of equations*

$$\begin{cases} \frac{D\rho}{Dt} + \rho\ \mathrm{div}v = 0 \\ \rho\frac{Dv}{Dt} = -\nabla\hat{p}(\rho) + \mu\Delta v + (\mu+\lambda)\nabla(\mathrm{div}v) \end{cases} \tag{1.5.76}$$

These are called compressible Navier-Stokes equations

Proof We need to replace (1.5.74) and (1.5.75) inside $\mathrm{div}(\sigma)$. We have

$$\begin{aligned} \mathrm{div}(-\hat{p}(\rho)_i &= -\partial_{x_i}\hat{p}(\rho) \\ [\mathrm{div}(Dv)]_i &= \Delta v_i \\ [\mathrm{div}(Dv^t)]_i &= \partial_{x_i}(\mathrm{div}(v)) \\ [\mathrm{div}(\mathrm{tr}Dv)Id]_i &= \partial_{x_i}(\mathrm{div}(v)) \\ \Rightarrow [\mathrm{div}(\sigma)]_i &= -\nabla_i\hat{p}(\rho) + \mu\Delta v_i + (\mu+\lambda)\nabla_i(\mathrm{div}v). \end{aligned}$$

Inserting this in the equation of motion we get the result. □
Note that if we set $\mu = \nu = 0$ we recover Euler-equations.

Incompressible fluid

In an incompressible fluid the volume occupied by the test volume U is independent of time $U(t) = U$ $\forall t$. By Lemma 6 this holds iff $\mathrm{div}(v) = 0$.

Lemma 20 *Assume $x(0,t) = X$, the fluid is isotropic and incompressible. Then the mass density is constant: $\rho(t,x) = \rho_0$ $\forall t, x$.*

Proof. Inserting $\mathrm{div}(v) = 0$ in the continuity equations we get $D_t\rho = 0$, which is equivalent to $\partial_t\rho_m = 0$. Then $\rho_m(t,X) = \rho_m(0,X)$ $\forall t$. By isotropy $\rho_m(t,X) = \frac{\rho_0}{\det Dx(t,X)}$, hence $\rho_m(0,X) = \rho_0$ $\forall X$ since $Dx(0,X) = Id$. □.
In the following we replace the continuity equation for an incompressible isotropic fluid just by $\mathrm{div}(v) = 0$. To enforce the incompressible constraint in the equation of motion we must replace $\hat{p}(\rho)$ by an independent function $p(t,x)$, the physical pressure. This can be seen as a Lagrange multiplier for the constraint $\mathrm{div}(v) = 0$.

Lemma 21 *An incompressible isotropic fluid with linear viscosity and no external force satisfies the system of equations*

$$\begin{cases} \mathrm{div}v = 0 \\ \frac{Dv}{Dt} = \nu\Delta v - \nabla\left(\frac{p}{\rho_0}\right) \end{cases} \tag{1.5.77}$$

where v, p are two unknown functions and $\nu = \frac{\mu}{\rho_0}$. These are called incompressible Navier-Stokes equations

If we set $\mu = 0$ we obtain

$$\begin{cases} \mathrm{div}v = 0 \\ \frac{Dv}{Dt} = -\nabla\left(\frac{p}{\rho_0}\right) \end{cases} \tag{1.5.78}$$

These are called incompressible Euler equations

2. Hydrodynamics

2.1 Introduction

The starting point are incompressible Navier-Stokes and Euler equations from the last chapter. We assume in the following frame indifference, isotropy, material symmetry group $SL(d)$ and linear viscosity. We consider

$$\begin{cases} \mathrm{div} v = 0 & (t,x) \in \mathcal{T} \\ \frac{Dv}{Dt} = \nu \Delta v - \nabla \left(\frac{p}{\rho_0}\right) & \end{cases} \tag{2.1.1}$$

where v, p are two unknown functions and $\nu = \frac{\mu}{\rho_0}$. We now need to insert boundary and initial conditions.

2.1.1 Boundary and initial conditions

The boundary may be external (the fluid is contained in the region Ω) or internal (the flow goes around some obstacle). At infinity we always assume the fluid has constant velocity (see Lecture 1). For finite boundary point we will consider two types of boundary conditions.

- Fluid with viscosity $\nu > 0$. One generally fixes the velocity at the boundary $v(t,x) = 0$ $\forall x \in \partial\Omega$. These are called *no slip* (or sticky) boundary conditions (the fluid sticks to the boundary).
- Fluid without viscosity $\nu = 0$. One generally fixes the normal velocity at the boundary $v(t,x) \cdot n(x) = 0$ $\forall x \in \partial\Omega$, where $n(x)$ is the vector orthogonal to the boundary at x. This means there is *no flux* through the boundary.

Remark Both boundary conditions imply that the boundary does not change: $\partial\Omega_t = \partial\Omega$ $\forall t$. That implies $\Omega_t = \Omega$, $\forall t$, i.e. $\mathcal{T} = \mathbb{R} \times \Omega$. Indeed we have $x(t,X) \in \partial\Omega_t$ iff $X \in \partial\Omega$.
Let $v_{|\partial\Omega_t} = 0$. Then $\partial_t x(t,X) = 0$ $\forall X \in \partial\Omega$. This means particles on the boundary do not move, hence the result.
Let $v(t,x) \cdot n(x) = 0$ $\forall x \in \partial\Omega$. Then the velocity is zero in the orthogonal directions and particles on the boundary must remain always on the boundary, though they may move along it.

[Lecture 11: 27.05]

Initial boundary value problem The classical initial boundary value problem consists in solving

$$\begin{cases} \partial_t v + v \cdot \nabla v = & \nu \Delta v - \nabla \left(\frac{p}{\rho_0}\right) \\ \mathrm{div}\ v = 0 & t \geq 0, x \in \Omega \end{cases} \tag{2.1.2}$$

with boundary condition $v_{|\partial\Omega} = 0$ if $\nu > 0$, and $(v \cdot n)_{|\partial\Omega} = 0$ if $\nu = 0$, and initial condition

$$v(0,x) = v_0(x) \quad \forall x \in \Omega.$$

This means we need to find a pair of functions (v,p), $v : [0,\infty) \times \Omega \to \mathbb{R}^d$, $p : [0,\infty) \times \Omega \to \mathbb{R}$, with some minimal regularity properties satisfying the equations above.

Remark 1 No initial condition is given on p

Remark 2 The function p cannot be unique. If $p(t,x)$ is a solution, then $p+h$ is also a solution for any function $h : [0,\infty) \to \mathbb{R}$ depending only on t.

Remark 3 A solution v is called stationary if $\partial_t v = 0$.

2.2 Vorticity formulation

Definition 25 *Let x be a motion and v the corresponding velocity field. The vorticity in $d = 3$ is defined by*

$$\begin{array}{rll} \omega : & \mathcal{T} \to & \mathbb{R}^3 \\ & (t,x) \to & \omega(t,x) = curl\ \omega = \nabla \times v \end{array}$$

In components $(curl\ \omega)_i = \sum_{jk} \epsilon^{ijk} \partial_{x_j} v_k$. The vorticity in $d = 2$ is defined by

$$\begin{array}{rll} \omega : & \mathcal{T} \to & \mathbb{R} \\ & (t,x) \to & \omega(t,x) = \partial_{x_1} v_2 - \partial_{x_2} v_1. \end{array}$$

The vorticity gives information on how much the flow is 'winding' locally. It is related to the velocity gradient as follows.

Lemma 22 *Let $W = \frac{Dv-(Dv)^t}{2}$, and $S = \frac{Dv+(Dv)^t}{2}$ the skew-symmetric (respectively symmetric) part of Dv. Then*

(i) we have $W^t = -W$, $S^t = S$, $Dv = W + S$,

(ii) when $d = 3$ it holds

$$\omega_i = -\sum_{jk} \epsilon^{ijk} W_{jk}, \qquad W_{ij} = -\frac{1}{2} \sum_k \epsilon^{ijk} \omega_k, \tag{2.2.3}$$

(iii) for any vector $V \in \mathbb{R}^3$ we have

$$(WV)_i = \frac{1}{2}(\omega \times V)_i.$$

Proof (i) : easy. (ii) : note that we have the relations

$$(a \times b)_i = \sum_{jk} \epsilon^{ijk} a_j b_k, \qquad (a \wedge b)_{ij} = \sum_k \epsilon^{ijk} (a \times b)_k \qquad \forall a,b \in \mathbb{R}^3.$$

Then

$$W_{ij} = \frac{1}{2}(Dv_{ij} - (Dv)_{ji}) = \frac{1}{2}(\partial_j v_i - \partial_i v_j) = -\frac{1}{2}(\nabla \wedge v)_{ij} = -\frac{1}{2} \sum_k \epsilon^{ijk} (\nabla \times v)_k = -\frac{1}{2} \sum_k \epsilon^{ijk} \omega_k.$$

Moreover

$$\omega_i = (\nabla \times v)_i = \frac{1}{2} \sum_{jk} \epsilon^{ijk} (\nabla \wedge v)_{jk} = -\sum_{jk} \epsilon^{ijk} W_{jk}.$$

(iii) :

$$(WV)_i = \sum_j W_{ij} V_j = -\frac{1}{2} \sum_{jk} \epsilon^{ijk} V_j \omega_k = \frac{1}{2} (\omega \times V)_i$$

since $a \times b = -b \times a$. □

We will need the following result from linear algebra.

Lemma 23 *Let $A \in \mathbb{R}^{3\times 3}$ some matrix s.t. $trA = 0$. Let $S = \frac{A+A^t}{2}$, $W = \frac{A-A^t}{2}$ and let $w \in \mathbb{R}^3$ the vector $\omega_i = -\sum_{jk} \epsilon^{ijk} W_{jk}$. Then the following identity holds*

$$\sum_{jk} \epsilon^{ijk} (A^2)_{kj} = -(S\omega)_i.$$

Proof exercise

2.2.1 Local structure of the flow

The matrices W (hence the vorticity) and S encode information of the local structure of the flow. Precisely W corresponds to local rotations while S to local dilatation (or stretching). To understand this let us consider two examples.

Example 1 [Jet flow] Assume $W(t,x) = 0$ and $S(t,x) = S_0$ constant for all (t,x). Then $Dv = S$ and $\partial_{x_i} v_j$ is constant, hence v is a linear function of x. Precisely there exists a function $v_0(t)$ such that

$$v(t,x) = v_0(t) + Sx$$

The trajectory $x(t,X)$ is a solution of the PDE

$$\partial_t x(t,X) = v(t, x(t,X)) = Sx(t,X) + v_0(t)$$

i.e.

$$x(t,X) = e^{tS} x(0,X) + \int_0^t e^{(t-\tau)S} v_0(\tau) d\tau.$$

We assume now that $v_0(t) = 0$ and $x(t,X) = X$. Then $v(t,x) = Sx$ and $x(t,X) = e^{tS}X$. Inserting these relations into N-S (2.1.1) and assuming $\rho_0 = 1$ we get

$$\begin{cases} \text{tr} S = 0 \\ S^2 x = -\nabla(p) \end{cases} \tag{2.2.4}$$

where we used

$$D_t v = (\partial_t V)_s, \quad \partial_t V(t,X) = \partial_t^2 x(t,X) = S^2 x(t,X),$$
$$\Delta v_i = \sum_j \partial_{x_j}^2 v_i = \sum_j \partial_{x_j} (Dv)_{ij} = \sum_j \partial_{x_j} S_{ij} = 0$$

since S is constant. The solution of $\partial_{x_i} p = -(S^2 x)_i$ is

$$p(x) = -\frac{1}{2} x \cdot S^2 x,$$

This solution is not unique since we can add any t dependent function. Finally S must satisfy $\text{tr}S = 0$. Since S is real symmetric we can diagonalize it performing a rotation. In the new basis $S = diag(\gamma_1, \gamma_2, \gamma_3)$. Inserting this expression in $v = Sx$, and (2.2.4) we obtain

$$v_i = \gamma_i x_i \Rightarrow x_i(t,X) = e^{t\gamma_i} X_i$$
$$\text{tr } S = \gamma_1 + \gamma_2 + \gamma_3 = 0$$
$$p(x) = -\frac{1}{2} \sum_{j=1}^3 \gamma_j x_j^2$$

If we take $\gamma_1 = \gamma_2 = \gamma < 0$, we must have $\gamma_3 = -2\gamma > 0$, hence $x_j(t) \to 0$, $j = 1,2$ and $x_3(t) \to \pm\infty$ as $t \to \infty$. The flow concentrates along the $z-$axis and is ejected at infinity. Note that $v_3(t) \to \infty$ as $t \to \infty$.

We must still insert the boundary conditions. Assume we are on the half-plane $x_3 \geq 0$. Then there is a boundary at $x_3 = 0$. The normal direction n is then n_3 and $v \cdot n = v_3 = \gamma_3 x_3$ is automatically zero at the boundary. So no-flux boundary conditions are automatically satisfied. On the contrary we cannot ensure no slip b.c. unless $\gamma = 0$, i.e. $S = 0$ and $v = 0$ everywhere.
A problem arises at $|x| \to \infty$ since the velocity diverges. This is physically impossible. Therefore this example can only describe a finite piece of fluid.

Example 2 [Vortex flow] Assume $S(t,x) = 0$ and $W(t,x) = W_0$ constant for all (t,x). Then $Dv = W$ and (as above) $\partial_{x_i} v_j$ is constant, hence v is a linear function of x. Precisely there exists a function $v_0(t)$ such that

$$v(t,x) = v_0(t) + Wx.$$

Assuming $v_0 = 0$ we get the trajectory

$$x(t,X) = e^{tW} X.$$

Since $W^T = -W$ we have

$$(e^{tW})^t e^{tW} = e^{tW^t} e^{tW} = e^{-tW)} e^{tW} = Id, \qquad \det e^{tW} = e^{\mathrm{tr} tW} = 1,$$

hence $e^{tW} \in SO(3)$, i.e. W is the infinitesimal generator for a rotation in $\mathbb{R}^3$. We can always rotate the basis in such a way that W generates rotations in the $x - y$ plane only. In that case we have

$$W = \begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.2.5}$$

Note that $W^2 = -\alpha^2 Id$, hence $W^{2n} = (-\alpha)^n Id$, $W^{2n+1} = (-\alpha)^n W$, and

$$\begin{aligned} e^{tW} &= \sum_{n\geq 0} \frac{t^n}{n!} W^n = \sum_{n\geq 0} \frac{t^{2n}}{2n!} W^{2n} + \sum_{n\geq 0} \frac{t^{2n+1}}{(2n+1)!} W^{2n+1} \\ &= \begin{pmatrix} 1_2 & 0 \\ 0 & 0 \end{pmatrix} \cos(\alpha t) + \frac{W}{\alpha} \sin(\alpha t) = \begin{pmatrix} \cos(t\alpha) & \sin(t\alpha) & 0 \\ -\sin(t\alpha) & \cos(t\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Then $x(t,X) =$ rotation around the $z-$axis with constant angular velocity α. Note that

$$v_1 = \alpha x_2, \quad v_2 = -\alpha x_1, \qquad v_3 = 0,$$

hence $\|v\| = \alpha \; dist(x, e_3) \to \infty$ as $dist(x, e_3) \to \infty$, i.e. the velocity grows if we are far from the $z-$axis, since the angular velocity is constant. To avoid the unphysical situation of having a diverging velocity as $dist(x, e_3) \to \infty$, we can take an infinite cylinder of radius R, i.e. $\Omega = \{(x, y, y) | x^2 + y^2 \leq R^2\}$. Note that the direction normal to the boundary at $x = (x_1, x_2, x_3)$ is $n_x = \frac{1}{R}(x_1, x_2, 0)$ hence

$$(n \cdot v) = (n \cdot Wx) = \alpha n \cdot (x_2, -x_1, 0) = 0.$$

So we have naturally no-flux boundary conditions.
Now, inserting these relations into N-S (2.1.1) and assuming $\rho_0 = 1$ we have as before $D_t v = W^2 x$, and $\Delta v = \sum_j \partial_{x_j} W_{ij} = 0$ since W is constant, hence

$$W^2 x = -\nabla(p) \tag{2.2.6}$$

where div $v = \mathrm{tr}(Dv) = \mathrm{tr}(W) = 0$ by construction since $W^t = -W$. The solution is

$$p(x) = -\frac{1}{2} x \cdot W^2 x = -\frac{1}{2}(x, W^2 x) = -\frac{1}{2}(W^t x, Wx) = \frac{1}{2}\|Wx\|^2$$

up to some arbitrary additive function $h(t)$. For W as in (2.2.5) we get $p(x) = \frac{\alpha}{2}[dist(x, e_3)]^2$.

Remark 1. Note that since W is skew-symmetric we have $x \cdot Wx = 0 \; \forall x \in \mathbb{R}^3$, but $(W^2)^t = W^2$ hence $x \cdot W^2 x$ need to be zero.

Remark 2. The portion of fluid far from the $z-$axis is hard to move since the pressure diverges as $dist(x, e_3) \to \infty$.

2.2.2 Vorticity formulation of N-S in $d = 3$

Theorem 8 *Let $d = 3$ and (p, v) be a solution of* (2.1.1) *with $\rho_0 = 1$ Then the vorticity $w = \mathrm{curl} v$ satisfies the equation*

$$D_t w = Sw + \nu \Delta w, \tag{2.2.7}$$

where $S = \frac{Dv + (Dv)^t}{2}$.

Note that we need at least $v \in C^2(I \times \Omega; \mathbb{R}^3)$ for some finite time interval.

Proof. Applying *curl* to both sides of N-S equation we get

$$curl \; [\partial_t v + v \cdot \nabla v - \nu \Delta v] = -curl \; [\nabla p] = 0$$

where in the last equality we used $curl \; \nabla p = \nabla \times \nabla p = 0$, by symmstry. Now exchanging derivatives we get

$$curl \; \partial_t v = \partial_t \omega, \qquad curl \; \Delta v = \Delta \omega.$$

Moreover

$$[curl \; v \cdot \nabla v]_i = \sum_{jk} \epsilon^{ijk} \partial_j (v \cdot \nabla) v_k = (v \cdot \nabla)\omega + \sum_{jkl} \epsilon^{ijk} \partial_j v_l \; \partial_l v_k = (v \cdot \nabla)\omega + \sum_{jkl} \epsilon^{ijk} (Dv)_{lj} (Dv)_{kl}$$

$$= (v \cdot \nabla)\omega + \sum_{jk} \epsilon^{ijk} (Dv)^2_{kl} \tag{2.2.8}$$

Since $\mathrm{div} v = \mathrm{tr}(Dv) = 0$ by Lemma 23 we have

$$[curl \; v \cdot \nabla v]_i = (v \cdot \nabla)\omega - (S\omega)_i.$$

Inserting this result in the equation above we get the result. □

[Lecture 12: 01.06]

2.2.3 Vorticity formulation of N-S in $d = 2$

Theorem 9 *Let $d = 2$ and (p, v) be a solution of* (2.1.1) *with $\rho_0 = 1$ Then the vorticity $w = \partial_{x_1} v_2 - \partial_{x_2} v_1$ satisfies the equation*

$$D_t w = \nu \Delta w, \tag{2.2.9}$$

Proof. exercise

Remark For $\nu = 0$ the vorticity is 'frozen'.

2.3 Vortex lines

Definition 26 *Let $d = 3$. A curve $\alpha \in C^1((0,1); \Omega)$ is called a vortex line at time t if there exists $\lambda : (0,1) \to \mathbb{R}$ s.t.*

$$\frac{d\alpha(s)}{ds} = \lambda(s) \omega(t, \alpha(s))$$

i.e. α is tangent to the vector field ω at time t.

Example In the case of the vortex flow with $S = 0$ and $W = \begin{pmatrix} 0 & \theta & 0 \\ -\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ we have already found $v_3 = 0$, $v_1 = \theta x_2$ and $v_2 = -\theta x_1$. The vorticity is given by $\omega = -e_3\|\omega\|$, where $\|\omega\| = 2\theta$ is the angular velocity. Note that ω is orthogonal to the plane of rotation. Then the function $\alpha(s) = se_3$ is a vortex line for any time t and in general $\alpha(s) = h(s)e_3$ is a vortex line for any time t for any function h at least C^1. Note that since ω is parallel to e_3 all vortex lines must be also parallel to e_3.

2.3.1 Vortex lines in an ideal fluid.

When $\nu = 0$, the vorticity in $d = 3$ satisfies the equation $D_t\omega = S\omega$. In particular vortex lines have a 'non mixing property', i.e they remain separated, as stated below.

Lemma 24 *Let (p, v) be a solution of (2.1.1) with $\nu = 0$. Then the vortex lines move with the fluid i.e. if $\alpha(s)$ is a vortex line at time $t = 0$, then $\alpha_t(s) := x(t, \alpha(s))$ is a vortex line at time t.*

Since the motion $x(t, X)$ is an invertible function, if we have two different vortex lines $\alpha_1(s) \neq \alpha_2(s)$ for some s then $(\alpha_1)_t(s) \neq (\alpha_2)_t(s)$ for all t.

Proof By definition $\frac{d\alpha}{ds}(s) = \lambda(s)\omega(0, \alpha(s))$ for some function $\lambda : (0, 1) \to \mathbb{R}$. To prove $\alpha_t(s)$ is a vortex line a time t we must find a function $\lambda_t : (0, 1) \to \mathbb{R}$ such that

$$\frac{d\alpha_t}{ds}(s) = \lambda_t(s)\omega(t, \alpha(s)).$$

Inserting $\alpha_t(s) := x(t, \alpha(s))$ and the definition of $\alpha(s)$ we get

$$\frac{d(\alpha_t)_i}{ds}(s) = \frac{\partial x_i(t, \alpha(s))}{\partial s} = \sum_j (Dx)_{ij}(t, \alpha(s))\frac{d\alpha}{ds}(s) = \lambda(s)\ [(Dx)(t, \alpha(s))\ \omega(0, \alpha(s))]_i.$$

We will prove below that

$$\omega_m(t, X) = Dx(t, X)\omega_m(0, X). \tag{2.3.10}$$

As a consequence

$$(Dx)(t, \alpha(s))\ \omega(0, \alpha(s)) = \omega_m(t, \alpha(s)) = \omega(t, x(t, \alpha(s))).$$

Inserting this result above we get

$$\frac{d\alpha_t}{ds}(s) = \lambda(s)\ \omega(t, x(t, \alpha(s))) = \lambda(s)\ \omega(t, \alpha_t(s)).$$

The result follows taking $\lambda_t(s) = \lambda(s)$. □
The proof of (2.3.10) is a corollary of the following theorem.

Theorem 10 *Let v be a smooth velocity field, $x(t, X)$ the corresponding motion and $h : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}^3$ some other smooth spatial field. Then we have*

$$D_t h(t, x) = (Dv)(t, x)h(t, x) \quad \Leftrightarrow \quad h_m(t, X) = Dx(t, X)h(0, X),$$

where we assumed $x(0, X) = X$, hence $h(0, x) = h(0, X) = (h_m)(0, X)$.

Proof (Exercise sheet)
($\Leftarrow$). Let $h_0(X) := h(0,X)$ and suppose $h_m(t,X) = Dx(t,X)h(0,X) = (Dx)h_0$. Then $\partial_t(h_m)_i = \sum_j \partial_t(Dx)_{ij}\,(h_0)_j$. Now

$$\begin{aligned}\partial_t(Dx)_{ij}(t,X) &= (\partial_t\partial_{X_j}x_i)(t,X) = \partial_{X_j}(v_i)_m(t,X) = \partial_{X_j}(v_i)(t,x(t,X))\\ &= \sum_k(\partial_{x_k}v_i)_m(t,X)(\partial_{X_j}x_k)(t,X) = [(Dv)_m(Dx)]_{ij}(t,X)\end{aligned} \tag{2.3.11}$$

Inserting this formula we get

$$\partial_t(h_m)_i = \sum_j \partial_t(Dx)_{ij}\,(h_0)_j = [(Dv)_m(Dx)h_0]_i = [(Dv)_m h_m]_i \Rightarrow \quad D_t h_i = [\partial_t(h_i)_m]_s = (Dv)h.$$

($\Rightarrow$). Suppose $D_t h = (Dv)h$. We want to prove $h_m(t,X) = (Dx)(t,X)h(0,X)$. Note that

$$h_m(t,X) = (Dx)(t,X)h_m(0,X) \quad \Leftrightarrow \quad [(Dx)^{-1}h_m](t,X) = h(0,X) \quad \text{independent of } t.$$

Fix $X \in \Omega$ and let $\psi(t) := ((Dx)^{-1}h_m)(t,X)$. It is enough to prove that $\psi'(t) = 0$. Now

$$\psi'(t) = [\partial_t(Dx)^{-1}]h_m + (Dx)^{-1}(\partial_t h_m) = (Dx)^{-1}[-(\partial_t Dx)(Dx)^{-1} + (Dv)_m]h_m$$

where we used $D_t h = (Dv)h$ and the formula $\partial_t(A^{-1}) = -(A^{-1})(\partial_t A)(A^{-1})$ which holds for any matrix-valued differentiable function $t \to A(t)$ such that $A(t)$ is invertible $\forall t$. From (2.3.11) above we have $\partial_t(Dx) = (Dv)_m Dx$, hence

$$-(\partial_t Dx)(Dx)^{-1} + (Dv)_m = -(Dv)_m(Dx)(Dx)^{-1} + (Dv)_m = 0.$$

This concludes the proof. □

Proof of (2.3.10) From (2.2.7) with $\nu = 0$, w satisfies

$$D_t w = Sw = (Dv - W)\omega = (Dv)\omega - \frac{1}{2}\omega \times \omega = (Dv)\omega,$$

where we used $Dv = S + W$ and Lemma 22*(iii)*. By theorem 10 with the field h replaced by ω we have $\omega_m(t,X) = (Dx)(t,X)\omega(0,X)$ hence the result. □

2.4 Local existence of strong solutions for N-S

We consider the incompressible N-S equation in infinite volume and small time interval:

$$\begin{cases} \partial_t v - \nu\Delta v = -v \cdot \nabla v - \nabla(p) & \\ \text{div } (v) = 0 & t \in (0,t_*), x \in \mathbb{R}^d \end{cases} \tag{2.4.12}$$

With initial condition $v(0,x) = v_0(x)$. Since we want to study small perturbations of a fluid at rest we take the boundary condition

$$\lim_{|x|\to\infty} v(t,x) = 0 \quad \forall t.$$

For the same reason we consider only initial velocity v_0 with compact support (localized initial perturbation) or at most in L^2. Note that in this case the kinetic energy at $t = 0$ is finite

$$E(0,\mathbb{R}^d) = \frac{1}{2}\int \rho(t,x)|v(t,x)|^2 dx = \frac{1}{2}\int |v_0(x)|^2 dx \leq \infty,$$

where we used $\rho = \rho_0 = 1$.

Regularity requirements. For the equations above to be well defined we need v to admit at least one time and two space derivatives. If the time interval is small enough we expect the solution to remain in $L^2(\mathbb{R}^d;\mathbb{R}^d)$. Note that differentiability plus L^2 guarantees that the boundary condition $\lim_{|x|\to\infty} v(t,x) = 0$ is satisfied.

[Lecture 13: 03.06]

2.4.1 Reorganizing the problem

Helmholtz decomposition

Definition 27 *Let L^2_σ and L^2_D the subspaces of $L^2(\mathbb{R}^d;\mathbb{R}^d)$ defined by*

$$L^2_\sigma = \{h \in L^2(\mathbb{R}^d;\mathbb{R}^d) \mid \text{div } h = 0 \text{ in } \mathcal{D}'\} \tag{2.4.13}$$

$$L^2_D = \{h \in L^2(\mathbb{R}^d;\mathbb{R}^d) \mid \exists q \in W^{1,2}_{loc}(\mathbb{R}^d) \text{ with } g = Dq\} \tag{2.4.14}$$

(2.4.15)

where $(Dq)_j = D_j q$ and D_j is the weak derivative wrt x_j. Finally div $h = 0$ in $\mathcal{D}'$ *means*

$$\int \vec{h} \cdot \vec{D}\xi \; dx = 0 \quad \forall \xi \in C^\infty_c(\mathbb{R}^d).$$

Theorem 11 *The sets L^2_σ and L^2_D are closed subspaces of $L^2(\mathbb{R}^d;\mathbb{R}^d)$, $L^2_\sigma \perp L^2_D$ and*

$$L^2(\mathbb{R}^d;\mathbb{R}^d) = L^2_\sigma \oplus L^2_D.$$

In particular, every $f \in L^2(\mathbb{R}^d;\mathbb{R}^d)$ has a unique decomposition

$$f = h + Dq, \quad h \in L^2_\sigma, Dq \in L^2_D.$$

This is called the Helmholtz decomposition.

The proof is divided in three steps.
Step 1: L^2_D is closed (hence $L^2(\mathbb{R}^d;\mathbb{R}^d) = L^2_D \oplus (L^2_D)^\perp$)
Step 2: $L^2_D \perp L^2_\sigma$ (hence $L^2_\sigma \subset (L^2_D)^\perp$)
Step 3: $L^2_D = (L^2_\sigma)^\perp$ (hence L^2_D is closed and $L^2(\mathbb{R}^d;\mathbb{R}^d) = L^2_\sigma \oplus L^2_D$)

Proof of Step 1. Let g_n be a Cauchy sequence in L^2_D. Then there exists $g \in L^2(\mathbb{R}^d;\mathbb{R}^d)$ such that $\|g_n - g\|_{L^2} \to 0$. We want to prove that $g \in L^2_D$, i.e. that there exists a function $q \in W^{1,2}_{loc}$ such that $g = Dq$. Since $g_n = D(q_n + c)$ for any constant $c \in \mathbb{R}$ we can assume that each q_n satisfies

$$\int_{B(0,1)} q_n dx = 0 \quad \forall n.$$

Then there exists a constant $c_R > 0$ such that the following Poincaré inequality applies (see exercise sheet)

$$\|q_n - q_m\|_{L^2(B(0,R))} \le c_R \|g_n - g_m\|_{L^2(B(0,R))}.$$

As a consequence q_n is a Cauchy-sequence in $W^{1,2}(B(0,R))$, hence $\exists q \in W^{1,2}(B(0,R))$ such that $Dq = g$ and $q_n \to q$ in $L^2(B(0,R))$. Repeating for all $R > 0$ we construct a function $q \in W^{1,2}_{loc}(\mathbb{R}^d)$ such that $Dq = g$.

Proof of Step 2. For all $h \in L^2_\sigma$ and for all $g = D\xi$ such that $\xi \in C^\infty_c(\mathbb{R}^d)$, we have

$$(h,g)_{L^2} = \int_{\mathbb{R}^d} \sum_j (h_j g_j)(x) dx = \int_{\mathbb{R}^d} \sum_j (h_j(x) D_j q(x)) dx = 0$$

To complete the argument it is enough to prove that $\{D\xi | \; \xi \in C^\infty_c(\mathbb{R}^d)\}$ is dense in L^2_D (exercise). Hint: take $q_R(x) = q(x)\phi_R(x)$, where $\phi_R(x) = \phi(|x|/R)$ and ϕ is a standard mollifier. Then consider $\phi_R * q_R$.

Proof of Step 3. Since we know from **2.** that $L^2_\sigma \subset (L^2_D)^\perp$ it is enough to prove $(L^2_D)^\perp \subset L^2_\sigma$. Let $h \in (L^2_D)^\perp$, then $\sum_j (h_j, g_j)_{L2} = 0$ for all $g \in L^2_D$. In particular this holds for all $g = D\xi$, with $\xi \in C^\infty_c(\mathbb{R}^d)$, then div $h = 0$ in $\mathcal{D}'$, hence $h \in L^2_\sigma$.

Application to Navier-Stokes Let $\mathcal{P}_\sigma$ be the orthogonal projection on L^2_σ. Applying this to NS we obtain

$$\mathcal{P}_\sigma \left(\partial_t v - \nu \Delta v\right) = -\mathcal{P}_\sigma \left((v \cdot D)v\right) - \mathcal{P}_\sigma \left(Dp\right)$$

Assuming $p \in W^{1,2}_{loc}(\mathbb{R}^d)$, we have $Dp \in L^2_D$, hence $\mathcal{P}_\sigma (Dp) = 0$. Moreover div $(v) = 0$ hence $v \in L^2_\sigma$ and $\mathcal{P}_\sigma v = v$. We will see below that $\mathcal{P}_\sigma$ commutes with ∂_t and D, hence

$$\partial_t v - \nu \Delta v = -\mathcal{P}_\sigma \left((v \cdot D)v\right),$$

where the p dependence has disappeared.

Strategy

(1) Weak formulation: we replace the space derivatives ∂_x with weak derivatives in the Sobolev space $W^{m,2}(\mathbb{R}^d; \mathbb{R}^d)$. Using Helmholtz decomposition we rewrite then (2.4.12) as

$$\begin{aligned} &\partial_t u - \nu \Delta u = F(u, u), \\ &u(0, x) = u_0(x), \quad u \in L^2_\sigma \end{aligned} \tag{2.4.16}$$

where, for any two functions $u \in W^{l,2}(\mathbb{R}^d; \mathbb{R}^d), v \in W^{m,2}(\mathbb{R}^d; \mathbb{R}^d)$, $l \geq 0, m \geq 1$ we define

$$F(u, v)_i := -\mathcal{P}_\sigma (\sum_j u_j D_j v_i). \tag{2.4.17}$$

Note that the p dependence has now disappeared.

(2) We will see that (2.4.16) is equivalent to solve the fixed point equation

$$u = \mathcal{G}_\nu u \tag{2.4.18}$$

where the operator $\mathcal{G}_\nu u$ acts on functions $u : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$ as

$$\mathcal{G}_\nu(u)(t, x) := \left[T_\nu(t) u_0\right](x) \int_0^t \left[T_\nu(t-s)\ F(u(s), u(s))\right](x) ds. \tag{2.4.19}$$

Here $T_\nu(t) = T(\nu t)$, where $T(t)$ is the heat-kernel (defined below in eq.(2.4.23)) and $u(s) : \mathbb{R}^d \to \mathbb{R}^d$ is defined by $u(t)(x) = u(t, x)$.

(3) The solution obtained in this way turns out to be more regular than expected, in particular weak derivatives can be replaced by ordinary derivatives.

Setting up the function spaces

We replace spatial derivatives by their weak version. Therefore we need to have

$$v(t, \cdot) \in W^{m,2}(\mathbb{R}^d; \mathbb{R}^d) = H^m(\mathbb{R}^d; \mathbb{R}^d) \quad m \geq 2, t > 0$$

(at least two space derivatives, functions in L^2). The next result ensures that the projection $\mathcal{P}_\sigma$ leaves $H^m(\mathbb{R}^d; \mathbb{R}^d)$ invariant.

Theorem 12 *Let $m \geq 1$, $f \in H^m(\mathbb{R}^d; \mathbb{R}^d)$. Then $\mathcal{P}_\sigma(f) \in H^m(\mathbb{R}^d; \mathbb{R}^d)$, and for all multiindices $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq m$ we have*

(i) $D^\alpha \mathcal{P}_\sigma f = \mathcal{P}_\sigma D^\alpha f$,

(ii) $\|D^\alpha \mathcal{P}_\sigma f\|_{L^2} \le \|D^\alpha f\|_{L^2}$, *in particular* $\|\mathcal{P}_\sigma f\|_{H^m(\mathbb{R}^d;\mathbb{R}^d)} \le \|f\|_{H^m(\mathbb{R}^d;\mathbb{R}^d)}$.

To prove this theorem we need the following result.

Lemma 25 *Let $p \in [1,\infty)$. For $h \in \mathbb{R}$, $h \neq 0$ and $i = 1,\dots,d$ we define the difference quotient $D_i^{(h)} \in Lin(L^p(\mathbb{R}^d); L^p(\mathbb{R}^d))$ by*

$$D_i^{(h)} := \frac{f(x+he_i) - f(x)}{h}.$$

The following are equivalent.

(i) $f \in W^{1,p}(\mathbb{R}^d))$

(ii) $\sup\{\|D_i^{(h)} f\|_{L^p} |\ h \neq 0, i \in \{1,\dots,d\}\} < \infty$

Moreover, if (i) or (ii) hold then

$$\lim_{h\to 0} D_i^{(h)} f = D_i f \quad \text{in } L^p.$$

Proof. homework

Proof of Thm 12. Assume first $m = 1$. By linearity $\mathcal{P}_\sigma D_i^{(h)} f = \frac{1}{h}\left[\mathcal{P}_\sigma \tau_{i,h} f - \mathcal{P}_\sigma f\right](x)$, where $\tau_{i,h} f(x) := f(x+he_i)$. By translation invariance $\mathcal{P}_\sigma \tau_{i,h} = \tau_{i,h}\mathcal{P}_\sigma$, hence $\mathcal{P}_\sigma D_i^{(h)} = D_i^{(h)}\mathcal{P}_\sigma$ for all i, h. Then

$$\|D_i^{(h)}\mathcal{P}_\sigma f\|_{L^2} = \|\mathcal{P}_\sigma D_i^{(h)} f\|_{L^2} \le \|D_i^{(h)} f\|_{L^2}.$$

where the last step holds since $\mathcal{P}_\sigma$ is an orthogonal projection. By Lemma 25 *(ii)* $f \in W^{1,2}$ implies

$$\sup\{\|D_i^{(h)} f\|_{L^2} |\ h \neq 0, i \in \{1,\dots,d\}\} < \infty \ \Rightarrow\ \sup\{\|D_i^{(h)}\mathcal{P}_\sigma f\|_{L^2} |\ h \neq 0, i \in \{1,\dots,d\}\} < \infty,$$

hence $\mathcal{P}_\sigma f \in W^{1,2}$. Finally, by Lemma 25 and continuity of $\mathcal{P}_\sigma$ we have $\lim_{h\to 0}\mathcal{P}_\sigma D_i^{(h)} f = \mathcal{P}_\sigma D f$ and $\lim_{h\to 0} D_i^{(h)}\mathcal{P}_\sigma f = D\mathcal{P}_\sigma f$ in L^2, hence $D^\alpha \mathcal{P}_\sigma f = \mathcal{P}_\sigma D^\alpha f$.
The case $m > 1$ is treated by induction. □

[Lecture 14: 8.06]

In the following we will see the function $u(t,x)$ in two different ways:

- as function of two variables taking values in $\mathbb{R}^d$
  $u : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$, i.e. $(t,x) \to u(t,x)$;
- as a function of one variable taking values in a function space
  $u : \mathbb{R}^+ \to H^m(\mathbb{R}^d;\mathbb{R}^d)$, i.e. $t \to u(t,\cdot)$

The relation between these two descriptions is the content of the next definition and lemma.

Definition 28 *Let X be a Banach space with norm $\|\cdot\|_X$ and $I \subset \mathbb{R}$ an interval. We denote by $C_b^0(I;X)$ the set of continuous bounded functions $f : I \to X$. We introduce the norm*

$$\|f\|_{C_b^0(I;X)} := \sup_{t\in I}\|f(t)\|_X. \tag{2.4.20}$$

Remarks: If I is compact then $C_b^0(I;X) = C^0(I;X)$. Moreover, we can define the X-valued integral $\int_a^b f(s)ds$. In particular we have $\| \int_a^b f(s)ds\|_X \leq \int_a^b \|f(s)\|_X ds$.

Lemma 26 *We have*

(i) If X is a Banach space then $C_b^0(I;X)$ is a Banach space wrt the norm $\|\cdot\|_{C_b^0(I;X)}$.

(ii) $C_b^0(I;\mathbb{R}^d)$ can be identified with a subspace of $C^0(I\times\mathbb{R}^d)$.

(iii) If $m \in \mathbb{N}$, $p \in [1,\infty]$, then any map $f \in C^0(I;W^{m,p}(\mathbb{R}^d))$ can be identified with a measurable map $f: I\times\mathbb{R}^d \to \mathbb{R}$.

(iv) If $m \in \mathbb{N}$, $p \in [1,\infty)$, and I is compact then $C_c^\infty(I\times\mathbb{R}^d)$ is a dense subset of $C^0(I;W^{m,p}(\mathbb{R}^d))$.

Proof. exercise sheet □

2.4.2 Nonlinear heat equation

We will consider first the the simpler situation when $\nu = 1$, the unknown function is a scalar function $u : \mathcal{T} \to \mathbb{R}$, and the function $F(u,u) = -\mathcal{P}_\sigma(u\cdot Du)$ is replaced by a function $f(u)$ $f:\mathbb{R}\to\mathbb{R}$. We study then

$$\begin{aligned} &\partial_t u - \Delta u = f(u), \\ &u(0,x) = u_0(x), \quad u \in L^2_\sigma. \end{aligned} \tag{2.4.21}$$

Heat kernel

Definition 29 *The heat kernel is the function $\Phi : \mathbb{R}^{d+1} \to \mathbb{R}$ defined by*

$$\Phi(t,x) = \begin{cases} (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}} & t > 0 \\ 0 & t \leq 0 \end{cases} \tag{2.4.22}$$

For $t \geq 0$ we define the map $T(t) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$

$$\left(T(t)u\right)(x) := \begin{cases} \int_{\mathbb{R}^d} \Phi(t,x-y)u(y)dy & t > 0 \\ u(x) & t = 0 \end{cases} \tag{2.4.23}$$

Remark. Φ satisfies $\|\Phi(t,\cdot)\|_{L^1} = \int_{\mathbb{R}^d}\Phi(t,x)dx = 1$ $\forall t > 0$.

Properties of T (see Introduction to PDEs):

(i) smoothing: $(t,x) \to (T(t)u)(x) \in C^\infty((0,\infty)\times\mathbb{R}^d)$ for all $u \in L^2(\mathbb{R}^d)$;

(ii) continuity in t: $\lim_{s\to t}\|T(s)u - T(t)u\|_{L^2(\mathbb{R}^d)} = 0$ for all $t, s \geq 0$.

(iii) boundedness: $\|T(t)u\|_{L^2(\mathbb{R}^d)} \leq \|u\|_{L^2(\mathbb{R}^d)}$ for all $t \geq 0$.

Let $t_* > 0$. We define $C^{1,2}((0,t_*)\times\mathbb{R}^d)$ = set of continuous functions with first oder derivative in t and second order derivative in x continuous. The following result was proved in *Introduction to PDEs*.

Theorem 13 (linear heat equation) *Let $g \in C^{1,2}((0,t_*)\times\mathbb{R}^d) \cap C_c^0(\mathbb{R}\times\mathbb{R}^d)$ and $u_0 \in L^2(\mathbb{R}^d)$. The system*

$$\begin{aligned} &\partial_t u - \Delta u = g(x), \\ &u(0,x) = u_0(x), \end{aligned} \tag{2.4.24}$$

has unique solution

$$u(t)(\cdot) = T(t)u_0(\cdot) + \int_0^t T(t-s)g(s,\cdot)ds. \tag{2.4.25}$$

We can now state the main result of this subsection.

Theorem 14 *(nonlinear heat equation) Let $f \in C^1(\mathbb{R}) \cap Lip(\mathbb{R})$ with $f(0) = 0$. Then $\exists t_* > 0$ (dependent on f) s.t for any initial condition $u_0 \in C^0_c(\mathbb{R}^d)$ the system*

$$\begin{aligned} &\partial_t u - \Delta u = f(u), \\ &u(0,x) = u_0(x), \end{aligned} \tag{2.4.26}$$

has unique solution $u \in C^{1,2}((0,t_*) \times \mathbb{R}^d) \cap C^0(\mathbb{R} \times \mathbb{R}^d)$.

Proof. Step 1. We rewrite the system (2.4.26) as a fixed point equation. Let $I = [0, t_*]$, $t_* > 0$. Similarly to the solution (2.4.25) of the linear heat equation, for any measurable map $u : I \times \mathbb{R}^d \to \mathbb{R}$ we define the operator $\mathcal{G}_0(u) : I \times \mathbb{R}^d \to \mathbb{R}$ by

$$\mathcal{G}_0(u) := T(t)u_0 + \int_0^t T(t-s)f(u(s,\cdot))ds \tag{2.4.27}$$

Let $\mathcal{B} := C^0(I; L^2(\mathbb{R}^d))$. Then $\mathcal{B}$ is a Banach space wrt the norm $\|f\|_{\mathcal{B}} = \sup_{t \in I} \|f(t)\|_{L^2(\mathbb{R}^d)}$. We will show that

- $\mathcal{G}_0 : \mathcal{B} \to \mathcal{B}$
- $\mathcal{G}_0(u)$ satisfies $\begin{cases} (\partial_t - \Delta)\mathcal{G}_0(u) = f(u) \\ \mathcal{G}_0(u)(0,x) = u_0(x), \end{cases}$
- $\mathcal{G}_0$ is a contraction i.e. $\exists 0 < \lambda < 1$ sucht that $\|\mathcal{G}_0 u - \mathcal{G}_0 v\|_{\mathcal{B}} \le \lambda \|u - v\|_{\mathcal{B}}$ $\forall u, v \in \mathcal{B}$. From this one can prove that there exists a unique $u \in \mathcal{B}$ such that $u = \mathcal{G}_0 u$, hence a unique solution for (2.4.24).

Step 2. We prove that $\mathcal{G}_0 : \mathcal{B} \to \mathcal{B}$. The first term $(t,x) \to (T(t)u_0)(x)$ is $C^\infty((0,\infty) \times \mathbb{R}^d)$, and is continuous in t on $[0,\infty)$. Therefore we need to study only the second term.
For this purpose note that since $f \in Lip(\mathbb{R})$, we have
$\|f(u(s,\cdot)) - f(u(t,\cdot))\|_{L^2} \le Lip(f)\|(u(s,\cdot) - u(t,\cdot)\|_{L^2} \to 0$ as $s \to t$.
Morever since $f(0) = 0$ we have

$$|f(s)| = |f(s) - f(0)| \le Lip(f)\ |s| \quad \forall s \in \mathbb{R} \qquad \Rightarrow \|f(u(s,\cdot))\|_{L^2} \le Lip(f)\|(u(s,\cdot)\|_{L^2}$$

hence $s \to f(u(s,\cdot)) \in \mathcal{B}$, for all $u \in \mathcal{B}$, and

$$\int_0^t \|T(t-s)f(u(s))\|_{L^2} ds \le t \sup_{s \in [0,t]} \|f(u(s))\|_{L^2} \le t\ Lip(f)\ \|u\|_{\mathcal{B}}.$$

This implies $\int_0^t T(t-s)f(u(s))ds \in L^2(\mathbb{R}^d)$ for all $t \in I$. Finally, $s \to T(t-s)f(u(s))$ is continuous since

$$\begin{aligned} &\|T(t-s)f(u(s)) - T(t-s')f(u(s'))\|_{L^2} \le \|[T(t-s) - T(t-s')]f(u(s))\|_{L^2} + \|T(t-s')[f(u(s)) - f(u(s'))]\|_{L^2} \\ &\quad \le \|[T(t-s) - T(t-s')]f(u(s))\|_{L^2} + Lip(f)\|u(s)) - u(s')\|_{L^2} \to_{s' \to s} 0. \end{aligned}$$

Similar arguments show that $t \to \int_0^t T(t-s)f(u(s))ds$ is continuous.

Step 3. $\mathcal{G}_0$ is a contraction i.e. $\exists t_* > 0, 0 < \lambda < 1$ sucht that $\|\mathcal{G}_0 u - \mathcal{G}_0 v\|_{\mathcal{B}} \le \lambda \|u - v\|_{\mathcal{B}}$ $\forall u, v \in \mathcal{B}$.
To prove this note that,

$$\begin{aligned} &\|\mathcal{G}_0 u - \mathcal{G}_0 v\|_{\mathcal{B}} = \sup_{t \in I} \|\mathcal{G}_0 u(t,\cdot) - \mathcal{G}_0 v(t,\cdot)\|_{L^2} = \sup_{t \in I} \|\int_0^t T(t-s)[f(u(s,\cdot)) - f(v(s,\cdot))]ds\|_{L^2} \\ &\le \sup_{t \in I} \int_0^t \|T(t-s)[f(u(s,\cdot)) - f(v(s,\cdot))]\|_{L^2} ds \\ &\le \sup_{t \in I} Lip(f) \int_0^t \|(u(s,\cdot)) - v(s,\cdot))]\|_{L^2} ds \le t_* Lip(f)\|u - v\|_{\mathcal{B}}. \end{aligned}$$

Setting $\lambda = t_* Lip(f)$ we have a contraction for any $t_* < \frac{1}{Lip(f)}$.

Step 4. Conclusion: existence and uniqueness.
By the smoothing properties of T and the solution of linear heat equation we have $\mathcal{G}_0 u \in C^{1,2}((0,t_*)\times\mathbb{R}^d)\cap C^0(\mathbb{R}\times\mathbb{R}^d)$ and $\mathcal{G}_0(u)$ satisfies $\begin{cases} (\partial_t - \Delta)\mathcal{G}_0(u) = f(u), \\ \mathcal{G}_0(u)(0,x) = u_0(x) \end{cases}$ for any $u \in \mathcal{B}$.
Moreover, if u is a solution of (2.4.26), then $w := u - \mathcal{G}_0(u)$ is a solution of the linear heat equation (2.4.24) with $g = 0$ and $u_0 = 0$. The unique solution is then $w = 0$. Hence u is a solution of (2.4.26) iff u is a fixed point i.e. $u = \mathcal{G}_0(u)$.
Finally, to construct a fixed point, let $(w_n)_{n\in\mathbb{N}}$ be the sequence in $\mathcal{B}$ defined by $w_0(t)(x) = u_0(x)$ $\forall t$, and $w_n = \mathcal{G}_0(w_{n-1})$, $\forall n \geq 1$. To prove this is a Cauchy sequence note that

$$\|w_{n+1} - w_n\|_{\mathcal{B}} = \|\mathcal{G}_0 w_n - \mathcal{G}_0 w_{n-1}\|_{\mathcal{B}} \leq \lambda \|w_n - w_{n-1}\|_{\mathcal{B}} \leq \lambda^n \|w_1 - w_0\|_{\mathcal{B}},$$

and hence for any $n > m$

$$\|w_n - w_m\|_{\mathcal{B}} \leq \sum_{k=m}^{n-1} \|w_{k+1} - w_k\|_{\mathcal{B}} \leq \|w_1 - w_0\|_{\mathcal{B}} \frac{\lambda^m - \lambda^n}{1-\lambda} \to_{m,n\to\infty} 0.$$

Finally, to prove unicity, let $w = \mathcal{G}_0 w, \bar{u} = \mathcal{G}_0 \bar{w}$ two fixed points. Then

$$\|w - \bar{w}\|_{\mathcal{B}} = \|\mathcal{G}_0 w - \mathcal{G}_0 \bar{w}\|_{\mathcal{B}} \leq \lambda \|w - \bar{w}\|_{\mathcal{B}} < \|w - \bar{w}\|_{\mathcal{B}}.$$

That is impossible hence $w = \bar{w}$. □

[Lecture 15: 11.06]

2.4.3 Navier-Stokes: preliminary results

In the case of NS equation, the nonlinear term $f(u)$ becomes $F(u,v) = -\mathcal{P}_\sigma(u\cdot Dv)$ defined in (2.4.17). The following results collect some important properties of this expression.

Lemma 27 *Assume $d = 2, 3$. We have*

(i) *If $l \leq m$, $m \geq 2$, $f \in H^m(\mathbb{R}^d)$, $g \in H^l(\mathbb{R}^d)$, then $fg \in H^l(\mathbb{R}^d)$ and*

$$\|fg\|_{H^l} \leq c\ \|f\|_{H^m}\ \|g\|_{H^l},$$

for some constant c.

(ii) *If $m \geq 2$, $u \in H^m(\mathbb{R}^d;\mathbb{R}^d)$, $v \in H^{m+1}(\mathbb{R}^d;\mathbb{R}^d)$, then $F(u,v) \in H^m(\mathbb{R}^d;\mathbb{R}^d)$ and*

$$\|F(u,v)\|_{H^m} \leq C\ \|u\|_{H^m}\|v\|_{H^{m+1}}$$

(iii) *If $u \in H^2(\mathbb{R}^d;\mathbb{R}^d)$, and $v \in H^2(\mathbb{R}^d;\mathbb{R}^d)$, or $u \in H^1(\mathbb{R}^d;\mathbb{R}^d)$, and $v \in H^3(\mathbb{R}^d;\mathbb{R}^d)$, then $F(u,v) \in H^1(\mathbb{R}^d;\mathbb{R}^d)$, and*

$$\|F(u,v)\|_{H^1} \leq \begin{cases} C\ \|u\|_{H^2}\|v\|_{H^2} \\ C\ \|u\|_{H^1}\|v\|_{H^3} \end{cases}$$

We will see the proof only in the case $d = 3$. We will use the following tools

- Sobolev embedding: $\|f\|_{W^{l,q}(\mathbb{R}^d)} \leq C\ \|f\|_{W^{m,p}(\mathbb{R}^d)}$ if $l \leq m$, $l - \frac{d}{q} = m - \frac{d}{p}$.
- Morrey inequality: assume $d < p \leq \infty$ then $\|f\|_{C^{0,\gamma}(\mathbb{R}^d)} \leq C_{p,d}\|f\|_{W^{1,p}(\mathbb{R}^d)}$ with $\gamma = 1 - \frac{d}{p}$.

We will also use the following preliminary result.

Lemma 28 *Let $u : \mathbb{R}^3 \to \mathbb{R}^3$. We have*

(1) $\|f\|_{L^6} \leq C_1 \|f\|_{H^1}$

(2) $\|f\|_{L^\infty} \leq C_2 \|f\|_{W^{1,6}}$

(3) $\|f\|_{L^\infty} \leq C_3 \|f\|_{H^2}$

(4) $\|f\|_{L^4} \leq \|f\|_{L^2}^{\frac{1}{4}} \|f\|_{L^6}^{\frac{3}{4}}$

(5) $\|f\|_{L^4} \leq C_4 \|f\|_{H^1}$

Proof. (1) holds By Sobolev embedding. (2) holds by Morreys inequality. (3) is obtained by (1) and (2) as follows

$$\|f\|_{L^\infty} \leq C_2 \|f\|_{W^{1,6}} \leq C_2' [\|f\|_{L^6} + \|Df\|_{L^6}] \leq C_2' C_1 [\|f\|_{H^1} + \|Df\|_{H^1}] \leq C_3 \|f\|_{H^2},$$

where in the first inequality we used (2) and in the third (1).
To obtain (4) we use Cauchy-Schwarz ineq.

$$\|f\|_{L^4}^4 = \|f^4\|_{L^1} = \|f \ f^3\|_{L^1} \leq \|f\|_{L^2} \|f^3\|_{L^2} = \|f\|_{L^2} \|f\|_{L^6}^3.$$

To obtain (5) we use (1) and (4)

$$\|f\|_{L^4} \leq \|f\|_{L^2}^{\frac{1}{4}} \|f\|_{L^6}^{\frac{3}{4}} \leq \|f\|_{L^2}^{\frac{1}{4}} \|f\|_{H^1}^{\frac{3}{4}} \leq C \|f\|_{H^1}$$

where we used (4) in the first, (1) in the second inequality and finally $\|f\|_{L^2} \leq C \|f\|_{H^1}$. □

Proof of Lemma 27 for $d = 3$.

Proof of (i). Let $f \in H^m(\mathbb{R}^d)$, $g \in H^l(\mathbb{R}^d)$, with $l \leq m$, $m \geq 2$. We know that fg admits l weak derivatives and $D^\alpha(fg) \in L^1$ $\forall |\alpha| \leq l$. To prove **(i)** it is enough to show that $D^\alpha(fg) \in L^2$ $\forall |\alpha| \leq l$ and $\|D^\alpha(fg)\|_{L^2} \leq c \ \|f\|_{H^m} \ \|g\|_{H^l}$, for some constant c depending only on m. For this purpose we write $D^\alpha(fg) = \sum_{\beta \leq \alpha} D^\beta f D^{\alpha-\beta} g$, where $\beta \leq \alpha$ if $0 \leq \beta_i \leq \alpha_i$ for all $i = 1, \ldots, d$. Then

$$\|D^\alpha(fg)\|_{L^2}^2 \leq C \sum_{\beta \leq \alpha} \int_{\mathbb{R}^3} |D^\beta f|^2 |D^{\alpha-\beta} g|^2 dx$$

for some constant $C > 0$. We distinguish now three cases.

Case 1: $|\beta| \leq m - 2$. Then

$$\begin{aligned}\int_{\mathbb{R}^3} |D^\beta f|^2 |D^{\alpha-\beta} g|^2 dx &\leq \|D^\beta f\|_{L^\infty}^2 \|D^{\alpha-\beta} g\|_{L^2}^2 \leq C \|D^\beta f\|_{H^2}^2 \|D^{\alpha-\beta} g\|_{L^2}^2 \\ &\leq C \|f\|_{H^{2+|\beta|}}^2 \|g\|_{H^{|\alpha-\beta|}}^2 \leq C \|f\|_{H^m}^2 \|g\|_{H^l}^2\end{aligned}$$

where in the second inequality we used Lemma 28 *(3)* and in the last we used $2 + |\beta| \leq m$ and $|\alpha - \beta| \leq l$.

Case 2: $|\beta| = m - 1$. Then

$$\begin{aligned}\int_{\mathbb{R}^3} |D^\beta f|^2 |D^{\alpha-\beta} g|^2 dx &\leq \left(\int_{\mathbb{R}^3} |D^\beta f|^4 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |D^{\alpha-\beta} g|^4 dx \right)^{\frac{1}{2}} \\ &= \|D^\beta f\|_{L^4}^2 \|D^{\alpha-\beta} g\|_{L^4}^2 \leq C \|D^\beta f\|_{H^1}^2 \|D^{\alpha-\beta} g\|_{H^1}^2 \\ &\leq C \|f\|_{H^{1+|\beta|}}^2 \|g\|_{H^{1+|\alpha-\beta|}}^2 = C \|f\|_{H^m}^2 \|g\|_{H^{1+|\alpha-\beta|}}^2\end{aligned}$$

where in the first line we used Cauchy-Schwarz, in the second Lemma 28 *(5)*. To estimate $|\alpha-\beta|$ we consider all cases when $|\beta| = m-1$ can be realized. Note that $|\beta| \leq |\alpha| \leq l \leq m$, and $m \geq 2$, hence $m-1 \geq 1$. Then $|\beta| = m-1$ only if $l = m-1$ or $l = m$. In the first case $l \geq 1$ and $|\alpha| = l = |\beta| = m-1 \geq 1$. Hence $1 + |\alpha - \beta| = 1 \leq l$. In the second case $l \geq 2$ and one of the following two situation may happen: $|\alpha| = l = m$ then $1 + |\alpha-\beta| = 2 \leq l$, or $|\alpha| = l-1 = |\beta| = m-1$ then $1 + |\alpha - \beta| = 1 \leq l$. Finally $\|D^\beta f D^{\alpha-\beta} g\|_{L^2}^2 \leq C\|f\|_{H^m}^2 \|g\|_{H^l}^2$.

Case 3: $|\beta| = m$. Then $|\beta| = |\alpha| = l = m \geq 2$, $\alpha - \beta = 0$ and

$$\int_{\mathbb{R}^3} |D^m f|^2 |g|^2 dx \leq \|D^m f\|_{L^2}^2 \|g\|_{L^\infty}^2 \leq C\|f\|_{H^m}^2 \|g\|_{H^2}^2 \leq C\|f\|_{H^m}^2 \|g\|_{H^l}^2$$

where in the second inequality we used Lemma 28 *(3)* and in the last $m = l \geq 2$.

Proof of (ii). Let $u \in H^m(\mathbb{R}^d;\mathbb{R}^d)$, $v \in H^{m+1}(\mathbb{R}^d;\mathbb{R}^d)$, with $m \geq 2$. Remember that

$$F(u,v)_i = -\mathcal{P}_\sigma(\vec{u}\cdot\vec{D}v)_i = -\sum_{j=1}^d \mathcal{P}_\sigma(u_j D_j v_i)$$

Let $f = u_j$, $g = D_j v_i$. Then since $f \in H^m, g \in H^{m+1}$ we have $fg \in H^m$, hence by Thm. 12 $\mathcal{P}_\sigma(fg) \in H^m$ and

$$\|\mathcal{P}_\sigma(fg)\|_{H^m} \leq \|fg\|_{H^m} \leq C\|f\|_{H^m}\|g\|_{H^m} \leq C\|u_j\|_{H^m}\|v_i\|_{H^{m+1}}$$

where in the second inequality we used **(i)**. The result follows.

Proof of (iii). Let $u \in H^2(\mathbb{R}^d;\mathbb{R}^d)$, and $v \in H^2(\mathbb{R}^d;\mathbb{R}^d)$. Then $f = u_j \in H^2$ and $g = D_j v \in H^1$. Applying **(i)** with $m = 2$, $l = 1$ we get $fg \in H^1$, hence $\mathcal{P}_\sigma(fg) \in H^1$ and

$$\|\mathcal{P}_\sigma(fg)\|_{H^1} \leq \|(fg)\|_{H^1} \leq C\|f\|_{H^1}\|g\|_{H^2} \leq C\|u_j\|_{H^2}\|v_i\|_{H^2}.$$

Similar arguments work in the case $u \in H^1(\mathbb{R}^d;\mathbb{R}^d)$, and $v \in H^3(\mathbb{R}^d;\mathbb{R}^d)$. □

Finally, to set up a Banach fixed point argument we need to define $\int_0^t F(s)ds$ with $F(s)$ some function taking values in H^m. Therefore we need to define the notion of L^p space in this context.

Definition 30 *Let X be a Banach space, $I \subset \mathbb{R}$ a bounded interval, $C_b^0(I;X)$ the set of bounded continuous functions $f : I \to X$. let $1 \leq q < \infty$. We define*

$$\|f\|_{L^q(I;X)} = \| \, \|f(t)\|_X \, \|_{L^q(I)} = \left[\int_I \|f(t)\|_X^q dt\right]^{\frac{1}{q}}$$

Moreover, we define $L^q(I;X)$ as the closure of $C_b^0(I;X)$ in X^I w.r.t. this norm (modulo maps that coincide a.e.).

Lemma 29 *The following statements hold.*

(i) *$L^q(I;X)$ is a Banach space.*

(ii) *Let $m \in \mathbb{N}$, $f \in L^q(I;W^{m,p}(\mathbb{R}^d))$ with $1 \leq p \leq \infty$, $1 \leq q < \infty$. Then $f(t)(x)$ defines a measurable function on $I \times \mathbb{R}^d$*

(iii) *If I is compact then $C_c^\infty(I \times \mathbb{R}^d)$ is a dense subset in $L^q(I;W^{m,p}(\mathbb{R}^d))$ for all $1 \leq p \leq \infty$, $1 \leq q < \infty$.*

(iv) *$\int_a^b f(t)dt$ is well defined.*

Proof Homework

[Lecture 16: 15.06]

2.4.4 Navier-Stokes: local solutions

Funtion spaces. Let $I \subset \mathbb{R}$ a bounded interval. We will work with the following spaces:

- $L^2_\sigma = \{h \in L^2(\mathbb{R}^d; \mathbb{R}^d) \mid \text{div } h = 0 \text{ in } \mathcal{D}'\}$, see (2.4.13),
- $H^m(\mathbb{R}^d; \mathbb{R}^d) = W^{m,2}(\mathbb{R}^d; \mathbb{R}^d)$
- $H^m_\sigma := H^m(\mathbb{R}^d; \mathbb{R}^d) = W^{m,2}(\mathbb{R}^d; \mathbb{R}^d) \cap L^2_\sigma$,
- $X^m = X^m(I) := C^0(I; H^m_\sigma)$ with the norm $\|u\|_{X^m} := \sup_{t \in I} \|u(t)\|_{H^m}$,
- $Y^m = Y^m(I) := L^1(I; H^m_\sigma)$ with the norm $\|u\|_{Y^m} := \int_I \|u(t)\|_{H^m} dt$,

Note that X^m, Y^m are Banach spaces w.r.t. the corresponding norms.

Theorem 15 *Let $d = 2, 3$, $m \geq 2$. There exists a constant c_m such that for any initial condition $u_0 \in H^m_\sigma$ and $\nu > 0$, setting the time interval $I = [0, t_*]$ with*

$$t_* = \frac{c_m \nu}{\nu \|u_0\|_{H^m} + \|u_0\|^2_{H^m}},$$

there exists a unique function $u \in X^m(I) \cap Y^{m+1}(I) \cap C^1(I; H^{m-2}_\sigma)$ satisfying

$$\begin{cases} \partial_t u - \nu \Delta u = F(u, u) \\ u(0, \cdot) = u_0(\cdot). \end{cases} \tag{2.4.28}$$

Proof. We consider first the case $\nu = 1$. We generalize to any ν at the end. The proof is separated into 6 steps.
Step 1: definition of the function space Z^m where we work, reformulation of the problem as a fixed point equation $u = \mathcal{G}u$ for an operator $\mathcal{G}$ acting on Z^m
Step 2: properties of $\mathcal{G}$.
Step 3: $\mathcal{G}$ is a contraction in the unit ball for Z^m.
Step 4: contruction of the fixed point.
Step 5: optimization of the time interval t_*.
Step 6: extension to general ν, conclusion.

Step 1 in the proof of Thm.15. Let $Z^m := X^m \cap Y^{m+1}$ with the norm

$$\|u\|_{Z^m} := \max\left\{\frac{\|u\|_{X^m}}{K}, \frac{\|u\|_{Y^{m+1}}}{L}\right\} \tag{2.4.29}$$

where the constants K, L will be chosen later (as functions of t_* and $\|u_0\|_{H^m}$) to ensure the correct estimates hold.
As in the nonlinear heat equation we define for each $u \in Z^m$ the map $\mathcal{G}(u) : I \times \mathbb{R}^d \to \mathbb{R}^d$ with

$$(\mathcal{G}u)(t, x) = (T(t)u_0)(x) + \int_0^t [T(t-s)F(u(s), u(s))](x) ds \tag{2.4.30}$$

where $F(u, v) = -\mathcal{P}_\sigma(Dv\ u)$. We will prove in Step 2 that $\mathcal{G}$ is well defined and $\mathcal{G} : Z^m \to Z^m$. Proving the theorem (for $\nu = 1$) is equivalent to prove existence and uniqueness of a fixed point $u \in Z^m$ for $\mathcal{G}$, i.e. $\mathcal{G}u = u$. This is a consequence of the next two lemmas.

Lemma 30 *Let $d = 2, 3$, $m \geq 2$, $u_0 \in H^m$, $f \in C^0([0, t_*]; H^{m-1})$. Let T be the heat semigroup and set*

$$(\mathcal{G}_0 f)t := T(t)u_0 + \int_0^t T(t-s)f(s) ds \tag{2.4.31}$$

where the x dependence is not explicitely written for clarity. Then the following hold.

(i) $(\mathcal{G}_0 f \in C^0([0,t_*];H^m) \cap C^1([0,t_*];H^{m-2})$ *and*

$$\begin{cases} (\partial_t - \Delta)(\mathcal{G}_0 f) = f & 0 < t < t_* \\ (\mathcal{G}_0 f)(0) = u_0. \end{cases} \tag{2.4.32}$$

(ii) *If a function* $w \in C^0([0,t_*];H^m) \cap C^1([0,t_*];H^{m-2})$ *satisfies*

$$\begin{cases} (\partial_t - \Delta)(w) = f & 0 < t < t_* \\ w(0) = u_0. \end{cases} \tag{2.4.33}$$

then we must have $w = \mathcal{G}_0 f$.

Proof. homework

Remark. The operator $\mathcal{G}$ defined above satisfies $\mathcal{G}u = \mathcal{G}_0 F(u,u)$.

Corollary 1 *Let* $d = 2,3$, $m \geq 2$, $I = [0,t_*]$. *The next two statements are equivalent.*

(a) $u \in Z^m$ *and* $u = \mathcal{G}u$.

(b) $u \in Z^m \cap C^1(I;H^{m-2})$ *and*

$$\partial_t u - \Delta u = F(u,u) \quad \forall 0 < t < t_*,$$

where the last equation holds in H^{m-2}

Proof. homework

Strong solutions. In the following we say that u is a strong solution of the N-S equation with $\nu = 1$ if *(a)* or *(b)* hold. For general ν we replace $\mathcal{G}$ with $\mathcal{G}_\nu$ defined by

$$\mathcal{G}_\nu u := T_\nu(t)u_0 + \int_0^t T_\nu(t-s)F(u(s),u(s))ds, \tag{2.4.34}$$

where $T_\nu(t) := T(\nu t)$. Finally the PDE in *(b)* is replaced by $\partial_t u - \nu\Delta u = F(u,u)$.

Step 2 in the proof of Thm.15. We want to prove that $\mathcal{G}$ is well defined and $\mathcal{G}: Z^m \to Z^m$. Precisely we will prove the two following claims.

- Claim 2.1: Let $\vec{0}$ be the function $\vec{0}(t,x) = 0\ \forall t,x$. Then $\mathcal{G}(\vec{0}) \in Z^m$ and

$$\|\mathcal{G}(\vec{0})\|_{X^m} \leq \|u_0\|_{H^m}$$
$$\|\mathcal{G}(\vec{0})\|_{Y^{m+1}} \leq C_1(t_* + \sqrt{t_*})\|u_0\|_{H^m}.$$

  where $C_1 > 0$ is some m-dependent constant.

- Claim 2.2: $\forall u, v \in Z^m$ we have $\mathcal{G}u - \mathcal{G}v \in Z^m$ and

$$\|\mathcal{G}u - \mathcal{G}v\|_{X^m} \leq C_2\Delta_{uv}$$
$$\|\mathcal{G}u - \mathcal{G}v\|_{Y^{m+1}} \leq C_1C_2(t_* + \sqrt{t_*})\Delta_{uv}.$$

  where $C_2 > 0$ is some m-dependent constant, and

$$\Delta_{uv} := \|u-v\|_{X^m}\|u\|_{Y^{m+1}} + \|v\|_{X^m}\|u-v\|_{Y^{m+1}}. \tag{2.4.35}$$

To prove these claims we need the following preliminary lemma.

Lemma 31 *We have*

(a) $t \to T(t)u_0 \in C^0(I;H^m_\sigma)$ *and* $\|T(t)u_0\|_{H^m} \leq \|u_0\|_{H^m}$,

(b) $\|T(t)u_0\|_{H^{m+1}} \leq \frac{C_1}{2}(1 + t^{-\frac{1}{2}})\|u_0\|_{H^m}$, $\forall t > 0$, $u_0 \in H^m$.

where the constant C_1 is the same as in the Claims 2.1 and 2.2 above.

Proof of (a). By Lemma 30 $t \to T(t)u_0 \in C^0(I; H^m_\sigma)$. Moreover for any $t > 0$ and $|\alpha| \le m$ $\|D^\alpha T(t)u_0\|_{H^m} = \|T(t)D^\alpha u_0\|_{H^m} \le \|D^\alpha u_0\|_{H^m}$, hence $\|T(t)u_0\|_{H^m} \le \|u_0\|_{H^m}$. Finally div $T(t)u_0 = T(t)$div $u_0 = 0$ in distribution since $u_0 \in L^2_\sigma$. It is easy to see that the case $t = 0$ holds too.

Proof of (b). Note that for $t > 0$

$$|\partial_{x_i}\Phi(t, x-y)| = \frac{|(x_i - y_i)|}{2t}\Phi(t, x-y) \le \frac{C}{\sqrt{t}}\Phi(2t, x-y),$$

for some constant $C > 0$. Hence

$$\begin{aligned}\|T(t)u_0\|_{H^{m+1}} &\le C'\left[\|T(t)u_0\|_{H^m} + \sum_{i=1}^d \sum_{|\alpha|=m} \|\partial_{x_i} D^\alpha T(t)u_0\|_{L^2}\right]\\ &\le C''\left[\|T(t)u_0\|_{H^m} + \frac{1}{\sqrt{t}}\sum_{|\alpha|=m}\|D^\alpha T(t)u_0\|_{L^2}\right].\end{aligned}$$

This gives the result. □

Proof of Claim 2.1 By the definition we have $\mathcal{G}(\vec{0})(t) = T(t)u_0$. From Lemma 31 (a) we have $\mathcal{G}(\vec{0}) \in X^m$ and $\|\mathcal{G}(\vec{0})\|_{X^m} = \sup_{t\in I}\|T(t)u_0\|_{H^m} \le \|u_0\|_{H^m}$. Moreover

$$\|\mathcal{G}(\vec{0})\|_{Y^{m+1}} = \int_0^{t_*}\|\mathcal{G}(\vec{0})(t)\|_{H^m}dt \le \frac{C_1}{2}\|u_0\|_{H^m}\int_0^{t_*}(1 + t^{-\frac{1}{2}})dt \le C_1(t_* + \sqrt{t_*})\|u_0\|_{H^m},$$

where in the second inequality we applied Lemma 31 (b).

Proof of Claim 2.2 Since the function $F(u, v)$ is linear in the two arguments we can write

$$\begin{aligned}\mathcal{G}u(t) - \mathcal{G}v(t) &= \int_0^t T(t-s)\left[F(u(s), u(s)) - F(v(s), v(s))\right]ds\\ &= \int_0^t T(t-s)F(u(s) - v(s), u(s))ds + \int_0^t T(t-s)F(v(s), u(s) - v(s))ds\end{aligned}$$

Inserting the norms and using Lemma 27 (ii) we get for all $t \in I$

$$\begin{aligned}\|\mathcal{G}u(t) - \mathcal{G}v(t)\|_{H^m} &\le \int_0^t \|T(t-s)F(u(s) - v(s), u(s))\|_{H^m}ds + \int_0^t \|T(t-s)F(v(s), u(s) - v(s))\|_{H^m}ds\\ &\le \int_0^t \|F(u(s) - v(s), u(s))\|_{H^m}ds + \int_0^t \|F(v(s), u(s) - v(s))\|_{H^m}ds\\ &\le C_2\int_0^t\left[\|u(s) - v(s)\|_{H^m}\|u(s)\|_{H^{m+1}} + \|v(s)\|_{H^m}\|u(s) - v(s)\|_{H^{m+1}}\right]ds\\ &\le C_2\left[\|u - v\|_{X^m}\|u\|_{Y^{m+1}} + \|v\|_{X^m}\|u - v\|_{Y^{m+1}}\right] = C_2\Delta_{uv}.\end{aligned}$$

Then $\|\mathcal{G}u - \mathcal{G}v\|_{X^m} \le C_2\Delta_{uv}$. Finally using the same estimates as above plus Lemma 31 (b).

$$\begin{aligned}&\|\mathcal{G}u(t) - \mathcal{G}v(t)\|_{H^{m+1}} \le \int_0^t \|T(t-s)\left[F(u(s) - v(s), u(s)) + F(v(s) - v(s), v(s))\right]\|_{H^{m+1}}ds\\ &\quad\le \frac{C_1}{2}\int_0^t (1 + (t-s)^{-\frac{1}{2}})\|F(u(s) - v(s), u(s)) + F(v(s), u(s) - v(s))\|_{H^m}ds\\ &\le \frac{C_1C_2}{2}\left[\|u - v\|_{X^m}\int_0^t(1 + (t-s)^{-\frac{1}{2}})\|u(s)\|_{H^{m+1}}ds + \|v\|_{X^m}\int_0^t(1 + (t-s)^{-\frac{1}{2}})\|u(s) - v(s)\|_{H^{m+1}}ds\right]\end{aligned}$$

hence

$$\|\mathcal{G}u - \mathcal{G}v\|_{Y^{m+1}} = \int_0^{t_*} \|\mathcal{G}u(t) - \mathcal{G}v(t)\|_{H^m} dt \le C_1 C_2 (t_* + \sqrt{t_*}) \Delta_{uv},$$

where we used

$$\int_0^{t_*}\int_0^t (1+(t-s)^{-\frac{1}{2}})\|u(s)\|_{H^{m+1}} ds dt = \int_0^{t_*} \|u(s)\|_{H^{m+1}} \int_0^{t_*-s} (1+t^{-\frac{1}{2}}) dt ds \le (t_*+2\sqrt{t_*})\|u\|_{Y^{m+1}}$$

and the same argment holds for u replaced by $u-v$. Finally, using similar arguments as for $\mathcal{G}(\vec{0})$ one can prove $\mathcal{G}u - \mathcal{G}v \in C^0(I; H^m_\sigma)$. This completes the proof of Step 2.

[Lecture 17: 17.06]

Step 3 in the proof of Thm.15. $\mathcal{G}$ is a contraction in the unit ball. Remember the two constants K, L in the definition (2.4.29) of $\|\cdot\|_{Z^m}$. We will prove the following Claim.

Claim 3.1: if we choose $L = KC_1(t_* + \sqrt{t_*})$, then

$$\|\mathcal{G}u - \mathcal{G}v\|_{Z^m} \le \frac{\lambda}{2}\|u-v\|_{Z^m}\left(\|u\|_{Z^m} + \|v\|_{Z^m}\right), \qquad \text{with } \frac{\lambda}{2} := C_1 C_2 K (t_* + \sqrt{t_*}). \tag{2.4.36}$$

Note that, once L is fixed, the only free parameters remaining are t_* and K. The claim above implies that, if we choose K and t_* such that $\lambda < 1$ we have

$$\|\mathcal{G}u - \mathcal{G}v\|_{Z^m} \le \lambda \|u-v\|_{Z^m} \qquad \forall u, v \in Z^m \quad \text{s.t. } \|u\|_{Z^m} \le 1 \text{ and } \|v\|_{Z^m} \le 1.$$

i.e. $\mathcal{G}$ is a contraction in the unit ball for Z^m.

Proof of Claim 3.1 Using Claim 2.2 we have

$$\|\mathcal{G}u - \mathcal{G}v\|_{Z^m} \le \max\left\{\frac{\|\mathcal{G}u - \mathcal{G}v\|_{X^m}}{K}, \frac{\|\mathcal{G}u - \mathcal{G}v\|_{Y^{m+1}}}{L}\right\} \le C_2 \Delta_{uv} \max\left\{\frac{1}{K}, \frac{C_1(t_* + \sqrt{t_*})}{L}\right\} = \frac{C_2 \Delta_{uv}}{K}$$

where in the last equality we used $L = KC_1(t_* + \sqrt{t_*})$. Now using

$$\|u-v\|_{X^m} \le K\|u-v\|_{Z^m}, \quad \|u-v\|_{Y^{m+1}} \le L\|u-v\|_{Z^m},$$

we get $\Delta_{uv} \le KL[\|u\|_{Z^m} + \|v\|_{Z^m}]\ \|u-v\|_{Z^m}$, hence

$$\|\mathcal{G}u - \mathcal{G}v\|_{Z^m} \le C_2 L \|u-v\|_{Z^m}\ [\|u\|_{Z^m} + \|v\|_{Z^m}].$$

Step 4 in the proof of Thm.15. Construction of the fixed point. Let $(U_n)_{n\in\mathbb{N}}$ the sequence in Z^m defined by

$$U_0 := \mathcal{G}\vec{0}, \quad U_{n+1} = \mathcal{G}U_n \ \forall n \ge 0.$$

Claim 4.1: For any $0 < \epsilon < 1$ there exists $0 < \lambda_0(\epsilon) < 1$ such that if $\|U_0\|_{Z^m} \le \epsilon$ and $\lambda \le \lambda_0$ we have $\|U_n\|_{Z^m} < 1\ \forall n \ge 0$.

Proof of Claim 4.1 By induction. By hypothesis the claim holds for U_0. Assume it holds for all $0 \le k \le n$. Using (2.4.36) and $\|U_k\|_{Z^m} < 1$ for all $0 \le k \le n$ we have

$$\|U_{k+1} - U_k\|_{Z^m} = \|\mathcal{G}U_k - \mathcal{G}U_{k-1}\|_{Z^m} \le \lambda \|U_k - U_{k-1}\|_{Z^m} \le \lambda^k \|U_1 - U_0\|_{Z^m}, \quad k = 1, \ldots, n.$$

Hence

$$\|U_{n+1}\|_{Z^m} \le \|U_0\|_{Z^m} + \sum_{k=0}^{n} \|U_{k+1} - U_k\|_{Z^m} \le \|U_0\|_{Z^m} + \sum_{k=0}^{n} \lambda^k \|U_1 - U_0\|_{Z^m} \le \|U_0\|_{Z^m} + \frac{1}{1-\lambda}\|U_1 - U_0\|_{Z^m}.$$

Note that $\|U_1 - U_0\|_{Z^m} = \|\mathcal{G}U_0 - \mathcal{G}\vec{0}\|_{Z^m} \le \frac{\lambda}{2}\|U_0\|^2_{Z^m}$, hence

$$\|U_{n+1}\|_{Z^m} \le \|U_0\|_{Z^m}\left(1 + \frac{\lambda\|U_0\|_{Z^m}}{2(1-\lambda)}\right) \le \epsilon\left(1 + \frac{\lambda\epsilon}{2(1-\lambda)}\right) < 1$$

if we choose λ small enough, depending on ϵ. This completes the proof of Claim 4.1.

Consequence of Claim 4.1 If $\|U_0\|_{Z^m} \le \epsilon$ and $\lambda \le \lambda_0$, then U_n is a Cauchy sequence in the unit ball of Z^m, hence $\exists U \in Z^m$ with $\|U\|_{Z^m} \le 1$ such that $U = \mathcal{G}U$.
The fixed point U is unique in the unit ball, since if U, U' are two fixed points with $\|U\|_{Z^m} \le 1$ and $\|U'\|_{Z^m} \le 1$, we obtain a contradiction:

$$\|U - U'\|_{Z^m} = \|\mathcal{G}U - \mathcal{G}U'\|_{Z^m} \le \lambda\|U - U'\|_{Z^m} < \|U - U'\|_{Z^m}.$$

It remains to prove that we can ensure $\|U_0\|_{Z^m} < 1$. Indeed

$$\|U_0\|_{Z^m} = \|\mathcal{G}\vec{0}\|_{Z^m} = \max\left\{\frac{\|\mathcal{G}\vec{0}\|_{X^m}}{K}, \frac{\|\mathcal{G}\vec{0}\|_{Y^{m+1}}}{L}\right\} \le \frac{\|u_0\|_{H^m}}{K}$$

where we used Claim 2.1 and $L = KC_1(t_* + \sqrt{t_*})$ If we choose $K = 2\|u_0\|_{H^m}$ then we can replace $\epsilon = \frac{1}{2}$ in Claim 4.1, hence $\lambda \le \lambda_0(1/2)$.

Step 5 in the proof of Thm.15. Optimization of t_*. Let us summarize the relations we obtained. C_1, C_2 are fixed constants. The parameter K is fixed to $K = 2\|u_0\|_{H^m}$ to ensure $\|U_0\|_{Z^m} < 1$. The parameter L is fixed to $L = KC_1(t_* + \sqrt{t_*}) = 2C_1\|u_0\|_{H^m}(t_* + \sqrt{t_*})$. Finally

$$\lambda = 2C_1C_2K(t_* + \sqrt{t_*}) = 4C_1C_2\|u_0\|_{H^m}(t_* + \sqrt{t_*})$$

and to ensure we have a contraction we must have $\lambda \le \lambda_0 = \lambda_0(1/2)$. Set $\lambda = \lambda_0$, i.e. the largest possible choice. Hence t_* must satisfy

$$(t_* + \sqrt{t_*}) = \frac{1}{a} \qquad \text{with} \qquad a := \frac{4C_1C_2\|u_0\|_{H^m}}{\lambda_0},$$

then $t_* \ge (2a)^{-1}$ or $\sqrt{t_*} \ge (2a)^{-1}$ i.e.

$$t_* \ge \min\left\{\frac{1}{2a}, \frac{1}{(2a)^2}\right\} = \frac{1}{\max\{2a, (2a)^2\}} \ge \frac{1}{2a + (2a)^2} \ge \frac{c_m}{\|u_0\|_{H^m} + \|u_0\|^2_{H^m}}$$

for some constant c_m. Note that if u_0 is very regular then $\|u_0\|_{H^m}$ is small and the time interval gets very large.

Step 6 in the proof of Thm.15. General ν. Let $U(t,x)$ be the unique solution of

$$\begin{cases} (\partial_t - \Delta)U = F(U,U) \\ U(0) = U_0. \end{cases} \tag{2.4.37}$$

on the time interval $[0, t_*^{(1)}]$ with $t_*^{(1)} = \frac{c_m}{\|u_0\|_{H^m} + \|u_0\|^2_{H^m}}$. Then the function $u(t,x) := \nu U(\nu t, x)$ is solution of

$$\begin{cases} (\partial_t - \nu\Delta)u = F(u,u) \\ u(0) = \nu U_0 = u_0. \end{cases} \tag{2.4.38}$$

on the time interval $\nu t \leq t_*^{(1)}$, i.e.

$$t_*^{(\nu)} = \frac{1}{\nu} \frac{c_m}{\|U_0\|_{H^m} + \|U_0\|_{H^m}^2} = \frac{c_m \nu}{\nu \|u_0\|_{H^m} + \|u_0\|_{H^m}^2}, \tag{2.4.39}$$

where we replaced $U_0 = \frac{u_0}{\nu}$. This concludes the proof of Thm.15 □

2.4.5 Local solutions for Euler-equation.

Our goal is to construct a local strong solution for Euler equation by letting $\nu \to 0$ in the local strong solution we obtained for N-S in the previous section. The two main obstacles in this program are:

(1) the time interval $[0, t_*]$ where we proved existence and uniqueness of N-S solution shrinks to zero as $\nu \to 0$ (see (2.4.39)). We will prove below that the time interval can be extended to $[0, T_*]$ where T_* still depends on the initial condition u_0, but is now independent of ν;

(2) we need to justify the limit $\nu \to 0$ of the solution of N-S in the appropriate function space.

Here we will consider only the solution of Problem (1) (extending the time interval). We will restrict to $d = 3$. In $d = 2$ one can obtain even stronger estimates. The main result of this section is summarized in the following theorem.

Theorem 16 *Let $d = 3$, $m \geq 3$ and $M > 0$ some constant. Then there exists a constant $\tilde{c}_m$, depending only on m, such that, for any $\nu > 0$ and $u_0 \in H_\sigma^m$ satisfying $\|u_0\|_{H^m} \leq M$ there exists a unique solution $u \in Z^m([0, T_*])$ of N-S with $u(0) = u_0$ on the time interval $[0, T_*]$ with*

$$T_* := \frac{1}{2\tilde{c}_m M}. \tag{2.4.40}$$

Preliminary results: energy estimates

Remember that the total kinetic energy of our fluid is given by $E_k(t) = \int_{\mathbb{R}^3} \rho(t,x) \frac{|u(t,x)|^2}{2} = \frac{1}{2}\|u(t)\|_{L^2}^2$, where $u(t,x)$ is the velocity field (in spatial coordinates) and we assumed incompressibility and homogeneity: $\rho(t,x) = \rho_0(x) = 1$. The first lemma states that the kinetic energy can only decrease in time.

Lemma 32 *Let $u \in Z^m$ with $m \geq 3 = d$ a solution of NS equation.*

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 = -\nu \|Du(t)\|_{L^2}^2 \leq 0.$$

Proof. We multiply NS equation $\partial_t u - \nu \Delta u + \mathcal{P}_\sigma(u \cdot Du) = 0$ by u and integrate over space

$$0 = \int_{\mathbb{R}^3} \sum_{j=1}^3 \left[u_j \partial_t u_j - u_j \Delta u_j + \sum_{k=1}^3 u_j \mathcal{P}_\sigma u_k \partial_k u_j \right] dx$$

Since $u \in C^1(I; H^{m-2})$, the first term can be reorganized as $\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |u(t,x)|^2 dx$. For the second note that for any $g \in C_c^\infty(\mathbb{R}^d)$ we have, integrating by parts,

$$\int_{\mathbb{R}^3} g \Delta g dx = -\int_{\mathbb{R}^3} |Dg|^2 dx.$$

By density this holds also for any $g \in H^2(\mathbb{R}^d)$. Hence $-\int_{\mathbb{R}^3} \sum_{j=1}^3 u_j \Delta u_j dx = \int_{\mathbb{R}^3} |Du|^2 dx$. Finally, for any $g \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$ with div $g = 0$ we have

$$\int_{\mathbb{R}^3} \sum_{k,j} g_j \mathcal{P}_\sigma g_k \partial_k g_j dx = \int_{\mathbb{R}^3} \sum_{k,j} g_j g_k \partial_k g_j dx = \frac{1}{2}\int_{\mathbb{R}^3} \sum_k g_k \partial_k |g|^2 dx = 0$$

where in the first step we used that $g-\mathcal{P}_\sigma g$ is a gradient and hence perpendicular to all divergence free fields, and in last step we used again that div $g=0$. By density the identity holds for $u\in H^2_\sigma$. Putting the integrals together we obtain the result. $\square$

[Lecture 18: 22.06]

Lemma 33 *Let $m\geq 3=d$ and $\alpha\in\mathbb{N}^d$ with $|\alpha|\leq m$. Then*

(i) $\forall u\in H^{m+1}_\sigma$ we have

$$|(D^\alpha u, D^\alpha F(u,u))_{L^2}|\leq c\|u\|^3_{H^m}.$$

(ii) $\forall u,v\in H^{m+1}_\sigma$ we have

$$|(D^\alpha u, D^\alpha F(v,u))_{L^2}|\leq c\|u\|^2_{H^m}\|v\|_{H^m}.$$

(iii) $\forall u\in H^{m+1}_\sigma$, $v\in H^{m+1}_\sigma$ we have

$$|(D^\alpha u, D^\alpha F(u,v))_{L^2}|\leq c\|u\|^2_{H^m}\|v\|_{H^{m+1}}.$$

Remark. Beware the changed order in (ii) and (iii) for the arguments of F!

Proof. Exercise sheet
A consequence of the estimates above are the following 'generalized' energy estimates.

Lemma 34 *Let $u\in Z^m([0,T_*])$ a solution of NS for $m\geq 3=d$ on some time interval $[0,T_*]$. Then there exists a constant $\tilde{c}_m>0$ depending only on m such that*

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{H^m}+\nu\|Du(t)\|^2_{H^m}\leq\tilde{c}_m\|u(t)\|^3_{H^m}\quad\forall 0<t<T_*. \tag{2.4.41}$$

In particular

$$\frac{d}{dt}\|u(t)\|_{H^m}\leq\tilde{c}_m\|u(t)\|^2_{H^m}\quad\forall 0<t<T_*. \tag{2.4.42}$$

Remark. Note that since $u\in C^1([0,T_*];H^{m-2}_\sigma)$ we know that $\|u(t)\|_{L^2}$ is differentiable in t, but it is not clear why $\|u(t)\|_{H^m}$ should be differentiable in t. The solution comes from the following corollary (which we give without proof).

Corollary 2 (higher regularity) *Let $d=2,3$, $m\geq 2$, $I=[0,T_*]$. If $u\in C^0(I;H^m_\sigma)\cap L^1(I;H^{m+1}_\sigma)\cap C^1(I;H^{m-2}_\sigma)$ is a solution of NS then $\forall k,l\in\mathbb{N}$, $\forall\epsilon>0$ we have $u\in C^k([\epsilon,T_*];H^l_\sigma)$.*
In particular $u\in C^\infty((0,T_*)\times\mathbb{R}^d)$ and $\partial_t u-\nu\Delta u=F(u,u)$ holds pointwise.

In the present case $u\in Z^m$ hence $u\in C^1([\epsilon,T_*];H^m_\sigma)$, hence the H^m norm is differentiable in t.

Proof of Lemma 34.

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{H^m}&=\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|\leq m}c_\alpha\int_{\mathbb{R}^3}D^\alpha u\cdot D^\alpha u\,dx=\sum_{|\alpha|\leq m}c_\alpha\int_{\mathbb{R}^3}D^\alpha u\cdot D^\alpha\partial_t u\,dx\\&=\sum_{|\alpha|\leq m}c_\alpha\int_{\mathbb{R}^3}D^\alpha u\cdot D^\alpha[\Delta u+F(u,u)]dx\\&=\sum_{|\alpha|\leq m}c_\alpha\left[(D^\alpha u,\Delta D^\alpha u)_{L^2}+(D^\alpha u,D^\alpha F(u,u))_{L^2}\right],\end{aligned}$$

where c_α is some combinatorial coefficient. Integrating by parts $\sum_{|\alpha|\le m} c_\alpha (D^\alpha u, \Delta D^\alpha u)_{L^2} = -\sum_{|\alpha|\le m} c_\alpha |D^\alpha Du|_{L^2}^2 = -\|Du\|_{H^m}^2$. Finally by Lemma 33 (i) $|(D^\alpha u, D^\alpha F(u,u))_{L^2}| \le c\|u\|_{H^m}^3$. This proves the first claim. To prove the second claim note that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{H^m}^2 = \|u(t)\|_{H^m}\frac{d}{dt}\|u(t)\|_{H^m} \le \tilde{c}_m\|u(t)\|_{H^m}^3.$$

□

Proof of Theorem 16. Step 1: preliminary result. Let $f \in C^0([0,T]) \cap C^1((0,T))$ such that

(a) $f(t) \ge 0 \ \forall t$,

(b) $f'(t) \le Kf(t)^2 \ \forall 0 < t < T$ for some fixed constant K,

(c) $T \le \frac{1}{2Kf(0)}$.

Then we have $f(t) \le 2f(0)$ for all $t \in [0,T]$.

To prove this fact assume first $f(t) > 0$ for all t. Then

$$f'(t) \le Kf(t)^2 \quad \text{iff} \quad \frac{f'(t)}{f(t)^2} \le K \quad \text{iff} \quad \left(\frac{1}{f'(t)}\right) \ge -K.$$

Integrating over t we get $\frac{1}{f(t)} - \frac{1}{f(0)} \ge -Kt \ge -KT \ge -\frac{1}{2f(0)}$, where in the last step we used assumption (c). Hence $\frac{1}{f(t)} \ge \frac{1}{2f(0)}$.
In the case $f(t) \ge 0$ we set $\epsilon > 0$ and $f_\epsilon(t) := f(t) + \epsilon > 0$ for all t. Note that $f_\epsilon'(t) = f'(t) \le Kf(t)^2 \le Kf_\epsilon(t)^2$. If we restrict to the interval $[0, T_\epsilon]$, with $T_\epsilon \le T$ such that $T_\epsilon \le \frac{1}{2Kf_\epsilon(0)}$, then $f_\epsilon(t) \le 2f_\epsilon(0)$. The result follows by continuity taking $\epsilon \to 0$.

Step 2: estimate of the time interval. We prove the result by contradiction. Fix $\nu > 0$ and let

$$T_1 := \sup\{\tilde{t} > 0 | \ \exists u \text{ strong solution of NS with } u(0) = u_0\}$$

be the maximal time interval where the solution for NS exists. Let $\|u_0\|_{H^m} \le M$. We want to show that $T_1 \ge T_* = \frac{1}{2\tilde{c}_m M}$. By constradiction assume $T_1 < T_*$. We will show that we can construct a solution in the interval $[T_1, T_1 + \delta]$ and paste it to the one on $[0, T_1]$. As a result we will obtain a solution on $[0, T_1 + \delta]$ which contradicts the definition of T_1.
To construct the extended solution let $T_2 = T_1 - \epsilon > 0$ with $0 < \epsilon \ll 1$. Since $T_2 < T_1$ there exists a unique strong solution for NS with $u(0) = u_0$. This solution must satisfy the energy estimate (2.4.42)

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{H^m} \le \tilde{c}_m\|u(t)\|_{H^m}^2 \quad \forall 0 < t < T_2.$$

Now set $f(t) = \frac{d}{dt}\|u(t)\|_{H^m}$. Then $f \in C^0([0,T_2]) \cap C^1((0,T_2))$, $f(t) \ge 0$ and $f'(t) \le \tilde{c}_m f(t)^2$. Moreover since $T_2 < T_1 < T_* = \frac{1}{2M\tilde{c}_m}$ we have $\tilde{c}_m \le \frac{1}{2T_2 f(0)}$. Then we can apply Step 1. with $K = \tilde{c}_m$, i.e. $\|u(t)\|_{H^m} \le 2\|u_0\|_{H^m} \le 2M = \tilde{M}$.
We consider now NS on the interval $[T_2, T_2 + t_*]$ with new initial condition $\tilde{u}_0(x) = u(T_2, x)$, and $t_* = \frac{c_m\nu}{\nu\|\tilde{u}_0\|_{H^m} + \|\tilde{u}_0\|_{H^m}^2}$. Since $\|\tilde{u}_0\|_{H^m} \le 2M$ we have

$$t_* \ge \frac{\nu c_m}{\nu 2M + 4M^2} \ge 2\epsilon$$

if ϵ is taken small enough. Then $T_2 + t_* > T_1$ and we obtain a contradiction. This ends the proof.
□

3. Calculus of variations and elasticity theory

[Lecture 19: 24.06]

3.1 Introduction: equilibrium configurations in a hyperelastic solid

Let $x(t, X)$ describe the deformation of an hyperelastic solid subject to some external time-independent force. After a long time the solid will stabilize into some new equilibrium configuration, described by a time-intependent function $x(X)$. We will see that this function $x(X)$ is the solution of a certain PDE, and can be related to the minimizer of a functional integral.

3.1.1 Reminders

Remember the equation of motion in material coordinates (1.4.47):

$$\rho_0(X)\partial_t^2 x(t, X) = (\det Dx)(t, X) f_m(t, X) + \mathrm{DIV}(\mathcal{S})(t, X)$$

where $f(t, x)$ is the force density and has the dimension $[force]/[volume]$. In the following it will be more convenient to replace f by

$$f(t, x) = \rho(t, x)b(t, x), \quad \rho = \text{mass density}, \; b : \mathbb{R} \times \Omega \to \mathbb{R}^d = [\text{force}]/[\text{mass}]$$

In material coordinates, using (1.4.46), we have

$$f_m(t, X) = \rho_m(t, X)b_m(t, X) = \frac{\rho_0(X)}{\det Dx(t, X)} b_m(t, X),$$

hence the equation of motion becomes

$$\rho_0(X)\partial_t^2 x(t, X) = \rho_0(X)b_m(t, X) + \mathrm{DIV}(\mathcal{S})(t, X). \tag{3.1.1}$$

We will make the following assumptions.

- The material is elastic and satisfies frame indifference (see Sect.1.5.2), hence $\mathcal{S}(t, X) = \hat{\mathcal{S}}(Dx)$, where $\hat{\mathcal{S}}$ is the constitutive law of the Piola-Kirchoff tensor $\hat{\mathcal{S}} : GL_+(d) \to \mathbb{R}^{d\times d}$.
- We have isotropy: $\rho_0(X) = \rho_0 = 1 \; \forall X$.
- The external force is time independent $b_m = b_m(X)$.

An equilibrium solution is a solution of (3.1.1) that is independent of time $x : \Omega \to \mathbb{R}^d$. Hence $\partial_t x = 0$ and $x(X)$ must satisfy

$$b_m(X) + \mathrm{DIV}(\hat{\mathcal{S}})(Dx(X)) = 0 \tag{3.1.2}$$

3.1.2 Stored energy

We assume the material is hyperelastic then (see Def.19 in Chapter 1) there is a function $\hat{W} \in C^1(GL_+(d);\mathbb{R})$ such that

$$\hat{S}_{ij}(F) = \frac{\partial}{\partial F_{ij}} \hat{W}(F).$$

The function $\hat{W}$ is called the stored energy and corresponds to the internal energy of the material. To see this, remember the expression for the total energy (1.4.54) $E(t, U) = \int_{U(t)} \rho \left[\frac{|v|^2}{2} + \epsilon\right] dx$, where ϵ is a C^1 spatial field (the internal energy density). The energy conservation equation (1.4.57) reads $\rho \frac{D\epsilon}{Dt} = \mathrm{tr}(\sigma^t Dv)$, where we assumed there is no heat exchange at equilibrium. In material coordinates this equation becomes

$$\det Dx \ \rho_m \partial_t \epsilon_m = \det Dx \ \mathrm{tr} \sigma_m^t Dv_m$$

Using (1.4.46), the left-hand side becomes $\det Dx \ \rho_m \partial_t \epsilon_m = \rho_0(X) \partial_t \epsilon_m$. For the right-hand side remember that, from (1.4.48), we have $\sigma_m = \frac{1}{\det Dx} \hat{\mathcal{S}} Dx^t$, and

$$(Dv)_{ij} = \partial_{x_j} v_i = \sum_k \frac{\partial X_k}{\partial x_j} \frac{\partial (v_m)_i}{\partial X_k} = \sum_k \frac{\partial X_k}{\partial x_j} \partial_t \frac{\partial x_i}{\partial X_k} = \sum_k [\partial_t (Dx)]_{ik} (Dx)^{-1}_{kj}$$

where we used $v_m = \partial_t x$ and $\frac{\partial X_k}{\partial x_j} = (Dx)^{-1}_{kj}$. Hence $(Dv)_m = [\partial_t (Dx)](Dx)^{-1}$, and

$$\det Dx \ \mathrm{tr}[\sigma_m^t Dv_m] = \mathrm{tr}[\hat{\mathcal{S}}^t \partial_t (Dx)] = \sum_{ij} \hat{\mathcal{S}}_{ij} \partial_t (Dx)_{ij} = \sum_{ij} \partial_t (Dx)_{ij} \frac{\partial \hat{W}}{\partial F_{ij}} (Dx) = \partial_t \hat{W}(Dx).$$

Inserting all this in the equation above we get

$$\partial_t [\rho_0 \epsilon_m] = \partial_t \hat{W}(Dx) \quad \Rightarrow \quad \rho_0(X) \epsilon_m(t, X) = \hat{W}(Dx(t, X)) + E_0(X)$$

where $E_0(X)$ is some initial energy (independent of time). Hence W = internal elastic energy.

3.1.3 Equilibrium solution and functional integrals

Claim. Solving (3.1.2) is related to find a minimizer of the function

$$I(x(X)) := \int_\Omega [\hat{W}(Dx) - b_m \cdot x] dX \tag{3.1.3}$$

We will prove this claim rigorously in the next sections. A non-rigorous justification of this fact is obtained as follows. Let $x : \Omega \to \mathbb{R}^d$, and $y : \Omega \to \mathbb{R}^d$, two fixed functions, set $x_\tau(X) := x(\tau) + \tau y(X)$, $\tau \in \mathbb{R}$ and consider $\tau \to h(\tau) := I(x_\tau)$. This gives the variation of the function $I(x)$ along the direction y. A necessary (not sufficient) condition for h to be minimal at $\tau = 0$ is that the first derivative vanishes $h'(0) = 0$, for any choice of the direction y. Inserting this in the expression for I, without worrying if the operations are justified, we get

$$\begin{aligned} \frac{d}{d\tau} \int_\Omega \left[\hat{W}(Dx_\tau) - b_m \cdot x_\tau\right] dX &= \int_\Omega \left[\sum_{ij} (Dy)_{ij} \frac{\partial \hat{W}}{\partial F_{ij}} (Dx_\tau) - b_m \cdot y\right] dX \\ &= -\int_\Omega \sum_i y_i \left[\mathrm{DIV}(\hat{\mathcal{S}})(Dx_\tau)_i + (b_m)_i\right] dX \end{aligned}$$

where in the last step we used $\hat{S}_{ij}(F) = \frac{\partial \hat{W}}{\partial F_{ij}}(F)$ and integration by parts. Setting $\tau = 0$ we get

$$\int_\Omega \sum_i y_i \left[\mathrm{DIV}(\hat{\mathcal{S}})(Dx)_i + (b_m)_i\right] dX = 0 \qquad \forall y, \quad \Rightarrow \mathrm{DIV}(\hat{\mathcal{S}})(Dx) + (b_m) = 0.$$

3.2 First and second variation

3.2.1 Setting up: minimizer

In the following we will look for minimizers of some functional of the form

$$I(u) := \int_{\Omega} f(x, u(x), Du(x))dx, \tag{3.2.4}$$

where

- $\Omega \subset \mathbb{R}^d$ is a bounded open set,
- $u : \Omega \to \mathbb{R}^n$, $n \geq 1$ is a vector-valued function in some Banach space $\mathcal{B}$ (typically we will take $W^{1,q}(\Omega; \mathbb{R}^n)$),
- depending on the problem we may have boundary conditions i.e. $u(x) = g(x)$ $\forall x \in \partial\Omega$.

The function f is called the **Lagrangian**.

Remark. In the case of a hyperelastic material $n = d$ and $u(x)$ is replaced by $x(X)$. Note that in the general case $Du(x) \in \mathbb{R}^{n \times d}$.

Definition 31 (minimizer) *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, f a function*

$$\begin{array}{lll} f: & \Omega \times \mathbb{R}^n \times \mathbb{R}^{n \times d} & \to \mathbb{R} \\ & (x, z, \xi) & \to f(x, z, \xi) \end{array}$$

satisfying $x \to f(x, u(x), Du(x)) \in L^1(\Omega)$ for all $u \in W^{1,q}(\Omega; \mathbb{R}^n)$, for some $q \geq 1$. Finally, let $I : W^{1,q}(\Omega; \mathbb{R}^n) \to \mathbb{R}$ the functional defined in eq.(3.2.4).
We say that $u \in W^{1,q}(\Omega; \mathbb{R}^n)$ is a minimizer of I with respect to its own boundary conditions, if

$$I(u + w) \geq I(u) \qquad \forall w \in W_0^{1,q}(\Omega; \mathbb{R}^n).$$

Remark. $w \in W_0^{1,p}(\Omega; \mathbb{R}^n)$ implies that $(u + w) = u$ on the boundary of Ω.

Example 1. Let $n = 1$, $X = \{u \in C^2(\bar{\Omega}) |\ u_{|\partial\Omega} = g\}$, for some $g \in C^0(\partial\Omega)$. We define

$$\begin{array}{ll} I: & X \to \mathbb{R} \\ & u \to I(u) = \int_{\Omega} \left[\frac{|Du(x)|^2}{2} - b(x)u(x) \right] dx \end{array}$$

where $b \in C^0(\bar{\Omega})$. Then $u_0 \in X$ is a minimizer for I, i.e. $I(u) \geq I(u_0)$ for all $u \in X$ iff u_0 is a solution of

$$\begin{cases} -\Delta u = b & \text{inside } \Omega \\ u_{|\partial\Omega} = g \end{cases}$$

where $\partial\Omega$ must be sufficiently regular.

Example 2. (minimal surface) Let $u : \Omega \to \mathbb{R}$. The set $\{(x, u(x))|\ x \in \Omega\}$ defines a surface. The corresponding area is given by $I(u) := \int_{\Omega} \sqrt{1 + |Du|^2} dx$. Then minimizing $I(u)$ corresponds to minimize the surface, w.r.t. some boundary conditions.

Example 3. (isoperimetric problem) Set $d = n = 1$ and $u(x) \geq 0$. Then $\{(x, u(x))| \; x \in \Omega\}$ is the surface generated by u, and $\{(x, y)| \; x \in \Omega, 0 \leq y \leq u(x)\}$ is the volume generated by u. Let $I(u)$ = surface - λ volume, i.e.

$$I(u) := \int_\Omega \sqrt{1 + u'(x)^2} dx - \lambda \int_\Omega u(x) dx,$$

with $\lambda > 0$ a parameter. To minimize I we need to minimize the surface and maximize the volume.

[Lecture 20: 29.06]

3.2.2 Directional derivatives

Definition 32 *let $u \in \mathcal{B}$ and $\omega \in C_c^\infty(\Omega; \mathbb{R}^n)$. We define the variation of $I(u)$ along the direction ω as the function*

$$\begin{aligned} h_{u,\omega} : \quad & \mathbb{R} \to \mathbb{R} \\ & \tau \to h_{u,\omega}(\tau) := I(u + \tau\omega) \end{aligned}$$

The next lemma gives some conditions on the Lagrangian f under which directional derivatives may be applied. To simplify notation we will collect all variables u_j and $(Du)_{ij}$ in a single big vector $X \in \mathbb{R}^N$. We replace then $f(x, u, Du)$ with $g(x, X)$.

Lemma 35 *Let*

$$\begin{aligned} g : \quad & \Omega \times \mathbb{R}^N \to \mathbb{R} \\ & (x, X) \to g(x, X) \end{aligned}$$

satisfying:

- *$\Omega \subset \mathbb{R}^d$ is a bounded open set, g is Borel measurable,*
- *$x \to g(x, 0) \in L^1(\Omega)$,*
- *$X \to g(x, X) \in C^p(\mathbb{R}^N)$, for some $p \geq 1$ and for all $x \in \Omega$,*
- *$\exists q \geq p$ and a constant $K > 0$ such that*
  $|D_X^l g(x, X)| \leq K(1 + |X|^{q-l})$ for all $x \in \Omega, X \in \mathbb{R}^N$ and $0 \leq l \leq p$.

Let

$$\begin{aligned} I : \quad & L^q(\Omega; \mathbb{R}^N) \to \mathbb{R} \\ & U \to I(U) := \textstyle\int_\Omega g(x, U(x)) dx. \end{aligned}$$

Then, for any fixed $U, W \in L^q(\Omega; \mathbb{R}^N)$, the function $h : \mathbb{R} \to \mathbb{R}$ defined by $\tau \to h(\tau) := I(U + \tau W)$, is differentiable p times and $\forall 0 \leq l \leq p$

$$\frac{d^l h(\tau)}{d\tau^l} = \int_\Omega \sum_{j_1, \ldots, j_l} W_{j_1}(x) \ldots W_{j_l}(x) \left[\frac{\partial}{\partial X_{j_1}} \cdots \frac{\partial}{\partial X_{j_l}} g \right] (x, U(x) + \tau W(x)) dx.$$

Remark 1. $x \to g(x, U(x)) \in L^1(\Omega)$, hence $I(U)$ is well defined. To see this write

$$\begin{aligned} g(x, U(x)) &= g(x, 0) + [g(x, U(x)) - g(x, 0)] = g(x, 0) + \int_0^1 \partial_s g(x, sU(x)) ds \\ &= g(x, 0) + \sum_j U_j(x) \int_0^1 \partial_{X_j} g(x, sU(x)) ds, \end{aligned}$$

where in the second equality we used $x \to g(x,X)$ is differentiable in X. Now $x \to g(x,0) \in L^1(\Omega)$, and

$$\left| \sum_j U_j(x) \int_0^1 \partial_{X_j} g(x, sU(x)) ds \right| \le K|U(x)| \int_0^1 (1 + |U(x)|^{q-1} s^{q-1}) ds \le K(|U(x)| + |U(x)^q|).$$

The result now follows from $U \in L^q(\Omega)$ and $|\Omega| < \infty$.

Remark 2. Note that in our case $X = U \in L^q$ corresponds to $u, Du \in L^q$, i.e $u \in W^{1,q}(\Omega;\mathbb{R}^n)$.

Proof of Lemma 35 We do the proof only for $p = 1$, the general case being similar. It is enough to study the derivative at $\tau = 0$ since for $\tau \neq 0$ we can study the derivative of $h(\tilde{\tau}) := I([U + \tau W] + \tilde{\tau} W)$ at $\tilde{\tau} = 0$. The finite variation is given by

$$\frac{h(\tau) - h(0)}{\tau} = \int_\Omega G_\tau(x) dx, \qquad \text{with} \quad G_\tau(x) := \frac{g(x, U(x) + \tau W(x)) - g(x, U(x))}{\tau}$$

We claim $\exists G \in L^1(\Omega)$ such that $|G_\tau(x)| \le G(x)$ $\forall x \in \Omega$ and $\forall |\tau| < 1$. Then, by dominated convergence $\lim_{\tau \to 0} \int_\Omega G_\tau(x) dx = \int_\Omega \lim_{\tau \to 0} G_\tau(x) dx$. Finally, since $X \to g(x,X) \in C^1$ we have $\lim_{\tau \to 0} G_\tau(x) = \sum_j W_j \partial_{X_j} g$. To prove the claim we use Taylor integral formula

$$G_\tau(x) = \frac{1}{\tau} \int_0^1 \partial_s g(x, U(x) + s\tau W(x)) ds = \int_0^1 \sum_j W_j(x) \partial_{X_j} g(x, U + s\tau W) ds$$

Hence

$$\begin{aligned} |G_\tau(x)| &\le |W(x)| \sup_{s \in [0,1]} |D_X g(x, U(x) + s\tau W(x))| \\ &\le K|W(x)|(1 + |U(x) + s\tau W(x)|^{q-1}) \le K'(|W(x)| + |W(x)|^q + |U(x)|^q), \end{aligned}$$

for some $K' > 0$. The result now follows from $U, W \in L^q(\Omega)$ and $|\Omega| < \infty$. □

3.2.3 First variation and Euler-Lagrange equation

The following result relates the minimizer to the first derivative.

Lemma 36 *Let*

$$\begin{aligned} f: \quad & \Omega \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \to \mathbb{R} \\ & (x, z, \xi) \to f(x, z, \xi) \end{aligned}$$

be a Lagrangian satisfying the assumptions of Lemma (35) with $p = 1$, i.e.

- $\Omega \subset \mathbb{R}^d$ *is a bounded open set, f is Borel measurable,*
- $x \to f(x,0,0) \in L^1(\Omega)$,
- $(z,\xi) \to g(x,z,\xi) \in C^1(\mathbb{R}^n \times \mathbb{R}^{n \times d})$, *for all $x \in \Omega$,*
- $\exists q \ge 1$ *and a constant $K > 0$ such that* $|D_z f| + |D_\xi f| \le K(1 + |z|^{q-1} + |\xi|^{q-1})$ *for all (x, z, ξ).*

Let $u \in W^{1,q}(\Omega;\mathbb{R}^n)$ be a minimizer of I w.r.t its own boundary conditions (see Def.31). Then for all $w \in W_0^{1,q}(\Omega;\mathbb{R}^n)$ we have

$$0 = \int_\Omega \left[(D_z f) \cdot w + (D_\xi f) \cdot Dw\right] dx \tag{3.2.5}$$

where $(D_z f) \cdot w = \sum_j w_j \partial_{z_j} f$, and $(D_\xi f) \cdot Dw = \sum_{jk} (Dw)_{jk} \partial_{\xi_{jk}} f$. In particular (3.2.5) can be reformulated as

$$div\ (D_\xi f) - D_z f = 0 \qquad \text{in distribution,} \tag{3.2.6}$$

where $div\ (D_\xi f)_i = \sum_k D_{x_k}(\partial_{\xi_{jk}} f)$. (3.2.6) is called the Euler-Lagrange equation for I.

Proof. Since $I(u+w) \geq I(u)$ for all $w \in W_0^{1,q}(\Omega;\mathbb{R}^n)$, the function $h(\tau) := I(u+\tau w)$ is minimal at $\tau = 0$, hence $h'(0) = 0$ for all w. The proof follows by Lemma 35 replacing U by (u, Du) and W by (w, Dw). □

Application: hyperelastic solid

Let us go back to the functional (3.1.3) $I(u) = \int_\Omega [\hat{W}(Du) - b \cdot u]dx$, where $n = d$, and $\hat{W} \in C^1(GL_+(d);\mathbb{R})$ is the stored energy. We assume $b \in C^0(\Omega;\mathbb{R}^d)$, and $\hat{W}(0) = 0$ (we can always translate the energy by a constant to ensure that). Then $f(x,z,\xi) = \hat{W}(\xi) - b(x) \cdot z$ satisfies $x \to f(x,0,0) = 0 \in L^1(\Omega)$, and $(z,\xi) \to f(x,z,\xi) \in C^1(\mathbb{R}^n \times \mathbb{R}^{n\times d})$.

[Lecture 21: 1.07]

3.2.4 Null Lagrangians

These are the analog of constant functions.

Definition 33 *A function $f \in C^1(\bar{\Omega} \times \mathbb{R}^n \times \mathbb{R}^{n\times d})$ is called* *<u>null Lagrangian</u>* *if any function $u \in C^2(\bar{\Omega})$ satisfies the Euler-Lagrange equation.*

Remark Since $f \in C^0(\bar{\Omega} \times \mathbb{R}^n \times \mathbb{R}^{n\times d})$ and both u and $Du \in C^0(\bar{\Omega})$, the integral $I(u)$ is well defined. Moreover, since $f \in C^1(\bar{\Omega} \times \mathbb{R}^n \times \mathbb{R}^{n\times d})$, we can compute directional derivatives, hence we can write the corresponding Euler-Lagrange equation.

Lemma 37 *Let $f \in C^1(\bar{\Omega} \times \mathbb{R}^n \times \mathbb{R}^{n\times d})$. Then f is a null Lagrangian iff $I(u) = I(v)$ for all $u, v \in C^2(\bar{\Omega})$ with $u = v$ on $\partial\Omega$.*

Proof. $\Rightarrow$ Assume f is a null Lagrangian. Let $u, v \in C^2(\bar{\Omega})$ with $u = v$ on $\partial\Omega$, and set $w = u - v$. Then $w \in C^2(\bar{\Omega})$ and $w = 0$ on $\partial\Omega$. Hence $u + \tau w = u$ on $\partial\Omega$ for all $\tau \in \mathbb{R}$. Let $h(\tau) := I(u+\tau w)$, Since f is a null Lagrangian, this implies $u + \tau w$ is a solution of Euler-Lagrange equation for all τ, i.e. $h'(\tau) = 0$ for all τ. Therefore $I(u + \tau w) = I(u)$ for all τ, i.e. $I(u) = I(v)$.

$\Leftarrow$ Assume $I(u) = I(v)$ for all $u, v \in C^2(\bar{\Omega})$ with $u = v$ on $\partial\Omega$. Again let $w = u - v$ and $h(\tau) := I(u + \tau w)$, hence we have $h(\tau) = h(0)$ for all τ. As a consequence $h'(\tau) = 0$ for all τ, i.e $u + \tau w$ satisfies the Euler-Lagrange equation for all τ. This is true in particular for u and v. □

Lemma 38 *Let $n = d$ and consider the Lagrangian $f(x,z,\xi) := \det \xi$. Then f is a null Lagrangian.*

Proof. Exercise sheet

3.2.5 Second variation

Lemma 39 *Let*

$$\begin{aligned} f: \quad & \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R} \\ & (x,z,\xi) \to f(x,z,\xi) \end{aligned}$$

be a Lagrangian satisfying the assumptions of Lemma (35) *with $p = 2$, i.e.*

- *$\Omega \subset \mathbb{R}^d$ is a bounded open set, f is Borel measurable,*
- *$x \to f(x,0,0) \in L^1(\Omega)$,*
- *$(z,\xi) \to g(x,z,\xi) \in C^2(\mathbb{R}^n \times \mathbb{R}^{n\times d})$, for all $x \in \Omega$,*

- *$\exists q \geq 2$ and a constant $K > 0$ such that*
  $|D_z f| + |D_\xi f| \leq K(1 + |z|^{q-1} + |\xi|^{q-1})$ *and*
  $|D_z^2 f| + |D_\xi D_z f| + |D_\xi^2| \leq K(1 + |z|^{q-2} + |\xi|^{q-2})$ *for all* (x, z, ξ).

Let $u \in W^{1,q}(\Omega; \mathbb{R}^n)$ be a minimizer of I w.r.t its own boundary conditions (see Def.31). Then for all $w \in W_0^{1,q}(\Omega; \mathbb{R}^n)$ we have

$$\int_\Omega \left[(w, (D_z^2 f)w) + 2(w, (D_z D_\xi f)Dw) + (Dw, (D_\xi^2 f)Dw) \right] dx \ \geq 0 \tag{3.2.7}$$

where

$$(w, (D_z^2 f)w) := \sum_{jk} w_j w_k (\partial_{z_j} \partial_{z_k} f), \quad (w, (D_z D_\xi f)Dw) := \sum_{jkk'} w_j (Dw)_{kk'} (\partial_{z_j} \partial_{\xi_{kk'}} f),$$
$$(Dw, (D_\xi^2 f)Dw) := \sum_{jj'kk'} (Dw)_{jj'} (Dw)_{kk'} (\partial_{\xi_{jj'}} \partial_{\xi_{kk'}} f).$$

Proof. If u is a minimizer, then $I(u + \tau w) \geq I(u)$ $\forall \tau$, hence $\frac{d^2}{d\tau^2} I(u + \tau w)_{|\tau=0} \geq 0$. Finally (3.2.7) follows from Lemma 35. □

Equations in second variation: scalar case.

We consider first the scalar case $n = 1$. Then $u : \Omega \to \mathbb{R}$ is a scalar, $Du : \Omega \to \mathbb{R}^d$ is a vector and $f : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$. The second variation becomes

$$\int_\Omega \left[w^2 (\partial_z^2 f) + 2 \sum_j w (\partial_j w) \partial_z \partial_{\xi_j} f + \sum_{jk} (\partial_j w)(\partial_k w)(\partial_{\xi_j} \partial_{\xi_k} f) \right] dx \ \geq 0$$

for all $w \in W_0^{1,q}(\Omega; \mathbb{R}^n)$.

Theorem 17 *Let f be as in Lemma 39, with $n = 1$ and let $u \in W^{1,q}(\Omega; \mathbb{R})$ be a minimizer w.r.t. its own b.c. Then*

$$\partial_\xi^2 f(x, u(x), Du(x)) \geq 0 \quad \text{as a quadratic form for a.e. } x \in \Omega, \tag{3.2.8}$$

i.e. $\forall b \in \mathbb{R}^d$ we have $\sum_{jk} b_j (\partial_{\xi_j} \partial_{\xi_k} f) b_k \geq 0$. This is called the second variation equation.

Proof. The strategy is to construct a sequence of functions w_δ such that $w_\delta \to 0$ and $Dw_\delta \not\to 0$ as $\delta \to 0$. This will be defined more precisely below. Let

$$w_\delta(x) := \delta_n \ \psi(x) \sin\left(\tfrac{b \cdot x}{\delta}\right), \quad 0 < \delta_n < 1, \ b \in \mathbb{R}^d, \ \psi \in C_c^\infty(\Omega; \mathbb{R}),$$

then $w_\delta \in C_c^\infty(\Omega; \mathbb{R})$. Moreover let

$$I_1(w) := \int_\Omega w(x)^2 \ \partial_z^2 f(x, u(x), Du(x)) \ dx$$
$$I_2(w) := 2 \sum_j \int_\Omega w(x) \partial_j w(x) \ \partial_z \partial_{\xi_j} f(x, u(x), Du(x)) \ dx$$
$$I_3(w) := \sum_{jk} \int_\Omega \partial_j w(x) \partial_k w(x) \ \partial_{\xi_j \xi_k}^2 f(x, u(x), Du(x)) \ dx$$

we will prove the following claim.

Claim. There exists a sequence $(\delta_n)_{n\in\mathbb{N}}$ with $\delta_n > 0\ \forall n$ and $\lim_{n\to\infty}\delta_n = 0$ such that

$$\lim_{n\to\infty} I_1(w_{\delta_n}) = \lim_{n\to\infty} I_2(w_{\delta_n}) = 0,$$
$$\lim_{n\to\infty} I_3(w_{\delta_n}) = \frac{1}{2}\int_\Omega \psi(x)^2 \left(b, \partial_\xi^2 f\, b\right) dx.$$

where $(b, \partial_\xi^2 f b) := \sum_{jk} b_j b_k \partial^2_{\xi_j\xi_k} f(x,u(x),Du(x))$.

Consequence of the Claim. Since we know that $\sum_{l=1}^3 I_l(w_\delta) \geq 0\ \forall\delta$ and ψ is an arbitrary function this implies $\partial_\xi^2 f(x,u(x),Du(x)) \geq 0$ for a.e. $x\in\Omega$.

Proof of the Claim. We have

$$|I_1(w_\delta)| \leq \|w_\delta\|^2_{L^\infty}\|\partial_z^2 f\|_{L^1}, \quad |I_2(w_\delta)| \leq \|w_\delta\|_{L^\infty}\|Dw_\delta\|_{L^\infty}\|\partial^2_{z\xi} f\|_{L^1}.$$

Since $u, Du \in L^q(\Omega)$, $|\Omega|$ finite and $|D_z^2 f| + |D_\xi D_z f| + |D_\xi^2| \leq K(1+|z|^{q-2}+|\xi|^{q-2})$, we have $x\to\partial_z^2 f(x,u(x),Du(x)) \in L^1(\Omega)$. The same holds for the other second order derivatives.
Moreover $\|w_\delta\|_{L^\infty} \leq \delta\|\psi\|_{L^\infty}$, and

$$\partial_j w_\delta(x) = \psi(x)\cos\left(\tfrac{b\cdot x}{\delta}\right) + \delta\partial_j\psi(x)\sin\left(\tfrac{b\cdot x}{\delta}\right) \qquad \Rightarrow \|Dw_\delta\|_{L^\infty} \leq C_1, \tag{3.2.9}$$

for some constant $C_1 > 0$ independent of $0<\delta<1$. Hence $\lim_{n\to\infty} I_1(w_{\delta_n}) = \lim_{n\to\infty} I_2(w_{\delta_n}) = 0$.
Finally, to prove the last limit, let j,k be fixed indices, set

$$h_\delta(x) := \partial_j w_\delta \partial_k w_\delta, \qquad \varphi(x) := \partial^2_{\xi_j\xi_k} f(x,u(x),Du(x)).$$

Therefore we need to study $\int_\Omega h_\delta(x)\varphi(x)dx$, where $\varphi\in L^1(\Omega)$ and $h_\delta \in L^\infty(\Omega)\ \forall\delta>0$.
Let $(\delta_n)_{n\in\mathbb{N}}$ be some sequence with $\delta_n>0\ \forall n$ and $\lim_{n\to\infty}\delta_n = 0$. From (3.2.9) $\sup_n \|h_{\delta_n}\|_{L^\infty} \leq C_1^2$, therefore there exists a subsequence δ_{n_k} and a function $h\in L^\infty(\Omega)$ such that $h_{\delta_{n_k}} \rightharpoonup^* h$ in $L^\infty(\Omega)$, i.e. $\forall\varphi\in L^1(\Omega)$

$$\int_\Omega h_{\delta_{n_k}}(x)\varphi(x)dx \to \int_\Omega h(x)\varphi(x)dx.$$

It remains to check that the limit is indeed $h(x) = \frac{\psi(x)}{2} b_j b_k$. This follows from

$$h_\delta = \psi(x)^2 b_j b_k\left[\cos\left(\frac{b\cdot x}{\delta}\right)\right]^2 + O(\delta) = \frac{1}{2}\psi(x)^2 b_j b_k + \frac{1}{2}\psi(x)^2 b_j b_k\cos\left(\frac{2b\cdot x}{\delta}\right) + O(\delta).$$

□

Remark. Let $H\in\mathbb{R}^{d\times d}$ a matrix, $a,b\in\mathbb{R}^d$ two vectors. Using $(a\otimes b)_{ij} = a_i b_j$, we can write

$$(a, Hb) = \mathrm{tr}(H[a\otimes b]^t) = H\cdot(a\otimes b).$$

Then positivity can be reexpressed as

$$(b, Hb) = \mathrm{tr}(H[b\otimes b]^t) = H\cdot(b\otimes b) \geq 0.$$

Equations in second variation: vector case.

We consider $u:\Omega\to\mathbb{R}^n$, $n>1$, hence the arguments of $f(x,z,\xi)$ satisfy $z\in\mathbb{R}^n$, $\xi\in\mathbb{R}^{n\times d}$.
To simplify the notations in the second derivatives, we define, for any two finite sets I_1, I_2,

$$\mathbb{R}^{I_1\times I_2} := \{M : (I_1\times I_2)\to\mathbb{R}\} \tag{3.2.10}$$

the set of matrices indexed by I_1 and I_2. In the following we will use the two following sets

$$I := I(n), \qquad I' = I(n) \times I(d) \quad \text{where } I(n) = \{1, \ldots, n\},\ I(d) = \{1, \ldots, d\}.$$

With these notations we can write

$$\partial_z^2 f \in \mathbb{R}^{I \times I}, \quad \partial_z \partial_\xi f \in \mathbb{R}^{I \times I'}, \quad \partial_\xi^2 f \in \mathbb{R}^{I' \times I'},$$

and the second variation (3.2.7) can be written as

$$\int_\Omega \left[(D_z^2 f) \cdot [w \otimes w] + 2(D_z D_\xi f) \cdot [w \otimes Dw] + (D_\xi^2 f) \cdot [Dw \otimes Dw] \right] dx \ \geq 0, \tag{3.2.11}$$

where $w \in \mathbb{R}^n, Dw \in \mathbb{R}^{n \times d}$, hence

$$w \otimes w \in \mathbb{R}^{I \times I}, w \otimes Dw \in \mathbb{R}^{I \times I'}, Dw \otimes w \in \mathbb{R}^{I' \times I'}.$$

Theorem 18 *Let f be as in Lemma 39, with $n > 1$ and let $u \in W^{1,q}(\Omega; \mathbb{R}^n)$ be a minimizer w.r.t. its own b.c. Then*

$$\partial_\xi^2 f(x, u(x), Du(x)) \cdot [(a \otimes b) \otimes (a \otimes b)] \geq 0 \quad \forall a \in \mathbb{R}^n, b \in \mathbb{R}^d, \quad \text{for } a.e.\ x \in \Omega. \tag{3.2.12}$$

Proof Analog to the proof of Thm.17 we introduce the sequence

$$w_\delta(x) := a\delta\ \psi(x) \sin\left(\frac{b \cdot x}{\delta}\right), \quad 0 < \delta < 1,\ a \in \mathbb{R}^n,\ b \in \mathbb{R}^d,\ \psi \in C_c^\infty(\Omega; \mathbb{R}).$$

Now the derivative becomes

$$(Dw_\delta)_{ij}(x) = a_i b_j \psi(x) \cos\left(\frac{b \cdot x}{\delta}\right) + O(\delta) = (a \otimes b)_{ij} \psi(x) \cos\left(\frac{b \cdot x}{\delta}\right) + O(\delta)$$

and

$$Dw_\delta \otimes Dw_\delta = (a \otimes b) \otimes (a \otimes b)\ \psi(x)^2 \cos^2\left(\frac{b \cdot x}{\delta}\right) + O(\delta).$$

The proof then works as in Thm.17. □

[Lecture 22: 6.07]

3.2.6 Second variation and convexity

Definition 34 *Let $X = \mathbb{R}^d$, or $X = \mathbb{R}^{n \times d}$. A function $f : X \to \mathrm{R} \cup \infty$ is <u>convex</u> if*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y)$$

for all $x, y \in X, \lambda \in [0, 1]$.

Remark 1. f is convex iff the function

$$\begin{aligned} h_{xy} :\ & \mathbb{R} && \to \mathbb{R} \\ & t && \to h_{xy}(t) := f(x + ty) \end{aligned} \tag{3.2.13}$$

is convex (in t) for all $x, y \in X$.

Remark 2. Let $f \in C^2(X)$. Then f is convex iff $\partial_x^2 f(x) \geq 0$ as a quadratic form $\forall x \in X$, i.e.

$$\partial_x^2 f(x) \cdot (b \otimes b) \geq 0 \quad \forall b \in \mathbb{R}^d, \quad \text{if } X = \mathbb{R}^d, \tag{3.2.14}$$

$$\partial_x^2 f(x) \cdot (B \otimes B) \geq 0 \quad \forall B \in \mathbb{R}^{n \times d}, \quad \text{if } X = \mathbb{R}^{n \times d}. \tag{3.2.15}$$

Convexity and second variation

Lemma 40 *Let $n = 1$ and assume $f \in C^2(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^d)$. Let $u \in W^{1,q}(\Omega; \mathbb{R})$ such that $\xi \to f(x, u(x), \xi)$ is convex in ξ, for almost all x. Then u is a solution of the second variation equation* (3.2.11).

Proof. Convexity implies $\partial^2_\xi f(x, u(x), \xi) \cdot (b \otimes b) \geq 0 \quad \forall b \in \mathbb{R}^d$. This is true in particular for $\xi = Du$. □

Lemma 41 *Let $n > 1$ and assume $f \in C^2(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^d)$. Let $u \in W^{1,q}(\Omega; \mathbb{R}^n)$ such that $\xi \to f(x, u(x), \xi)$ is convex in ξ, for almost all x. Then u is a solution of the second variation equation* (3.2.12).

Proof. Convexity implies $\partial^2_\xi f(x, u(x), \xi) \cdot (B \otimes B) \geq 0 \quad \forall B \in \mathbb{R}^{n \times d}$. This is true in particular for $\xi = Du$. and $B = (a \otimes b)$, $a \in \mathbb{R}^n$, $b \in \mathbb{R}^d$. □
Note that in the vector case covexity is too strong a condition. We only need rank-1 convexity.

Definition 35 *A function $f : \mathrm{R}^{n \times d} \to \mathrm{R} \cup \infty$ is rank-1 convex if*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y)$$

for all $\lambda \in [0, 1]$, and all $x, y \in \mathrm{R}^{n \times d}$ such that rank$[x - y] = 1$.

Remark 3 **(i)** f is convex $\Rightarrow$ f is rank-1 convex.

(ii) f rank-1 convex iff
$f(y+\lambda(x-y)) \leq f(y)+\lambda[f(x)-f(y)]$ for all $\lambda \in [0, 1]$, and all $x, y \in \mathrm{R}^{n \times d}$ such that rank$[x-y] = 1$, iff
the function h_{xy} defined in (3.2.13) is convex in t for all $x \in \mathrm{R}^{n \times d}$ and $y \in \mathrm{R}^{n \times d}$ of the form $y = a \otimes b$, with $a \in \mathrm{R}^n, b \in \mathrm{R}^d$.

(iii) Let $f \in C^2(\mathrm{R}^{n \times d})$. Then f is rank-1 convex iff

$$\partial^2_x f(x) \cdot (B \otimes B) \geq 0 \quad \forall B = a \otimes b, \quad a \in \mathrm{R}^n, b \in \mathrm{R}^d. \tag{3.2.16}$$

Lemma 42 *Let $n > 1$ and assume $f \in C^2(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^d)$. Let $u \in W^{1,q}(\Omega; \mathbb{R}^n)$ such that $\xi \to f(x, u(x), \xi)$ is rank-1 convex in ξ, for almost all x. Then u is a solution of the second variation equation* (3.2.12).

Proof. Convexity implies $\partial^2_\xi f(x, u(x), \xi) \cdot ((a \otimes b) \otimes (a \otimes b)) \geq 0 \quad \forall a \in \mathrm{R}^n, b \in \mathrm{R}^d$. This is true in particular for $\xi = Du$. □

Example. The determinant is rank-1 convex but not convex. To show this we study the function

$$\begin{aligned} f: \quad & \mathbb{R} \to \mathbb{R} \\ & t \to h(t) := \det(M + tN) \end{aligned} \tag{3.2.17}$$

where $M, N \in \mathrm{R}^{n \times n}$. This function is a polynome of degree n in t, hence C^∞. Without loss of generality, we can consider only the second derivative at $t = 0$ $h''(0)$. We can also assume $\det M \neq 0$. Then $f(t) = \det(M + tN) = \det(M) \det(I + tM^{-1}N)$. Note that, if N is rank-1, then also $M^{-1}N$ is rank-1. Let us consider the function $h(t) := \det(I + tN)$. If t is small enough we have

$$0 < \det(I + tN) = e^{\ln \det(I+tN)} = e^{\mathrm{tr} \ln(I+tN)} = e^{g(t)},$$

where $\ln(I+tN) := \sum_n c_n (tN)^n$, and c_n are the coefficients in the Taylor expansion for the function $\ln(1+x)$. The relation $\ln \det(I+tN) = \mathrm{tr} \ln(I+tN)$ can be proved first for diagonalisable matrices, then extended by density to any matrix, such that the sum above is convergent. Since $h(0) = 1$, the second derivative reduces to

$$h''(0) = \left[g''(0) + (g'(0))^2\right].$$

Using

$$g'(t) = \operatorname{tr}\ln(I + tN)' = \operatorname{tr}(I + tN)^{-1}N, \qquad g''(t) = -\operatorname{tr}(I + tN)^{-1}N(I + tN)^{-1}N,$$

we finally obtain

$$h''(0) = \left[(\operatorname{tr}N)^2 - \operatorname{tr}N^2\right].$$

When N is rank-1 we have $N = a \otimes b = ab^t$ for two vectors $a, b \in \mathrm{R}^n$, hence $\operatorname{tr}N = \operatorname{tr}(ab^t) = (b, a)$, and $\operatorname{tr}N^2 = \operatorname{tr}(ab^tab^t) = (b, a)^2$, where $(.,.)$ is the euclidean scalar product. Therefore $f''(0) = h''(0) = 0$. This proves the determinant is rank-1 convex. To show the function is not convex take for example

$$n = 2, \quad M = I, \quad N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $\operatorname{tr}N = 0, \operatorname{tr}N^2 = 2$, hence $f''(0) < 0$.

3.3 Existence of a minimizer: direct method of calculus of variation

Problem. Let X be some space of functions $u : \Omega \to \mathbb{R}^d$ (typically $X = W^{1,q}(\Omega; \mathbb{R}^d)$), $I : X \to \mathbb{R} \cup \{\infty\}$ and $g : \partial\Omega \to \mathbb{R}$ some fixed boundary condition. The goal is to prove that there exists a function $u_* \in X$ such that $u_* = g$ on $\partial\Omega$ and $I(u) \geq I(u_*)$ for all $u \in X$ with $u = u_*$ on $\partial\Omega$.

3.3.1 Strategy

Let $X_g = \{u \in X |\ u = g \text{ on } \partial\Omega\}$, and let

$$I_g := \inf_{u \in X_g} I(u) \tag{3.3.18}$$

If the set X_g is empty we write $I_g = +\infty$.

Step 1: existence of a minimizing sequence. Our goal is to prove:

$$I_g \neq \pm\infty. \tag{3.3.19}$$

If this is true, then there exists a sequence $\{u_k\}_{k \in \mathbb{N}}$ of functions in X_g such that

$$\lim_{k \to \infty} I(u_k) = I_g. \tag{3.3.20}$$

Note that the sequence u_k <u>may not have a limit</u> in X.

Step 2: compactness. Our goal is to prove that there exists a subsequence u_{k_l} such that $u_{k_l} \xrightarrow{\tau} u_*$ is a suitable topology τ. Note that $I_g = \lim_l I(u_{k_l})$ <u>may not coincide with</u> $I(u_*)$ unless the function I is continuous.

Step 3: lower semicontinuity. Our goal is to prove that I is at least lower semicontinuous in the topology τ i.e. given any sequence $\{v_j\}_{j \in \mathbb{N}}$ in X_g and a function $v \in X$ it holds

$$v_j \xrightarrow{\tau} v \quad \Rightarrow \quad \liminf_{j \to \infty} I(v_j) \geq I(v) \tag{3.3.21}$$

Conclusion. Putting together the three steps above we obtain

$$I_g = \lim_{l\to\infty} I(u_{k_l}) = \liminf_{l\to\infty} I(u_{k_l}) \geq I(u_*).$$

Hence $I(u_*) = I_g$ by definition of I_g.

[Lecture 23: 13.07]

3.3.2 Some examples.

Example 1. Let $n = 1$, $X = W^{1,2}(\Omega)$. Let $g : \partial\Omega \to \mathbb{R}$. Assume $X_g \neq \emptyset$, i.e. there exists at least a function $\tilde{g} \in X$ such that $\tilde{g} = g$ on $\partial\Omega$, otherwise we cannot even start the argument. Note that, if the function g is too 'bad', the set X_g may indeed be empty. Therefore we can write

$$X_g = \{u \in X|\ u - \tilde{g} \in W_0^{1,2}(\Omega)\} = \tilde{g} + W_0^{1,2}(\Omega).$$

Let

$$I(u) := \int_\Omega |Du|^2 dx = \|Du\|^2_{L^2(\Omega)}.$$

Step 1. Since $I(u) \geq 0$ we have $I_g > -\infty$. Moreover for $u = \tilde{g}$ we have $I(\tilde{g}) \leq \|\tilde{g}\|_{W^{1,2}} < \infty$ hence $I_g \neq \pm\infty$ and there exists a sequence $\{u_k\}_{k\in\mathbb{N}}$ of functions in X_g such that $\lim_{k\to\infty} I(u_k) = I_g$.
Step 2. Let $v_j := u_j - \tilde{g}$. Since I_g is finite we have

$$\sup_j \|Dv_j\|_{L^2(\Omega)} \leq \|D\tilde{g}\|_{L^2(\Omega)} + \sup_j \|Du_j\|_{L^2(\Omega)} < \infty.$$

Moreover, since $v_j \in W_0^{1,2}(\Omega)$, we can apply Poincaré inequality $\|v_j\|_{L^2(\Omega)} \leq C\|Dv_j\|_{L^2(\Omega)}\ \forall j$. Therefore $\sup_j \|v_j\|_{W_0^{1,2}(\Omega)} < \infty$. Note that $W_0^{1,2}$ is reflexive, hence the unit closed ball is sequentially compact in weak topology. Therefore, since our sequence is bounded, there exists a subsequence v_{j_l} and a function $v_* \in W_0^{1,2}$ such that $v_{j_l} \rightharpoonup v_*$ and $Dv_{j_l} \rightharpoonup Dv_*$ in $L^2(\Omega)$, i.e.

$$\int_\Omega v_{j_l}(x)h(x)dx \to_{l\to\infty} \int_\Omega v_*(x)h(x)dx$$

for all $h \in L^2(\Omega)$ and the same holds for Dv_{j_l}. Finally, letting $u_* = v_* + \tilde{g}$, we conclude that $u_{j_l} \rightharpoonup u_*$ in X.
Step 3. The norm $\|\cdot\|_{L_2}$ is not continuous in the weak topology. We will show later that this function is nevertheless lower semicontinuous. Then we can conclude the argument.

Example 2. Let $n = 1$, $X = L^2(\Omega)$. Let $g : \partial\Omega \to \mathbb{R}$ and assume $X_g \neq \emptyset$, i.e. there exists a function $\tilde{g} \in X$ such that $\tilde{g} = g$ on $\partial\Omega$. Let $b \in L^2(\Omega)$ and

$$I(u) := \int_\Omega u(x)b(x)dx = (u, b)_{L^2(\Omega)}.$$

For any constant $K > 0$ we can find a $u \in X_g$ such that $I(u) < -K$. Then $I_g = -\infty$ and Step 1 fails.

Example 3. We define X and X_g as in Example 1. Let $b \in L^2(\Omega)$ and

$$I(u) := \int_\Omega [|Du|^2(x) - u(x)b(x)]dx = \|Du\|^2_{L^2(\Omega)} - (u, b)_{L^2(\Omega)}$$

Step 1. Taking $u = \tilde{g}$ we have

$$I(\tilde{g}) \leq \|D\tilde{g}\|^2_{L^2(\Omega)} + \|\tilde{g}\|_{L^2(\Omega)}\|b\|_{L^2(\Omega)} < \infty,$$

hence $I_g < \infty$. To prove a lower bound note that

$$I(u) \geq \|Du\|_{L^2(\Omega)}^2 - \|u\|_{L^2(\Omega)}\|b\|_{L^2(\Omega)}.$$

As in Example 1, using $v = u - \tilde{g}$, and Poincaré inequality we have

$$\|u\|_{L^2(\Omega)} \leq \|\tilde{g}\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} \leq \|\tilde{g}\|_{L^2(\Omega)} + C\|Dv\|_{L^2(\Omega)} \leq \|\tilde{g}\|_{L^2(\Omega)} + C\|D\tilde{g}\|_{L^2(\Omega)} + C\|Du\|_{L^2(\Omega)}.$$

Then there exist two constants $C_1, C_2 > 0$ (depending on b and $\tilde{g}$) such that

$$I(u) \geq \|Du\|_{L^2(\Omega)}^2 - C_1\|Du\|_{L^2(\Omega)} - C_2 = \|Du\|_{L^2(\Omega)}\left(\|Du\|_{L^2(\Omega)} - C_1\right) - C_2 > -K \quad (3.3.22)$$

for some constant $K > 0$ independent of u. Hence $I_g \neq \pm\infty$. Then there exists a sequence u_j in X_g such that $\lim_{j\to\infty} I(u_j) = I_g$.
Step 2. From $\sup_j I(u_k) < \infty$ and (3.3.22) above, we have $\sup_j \|Du_j\|_{L^2(\Omega)} < \infty$. Moreover, from $v_j = u_j - \tilde{g}$ and Poincaré inequality we have $\sup_j \|v_j\|_{W_0^{1,2}(\Omega)} < \infty$, hence there exists a subsequence v_{j_l} and a function $v_* \in W_0^{1,2}$ such that $v_{k_l} \rightharpoonup v_*$ weakly in $W_0^{1,2}$. Hence $u_{k_l} \rightharpoonup u_* = \tilde{g} + v_*$, weakly in $W^{1,2}$.
Step 3. The map $u \to (u, b)_{L^2}$ is continuous wrt the weak topology and $\|Du\|_{L^2}$ is lower semicontinuous. Hence $I(u)$ is lower semicontinuous.

3.3.3 Convex Lagrangians

The following theorem gives a set of conditions sufficient (but not necessary) to ensure lower semicontinuity.

Theorem 19 *Let $\Omega \subset \mathbb{R}^d$ open bounded and with Lipschitz boundary. Let the Lagrangian*

$$\begin{aligned} f: \quad & \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R} \\ & (x, z, \xi) \to f(x, z, \xi) \end{aligned}$$

satisfy

- *f continuous on $\Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times d}$,*
- *$f \geq 0$,*
- *$\forall (x, z)$ $\xi \to f(x, z, \xi)$ is convex and differentible,*
- *$x \to f(x, u(x), v(x)) \in L^1(\Omega)$ for all $u \in L^q(\Omega; \mathbb{R}^d)$, $v \in L^q(\Omega; \mathbb{R}^{n\times d})$, for some $q \geq 1$.*

Then $I(u) = \int_\Omega f(x, u(x), Du(x))dx$ is lower semicontinuous in the weak $W^{1,q}(\Omega; \mathbb{R}^d)$ topology i.e. for any function $u_* \in W^{1,q}(\Omega; \mathbb{R}^d)$, and any sequence $u_j \in W^{1,q}(\Omega; \mathbb{R}^d)$ such that $u_j \rightharpoonup u_*$ in $W^{1,q}(\Omega; \mathbb{R}^d)$ we have

$$\liminf_{j\to\infty} I(u_j) \geq I(u_*).$$

Proof. Is is enough to prove that for *any* weakly converging sequence $u_j \rightharpoonup u_*$ there is a subsequence u_{j_l} such that $\liminf_{l\to\infty} I(u_{j_l}) \geq I(u_*)$.
Indeed, assume this is true and $I_0 := \liminf_{j\to\infty} I(u_j) < I(u_*)$. Since lim inf is an accumulation point, there is a subsequence $\tilde{u}_l := u_{j_l}$ such that $\lim_l I(\tilde{u}_l) = I_0$. Then there exists a subsequence $\tilde{u}_{l_k}$ such that $I_0 = \lim_k \tilde{u}_{l_k} = \liminf_k \tilde{u}_{l_k} \geq I(u_*)$. But this is in contradiction with $I_0 > I(u_*)$.
The proof of the theorem is then a consequence of Theorem 20 below. Indeed, inserting the subsequence u_{j_h} and the sets E_k from that theorem, we have

$$\begin{aligned} \liminf_h \int_\Omega f(x, u_{j_h}(x), Du_{j_h}(x))\, dx &\geq \liminf_h \int_{E_k} f(x, u_{j_h}(x), Du_{j_h}(x))\, dx \\ &\geq \int_{E_k} f(x, u_*(x), Du_*(x))\, dx \quad \forall k. \end{aligned}$$

Then

$$\liminf_h \int_\Omega f(x, u_{j_h}(x), Du_{j_h}(x))\ dx \geq \sup_k \int_{E_k} f(x, u_*(x), Du_*(x))\ dx = \int_\Omega f(x, u_*(x), Du_*(x))\ dx.$$

This concludes the proof of Theorem 19. □

Theorem 20 *Under the same assumptions as in Theorem 19, let u_j be a sequence in $W^{1,q}(\Omega;\mathbb{R}^d)$ such that $u_j \rightharpoonup u_*$ in $W^{1,q}(\Omega;\mathbb{R}^d)$. Then there exists a subsequence u_{j_h} and a family of sets $E_k \subset \Omega$, $k \in \mathbb{N}$ such that*

(a) $E_k \subseteq E_{k+1}\ \forall k$,

(b) $\cup_{k\in\mathbb{N}}\ E_k = \Omega$, *up to a set of measure zero,*

(c) $\liminf_h \int_{E_k} f(x, u_{j_h}(x), Du_{j_h}(x))\ dx\ \ \geq \int_{E_k} f(x, u_*(x), Du_*(x))\ dx$ *for all* $k \in \mathbb{N}$.

Proof. We distinguish two cases: f independent of z and general f.

Case 1. Assume f depends on (x,ξ) only. Since f is convex and differentiable in ξ we have

$$f(x,\xi_1) \geq f(x,\xi_0) + (\xi_1 - \xi_0)\cdot \partial_\xi f(x,\xi_0)$$

for all $x \in \Omega$, $\xi_0, \xi_1 \in \mathbb{R}^{n\times d}$, where $\cdot$ above indicates the matrix scalar product $M\cdot N = \mathrm{tr} MN^t$. In the following we write $v_j(x) := Du_j(x)$ and $v_* := Du_*$. Then, using the inequality above, we have

$$\int_{\Omega'} f(x, v_j(x))dx \geq \int_{\Omega'} f(x, v_*(x))dx + \int_{\Omega'} (v_j(x) - v_*(x))\cdot \partial_\xi f(x, v_*(x))dx$$

for any $\Omega' \subset \Omega$ <u>such that</u> $x \to (v_j(x) - v_*(x))\cdot\partial_\xi f(x, v_*(x)) \in L^1(\Omega')$. To construct such a subset note that $v_j - v_* \in L^q(\Omega;\mathbb{R}^{n\times d})$. It is then enough to find Ω' such that $\partial_\xi f(\cdot, v_*(\cdot)) \in L^p(\Omega';\mathbb{R}^{n\times d})$, with $\frac{1}{p} + \frac{1}{q} = 1$. Now, let $M > 0$ and define

$$\Omega_M := \{x \in \Omega |\, |\partial_\xi f(x, v_*(x))| \leq M\}, \tag{3.3.23}$$

where $|\cdot|$ denotes the matrix norm. Since $\partial_\xi f(\cdot, v_*(\cdot))$ is bounded on Ω_M, it belongs to $L^p(\Omega_M;\mathbb{R}^{n\times d})$. Moreover, since $v_j \rightharpoonup v_*$ weakly in $L^q(\Omega_M;\mathbb{R}^{n\times d})$, we have

$$\int_\Omega (v_j(x) - v_*(x))\cdot\varphi(x)dx \to_{j\to\infty} 0 \qquad \forall\varphi \in L^p(\Omega;\mathbb{R}^{n\times d}).$$

Applying this to $\varphi(x) = \mathbf{1}_{\Omega_M}(x)\partial_\xi f(x, v_*(x))$ we get

$$\liminf_j \int_{\Omega_M} (v_j(x) - v_*(x))\cdot\partial_\xi f(x, v_*(x))dx = \lim_j \int_{\Omega_M} (v_j(x) - v_*(x))\cdot\partial_\xi f(x, v_*(x))dx = 0.$$

Therefore

$$\begin{aligned}\liminf_j \int_{\Omega_M} f(x, v_j(x))dx &\geq \int_{\Omega_M} f(x, v_*(x))dx + \liminf_j \int_{\Omega_M} (v_j(x) - v_*(x))\cdot\partial_\xi f(x, v_*(x))dx \\ &= \int_{\Omega_M} f(x, v_*(x))dx,\end{aligned}$$

where we used $\liminf_j(a_j + b_j) \geq \liminf_j a_j + \liminf_j b_j$. Now, replacing M by a sequence M_k, with $M_{k+1} > M_k\ \forall k$ and $\lim_{k\to\infty} M_k = \infty$, the proof of Case 1 follows.

[Lecture 24: 15.07]

Case 2. We consider the general case, when f depends on x, z and ξ. We want to use Case 1. As above let $v_j := Du_j$ and $v_* := Du_*$. For any $\Omega' \subseteq \Omega$ we have

$$\int_{\Omega'} f(x, u_j(x), v_j(x))dx = \int_{\Omega'} f(x, u_*(x), v_j(x))dx + \int_{\Omega'} \left[f(x, u_j(x), v_j(x)) - f(x, u_*(x), v_j(x))\right] dx.$$

The first integral can be treated as Case 1 with $\tilde{f}(x, \xi) := f(x, u_*(x), \xi)$, hence

$$\liminf_j \int_{\Omega'} f(x, u_*(x), v_j(x))dx \geq \int_{\Omega'} f(x, u_*(x), v_*(x))dx \qquad \forall \Omega' \subseteq \Omega.$$

Our goal is to find a subsequence u_{j_h}, v_{j_h} and a sequence of subsets E_k such that the second integral converges to zero. The idea is to first get rid of the v_j dependence. For this purpose, analog to (3.3.23) let $M > 0$ and

$$A_{j,M} := \{x \in \Omega | \, |v_j(x)| \leq M\}. \tag{3.3.24}$$

Moreover let

$$g(x, j, M) := \sup_{|\xi| \leq M} |f(x, u_j(x), \xi) - f(x, u_*(x), \xi)| \,. \tag{3.3.25}$$

Then, for all $M > 0$ and $j \in \mathbb{N}$ we have

$$\left| \int_{A_{j,M}} \left[f(x, u_j(x), v_j(x)) - f(x, u_*(x), v_j(x))\right] dx \right| \leq \int_{A_{j,M}} g(x, j, M)dx.$$

We will use the next two claims (proved later).

<u>Claim 1</u>: there exists a constant $C > 0$ idependent of M and j such that $Vol(A_{j,M}) \leq \frac{C}{M} \; \forall j, M$.

<u>Claim 2</u>: for all $M > 0$ fixed we have

$$\lim_{j \to \infty} g(x, j, M) = 0 \qquad \text{for a.e.} \qquad x \in \Omega.$$

From the assumptions of the theorem $g(\cdot, j, M) \in L^1(\Omega)$, and from Claim 2 $g(\cdot, j, M)$ converges pointwise to zero for a.e. $x \in \Omega$. Hence $g(\cdot, j, M) \to 0$ in mass i.e.

$$Vol\{x \in \Omega| \; |g(x, j, M)| > \epsilon\} \to_{j \to \infty} 0 \qquad \forall \epsilon > 0.$$

More precisely, $\forall \epsilon > 0$ and $\forall \delta > 0$ $\exists j_{\epsilon,\delta} \in \mathbb{N}$ such that

$$Vol\{x \in \Omega| \; |g(x, j, M)| > \epsilon\} \leq \delta \qquad \forall j \geq j_{\epsilon,\delta}.$$

Here we set $h \in \mathbb{N}$, and we define

$$M = 2^h, \quad \epsilon = \frac{1}{h}, \quad \delta = 2^{-h}, \quad j_{\epsilon.\delta} := j_h.$$

Then

$$Vol\left\{x \in \Omega| \; |g(x, j, 2^h)| > \tfrac{1}{h}\right\} \leq 2^{-h} \qquad \forall j \geq j_h. \tag{3.3.26}$$

We are now ready to define the subsequence and the sets E_k.
Subsequence. We take the subsequence (u_{j_h}, v_{j_h}) where j_h is defined above.
Sets. We first introduce the sets A_h and B_h defined by

$$A_h := \{x \in \Omega| \, |v_{j_h}(x)| \leq 2^h\} = A_{j_h, 2^h}, \qquad B_h := \{x \in \Omega| \, |g(x, j_h, 2^h)| \leq \tfrac{1}{h}\}. \tag{3.3.27}$$

Finally we define

$$E_k := \cap_{h\geq k}(A_h \cap B_h). \tag{3.3.28}$$

This definition clearly satisfies *(a)* in Thm 20. To prove *(b)* note that

$$Vol(E_k^c) = Vol(\cup_{h>k}(A_h)^c \cup B_h^c) \leq \sum_{h\geq k} \left[Vol(A_h^c) + Vol(B_k^c)\right].$$

By Claim 1 and (3.3.26) we have

$$Vol(A_h^c) \leq C\ 2^{-h}, \quad Vol(B_h^c) \leq 2^{-h}.$$

Hence $Vol(E_k^c) \leq C' \sum_{h\geq k} 2^{-h} \to_{k\to\infty} 0$. Then $\cup_k E_k = \Omega$ up to a set of measure zero.
Finally, to prove *(c)* note that

$$\int_{E_k} g(x, j_h, 2^h)dx \leq \frac{Vol(E_k)}{h} \leq \frac{Vol(\Omega)}{h} \to_{h\to\infty} 0.$$

Therefore

$$\begin{aligned}\liminf_h \int_{E_k} f(x, u_{j_h}(x), v_{j_h}(x))dx &\geq \liminf_h \int_{E_k} f(x, u_*(x), v_{j_h}(x))dx \\ &\quad + \liminf_h \int_{E_k} \left[f(x, u_{j_h}(x), v_{j_h}(x)) - f(x, u_*(x), v_{j_h}(x))\right] dx \\ &\geq \int_{E_k} f(x, u_*(x), v_*(x))dx,\end{aligned}$$

where we used

$$\lim_{h\to\infty} \left| \int_{E_k} \left[f(x, u_{j_h}(x), v_{j_h}(x)) - f(x, u_*(x), v_{j_h}(x))\right] dx \right| \leq \int_{E_k} g(x, j_h, 2^h)dx = 0.$$

To conclude the proof of the theorem it remains to prove the two claims.

Proof of Claim 1. $v_j \rightharpoonup v_*$ weakly in $L^q(\Omega; \mathbb{R}^{n\times d}) \Rightarrow \sup_j \|v_j\|_{L^q} < \infty$, $\Rightarrow$ (since $q \geq 1$ and $Vol(\Omega) < \infty$) $S = \sup_j \|v_j\|_{L^1} < \infty$. Then $\forall j$ we have

$$S \geq \|v_j\|_{L^1} = \int_\Omega |v_j(x)|dx \geq \int_{A_{j,M}^c} |v_j(x)|dx \geq\ M\ Vol(A_{j,M}^c).$$

Hence the result.

Proof of Claim 2. Remember that if $\Omega \subset \mathbb{R}^d$ is open, bounded and with Lipschitz boundary, then the identity map $Id : W^{1,q}(\Omega) \to L^q(\Omega)$ is compact for $q \geq 1$. This means that for any bounded sequence in $W^{1,q}(\Omega)$ there exists a subsequence strongly convergent in $L^q(\Omega)$. Here $u_j \rightharpoonup u_*$ in $W^{1,q}(\Omega; \mathbb{R}^n)$ hence $\sup_j \|u_j\| < \infty$ hence there exists a subsequence converging strongly in $L^q(\Omega; \mathbb{R}^n)$. Finally, this implies there exists another subsequence (of the subsequence) converging pointwise a.e. in Ω. We restrict to this subsequence. Then for a.e. $x \in \Omega$ we have $u_j(x) \to u_*(x)$. The result is now a direct consequence of Lemma 43 below. This concludes the proof of Claim 2, hence of the theorem. □

Lemma 43 *Let $M > 0$, $z_j \to z_* \in \mathbb{R}^n$ and f continuous on $\Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times d}$. Then*

$$\lim_{j\to\infty} \sup_{|\xi\leq M|} |f(x, z_j, \xi) - f(x, z_*, \xi)| = 0. \tag{3.3.29}$$

Proof. By contradiction. Suppose the limit is different from zero. Then there exists a sequence ξ_j and $\delta > 0$ such that

$$|f(x, z_j, \xi_j) - f(x, z_*, \xi_j)| \geq \delta > 0 \quad \forall j.$$

Since the closed ball $\bar{B}_M = \{\xi |\ |\xi| \leq M\}$ is compact and $\xi_j \in \bar{B}_M$, there exists a subsequence ξ_{j_k} converging to some point ξ_* in $\bar{B}_M$. Then $(x, z_{j_k}, \xi_{j_k}) \to (x, z_*, \xi_*)$. Since f is continuous this implies

$$\lim_{k\to\infty} |f(x, z_{j_k}, \xi_{j_k}) - f(x, z_*, \xi_{j_k})| = f(x, z_*, \xi_*) - f(x, z_*, \xi_*) = 0.$$

Hence the contradiction. □