Partial Differential Equations and Modelling

Summer Term 2018
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Exercise Sheet 6

Due: June 6, 2018, in class.

Exercise 1 (Gagliardo-Nirenberg Interpolation) Assume that $p, q, r \in [2, \infty]$, $p < \infty$. Let $k, l \in \mathbb{N}$ with $0 < k < l$ and

$$\frac{1}{p} = \frac{l-k}{l}\frac{1}{q} + \frac{k}{l}\frac{1}{r}.$$

Show that there is a constant $C > 0$ depending on p, q, r, l, k, n such that

$$\|D^k u\|_{L^p(\mathbb{R}^n)} \leq C\|u\|_{L^q(\mathbb{R}^n)}^{\frac{l-k}{l}} \|D^l u\|_{L^r(\mathbb{R}^n)}^{\frac{k}{l}}$$

for all $u \in L^q(\mathbb{R}^n)$ with $D^l u \in L^r(\mathbb{R}^n)$. Proceed in the following steps:

(a) First consider $k = 1$ and $l = 2$, i.e., show that

$$\|Du\|_{L^p(\mathbb{R}^n)} \leq C\|u\|_{L^q(\mathbb{R}^n)}^{\frac{1}{2}} \|D^2 u\|_{L^r(\mathbb{R}^n)}^{\frac{1}{2}}$$

if $p^{-1} = (2q)^{-1} + (2r)^{-1}$.

Hint: By a density argument it is sufficient to consider $u \in C_c^\infty(\mathbb{R}^n)$. Rewrite the integral on the right hand side as $|Du|^q = Du \cdot (|Du|^{q-2} Du)$ and use integration by parts. Then use the generalized Hölder inequality.

(b) Use induction to derive the general statement.

Hint: For example you can first show by induction in k that

$$\|D^k u\|_{L^p(\mathbb{R}^n)} \leq C\|u\|_{L^q(\mathbb{R}^n)}^{\frac{1}{k+1}} \|D^{k+1} u\|_{L^r(\mathbb{R}^n)}^{\frac{k}{k+1}}$$

if $p^{-1} = ((k+1)q)^{-1} + k((k+1)r)^{-1}$ and then use induction in $l - k$.

(3+3 Punkte)

Exercise 2

(a) Show that for all $m \in \mathbb{N}_0$ there exists $c > 0$ such that for all $u, v \in L^\infty(\mathbb{R}^n) \cap H^m(\mathbb{R}^n)$ (recall $H^m(\Omega) := W^{m,2}(\Omega)$)

$$\|uv\|_{H^m} \leq c\left(\|u\|_{L^\infty}\|D^m v\|_{L^2} + \|D^m u\|_{L^2}\|v\|_{L^\infty}\right).$$

Hint: Use the product rule and then apply Exercise 1.

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(b) Show that for all $m \in \mathbb{N}_0$ there exists $c > 0$ such that for all $u \in H^m(\mathbb{R}^n)$ with $Du \in L^\infty(\mathbb{R}^n)$ and $v \in L^\infty(\mathbb{R}^n) \cap H^{m-1}(\mathbb{R}^n)$

$$\sum_{0 \le |\alpha| \le m} \|D^\alpha(uv) - uD^\alpha v\|_{L^2} \le c\left(\|Du\|_{L^\infty}\|D^{m-1}v\|_{L^2} + \|D^m u\|_{L^2}\|v\|_{L^\infty}\right)$$

(c) Show that for $m > n/2$ there is a $c > 0$ such that for all $u, v \in H^m(\mathbb{R}^n)$

$$\|uv\|_{H^m} \le c\|u\|_{H^m}\|v\|_{H^m}.$$

In particular $H^m(\mathbb{R}^n)$ is a Banach algebra for $m > n/2$.

(4+2+1 Punkte)

Exercise 3 (Elliptical Vortices I) In this and the following exercise we want to find a solution of the 2-dimensional Euler equation where the initial vorticity is given by

$$\omega(x, 0) = \begin{cases} \omega_0 & x \in E(a, b) \\ 0 & x \notin E(a, b) \end{cases} \tag{1}$$

where $E(a, b) = \{x \in \mathbb{R}^2 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1\}$, denotes an ellipse with semi-major axes a and semi-minor axes b.

Note that $\omega(\cdot, 0)$ is not continuous so we are actually looking for weak solutions of the Euler equation that will be discussed later in the course. However, we have seen that classical solutions of the 2-dimensional Euler equation just transport the vorticity along the velocity field, i.e., $\omega(t, \varphi_t(x)) = \omega(0, x)$ where $\partial_t \varphi_t(x) = v(t, \varphi_t(x))$. In particular, when $\omega(\cdot, 0) = \mathbb{1}_A$ is the indicator function of a bounded set A (we call this a vortex patch) we find $\omega(x, t) = \mathbb{1}_{\varphi_t(A)}$. In this exercise we find a vorticity ω and a corresponding velocity field v that has this property. It can be shown that this pair (ω, v) in fact is a weak solution of the Euler equation in vorticity form.

The solution can be found in two steps: The first step is to calculate the velocity field $v(x, 0)$ given that $\omega(x, 0)$ is as in (1). Then, in a second step, we use the velocity field to transport the vorticity which yields a solution of the Euler equation with initial value as in (1).

In this exercise we will do the second step. The first step will be done in the following exercise. The main result of the next exercise is that the velocity field $v(x, 0)$ is Lipschitz continuous and linear inside $E(a, b)$:

$$v(x) = U(a, b)x \quad \text{for } x \in E(a, b) \text{ with } U(a, b) = \frac{\omega_0}{a + b}\begin{pmatrix} 0 & -a \\ b & 0 \end{pmatrix}.$$

(a) Find the velocity field $v(x) = U(a, b, \theta)x$ inside the ellipse $R(\theta)E(a, b)$ if $\omega = \omega_0 \mathbb{1}_{R(\theta)E(a,b)}$ where $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ denotes the rotation by θ. It is sufficient to state the result as a product of 3 matrices.

Let us introduce the notation $E(a, b, \theta) = R(\theta)E(a, b)$ for the rotated ellipse. Since the velocity field is linear inside $E(a, b, \theta)$ when the vorticity is given by $\omega(x) = \omega_0 \mathbb{1}_{E(a,b,\theta)}(x)$ the shape of the vortex patch remains an ellipse (linear maps map ellipses to ellipses).

(b) Since the shape of the ellipse is conserved we find

$$\omega(\cdot, t) = \omega_0 \mathbb{1}_{\varphi_t(E(a,b))} = \omega_0 \mathbb{1}_{E(a(t),b(t),\theta(t))}$$

for some functions $a(t)$, $b(t)$, and $\theta(t)$ with $a(0) = a$, $b(0) = b$, $\theta(0) = 0$. Let $\boldsymbol{x}(t)$ be an integral curve of the velocity field, i.e., $\dot{\boldsymbol{x}}(t) = v(\boldsymbol{x}(t), t)$ with $\boldsymbol{x}(0) \in \partial E(a, b)$ (they exist since v is Lipschitz in x). Then $\boldsymbol{x}(t) \in \partial E(a(t), b(t), \theta(t))$. Show that this is equivalent to

$$\boldsymbol{x}(t) \cdot A(a(t), b(t), \theta(t))\boldsymbol{x}(t) = 1 \tag{2}$$

where $A(a, b, \theta) = R(\theta) \begin{pmatrix} a^{-2} & 0 \\ 0 & b^{-2} \end{pmatrix} R(-\theta)$.

(c) Differentiate (2) and find differential equations for $a(t)$, $b(t)$, and $\theta(t)$. What is the time evolution of the vortex patch?

Hint: You should obtain the equation $U^\top A + \dot{A} + AU = 0$. It is helpful to multiply this equation from left and right by $R(-\theta)$ and $R(\theta)$.

(2+2+3 Punkte)

Exercise 4 (Elliptical Vorticies II) In this exercise we calculate the velocity field v for ω as in (1). Recall that in the 2-dimensional case the velocity v can be recovered from the vorticity ω via $v^\perp = \nabla\psi$ where ψ solves $-\Delta\psi = \omega$.

This exercise is sligthly longer than usual. If you need help you can look at *Migual Furman, Compact Complex Expressions for the Electric Field of 2-D Elliptical Charge Distribution. Note that in the electrostatic setting the electric field $\boldsymbol{E}$ correponds to $\nabla\psi$.*

(a*) *Find an expression for $v(x)$ for a given vorticity $\omega \in L^\infty(\mathbb{R}^2)$ with compact support.*

Hint: Use the fundamental solution of the Laplace equation and then differentiate under the integral sign.

(b*) *To calculate the velocity for $\omega = \omega_0 \mathbb{1}_{E(a,b)}$ it is helpful to identify $\mathbb{R}^2$ with $\mathbb{C}$. Use the result from* **(a*)** *to express $\nabla\psi(z)$ through $\int_{E(a,b)} \frac{1}{z-z'} \mathrm{d}z'$.*

(c*) *Now we want to evaluate the integral*

$$\int_{E(a,b)} \frac{1}{z - z'} \mathrm{d}z'. \tag{3}$$

Use the change of variable $z' = x + iy = ar\cos\varphi + ibr\cos\varphi$ with $\varphi \in (0, 2\pi)$ and $r \in (0, 1)$. Then substitute $\zeta = e^{i\varphi}$ to rewrite the φ integral as a contour integral over $|\zeta| = 1$. The contour integral should have the form

$$\oint_{|\zeta|=1} \frac{1}{\zeta^2 + A\zeta + B} \mathrm{d}\zeta.$$

(d*) *To evaluate the contour integral in* **(c*)** *we first restrict to $r = 1$. Then the integral should simplify to*

$$\oint_{|\zeta|=1} \frac{1}{\zeta^2 - \frac{2z}{a+b}\zeta + \frac{a-b}{a+b}} \mathrm{d}\zeta. \tag{4}$$

To calculate the integral we have to analyse the positions of the poles

$$\zeta_\pm(z) = \frac{z \pm \sqrt{z^2 - a^2 + b^2}}{a+b}.$$

Show that if $z \in E(a,b)$ both poles lie in the unit circle while only one pole lies within the unit circle for $z \notin E(a,b)$.

Hint: Define the square root such that $z \to \sqrt{z^2 - a^2 + b^2}$ is a holomorphic function on the set $\mathbb{C} \setminus [-\sqrt{a^2-b^2}, \sqrt{a^2-b^2}]$ and agrees with the positive square root for $z \in \mathbb{R}$ with $z > \sqrt{a^2-b^2}$. Prove that for any $w \in \mathbb{C}$ there is at most one $z \in \mathbb{C}$ such that $\zeta_+(z) = w$ or $\zeta_-(z) = w$. Show that ζ_+ maps the boundary of the ellipse $E(a,b)$ to the unit circle. For this you might use that $\partial E(a,b) = \{a \cos\varphi + ib \sin\varphi\}$.

(e*) *Use the residue theorem to evaluate the integral (4).*

Hint: You should find that the integral vanishes if $z \in E(a,b)$.

(f*) *Generalize the previous results to $r \in (0,1)$ and calculate the integral in (3).*

Hint: You should find an integral of the form $\frac{r}{\sqrt{A - Br^2}}$. Be careful with the upper limit in the r integral. In **(e*)** we have seen that the φ integral vanishes if $z \notin E(ra, rb)$.

(g*) *Use the results of* **(b*)** *and* **(f*)** *to give an expression for $v(x)$ in the case that $x \in E(a,b)$.*

Hint: You should find that $v(x) = \frac{\omega_0}{a+b} \begin{pmatrix} -ax_2 \\ bx_1 \end{pmatrix}$. In particular the velocity is linear in x within the ellipse.

It is also possible to give an expression for v outside the ellipse and show that v is Lipschitz continuous (cf. the paper mentioned above).

(1* +2* +2* +3* +2* +2* +2* Punkte)

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