S4F5 - Graduate Seminar on Interacting Random Systems: Topics in Mean Field Spin Glasses

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University of Bonn, Fall 2023

Time: Tuesdays, 12.00 PM - 2.00 PM (tentatively)
Place: N 0.007

Content: Spin glasses are models originally introduced in statistical physics as simplified models for magnetic alloys, but have soon been realized to have interesting connetions to other research fields like e.g. theoretical computer and data science.

In this seminar, our goal is to understand various mathematical tools to study the limiting behavior of basic spin glass models such as the Sherrington-Kirkpatrick (SK) and perceptron models. The SK model is the paradigmatic example of a mean field spin glass. It describes a disordered spin system whose energies are determined by the Hamiltonian $H_N : \{-1,1\}^N \to \mathbb{R}$, given by

$$H_N(\sigma) = \sum_{1\leq i<j\leq N} \beta g_{ij}\sigma_i\sigma_j + \sum_{i=1}^N h_i\sigma_i.$$

The interaction couplings $\{g_{ij}\}$ are i.i.d. centered Gaussian random variables of variance N^{-1}, $\beta \geq 0$ tunes the interaction strength and $\mathbf{h} = (h_1,\dots,h_N)$ represents an external magnetic field. The Hamiltonian induces a (random) probability measure on the space of spin configurations $\{-1,1\}^N$, the Gibbs measure, whose expectation $\langle\cdot\rangle$ is defined by

$$\langle f\rangle = \frac{1}{Z_N}\sum_{\sigma\in\{-1,1\}^N} f(\sigma)e^{H_N(\sigma)}, \qquad Z_N = \sum_{\sigma\in\{-1,1\}^N} e^{H_N(\sigma)}.$$

According to the rules of statistical mechanics, the Gibbs measure determines the behavior of the system in thermal equilibrium and the goal in statistical mechanics is to understand the Gibbs measure qualitatively in the limit $N\to\infty$ of large particle number (or infinite volume), depending on the external parameters β and h. In particular, one would like to understand if the system undergoes a phase transition between qualitatively different macroscopic states and how to describe the different phases.

An important quantity in this context is the log-partition function $\log Z_N$ which determines basic observables like the magnetization vector

$$\mathbf{m} = (m_1,\dots,m_N) \equiv (\langle\sigma_1\rangle,\dots,\langle\sigma_N\rangle) = \nabla_{\mathbf{h}} \log Z_N \in \mathbb{R}^N$$

or the susceptibility matrix

$$\mathbf{M} \equiv \big(\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle\big)_{1\le i,j\le N} = \big(\partial_{h_i}\partial_{h_j}\log Z_N\big)_{1\le i,j\le N} \in \mathbb{R}^{N\times N}.$$

In the first part of the seminar, we introduce basic mathematical tools to verify simple predictions on the above objects in regimes which, loosely speaking, correspond to weak interactions. Discussing in particular the so called TAP approach, we will show that the magnetization vector $\mathbf{m}$ satisfies a mean field type equation of the form

$$m_i \approx \tanh\Big(h_i + \sum_{j\neq i} g_{ij}m_j - \beta^2(1-q)m_i\Big),$$

at least for small β. Similar identities can be derived for $\mathbf{M}$ and combined they imply

$$\frac{1}{N}\log Z_N \approx \log 2 + E\,\log\cosh(h+\beta\sqrt{q}Z) + \frac{\beta^2}{4}(1-q)^2.$$

Here, $q\in[0,1]$ denotes the unique solution of the equation $q = E\,\tanh^2(h+\beta\sqrt{q}Z)$, assuming for simplicity that $h_i\equiv h$ for some $h>0$ and for all $i\in[N]$.

The second part of the seminar deals with the application of the tools and ideas discussed in the first part to derive basic properties of the perceptron model, a model that describes a simple binary machine learning classifier. Here, one tries to understand the typical size of

$$\Big|\ \{-1,1\}^N \cap \bigcap_{k=0}^{\alpha N}\big\{\mathbf{x}\in\mathbb{R}^N : (\mathbf{x},\mathbf{g}_k)\ge 0\big\}\Big|,$$

where the vectors $\mathbf{g}_k$ are independent Gaussian random vectors. In other words, the question is to find the typical number of configurations that are contained in a large number αN (for $\alpha>0$) of half-spaces in $\mathbb{R}^N$. Surprisingly, this problem is very much related to the theory of mean field spin glasses if we observe that

$$\Big|\ \{-1,1\}^N \cap \bigcap_{k=0}^{\alpha N}\big\{\mathbf{x}\in\mathbb{R}^N : (\mathbf{g}_k,\mathbf{x})\ge 0\big\}\Big| = \lim_{\beta\to\infty}\sum_{\sigma\in\{-1,1\}}\exp\Big[-\beta\sum_{k=0}^{\alpha N}\mathbf{1}_{\big\{(\mathbf{x},\mathbf{g}_k)<0\big\}}(\sigma)\Big].$$

Prerequisites: The course deals with the analysis of disordered spin systems. A first course in measure theoretic probability should suffice as background; basic knowledge of stochastic processes is helpful, but not necessary. The seminar is self-contained, in particular no physics background is required (moreover, the discussion of background material can be integrated into the seminar depending on the needs and interests of the audience). For more on mathematical background, see the references below and for more on the physics and applications side, interested readers may look at e.g.

- *Spin Glass Theory and Beyond* by M. Mézard, G. Parisi and A. Virasoro
- *Information, Physics and Computation* by M. Mézard and A. Montanari (see `https://web.stanford.edu/~montanar/RESEARCH/book.html`)

Organization: Interested students sign up tentatively with their email address during the presentation meeting on July 5. About a month before the beginning of the fall semester, the students will receive via email a list of possible topics and will be asked for a definite confirmation of their participation.

References

[1] A. Adhikari, C. Brennecke, P. von Soosten, H.-T. Yau. Dynamical Approach to the TAP Equations for the Sherrington-Kirkpatrick Model. *J. Stat. Phys.* **183**, 35 (2021).

[2] E. Bolthausen. An Iterative Construction of Solutions of the TAP Equations for the Sherrington–Kirkpatrick Model. *Comm. Math. Phys.* **325** (2014), pp. 333-366.

[3] E. Bolthausen. A Morita Type Proof of the replica-symmetric Formula for SK. In: *Statistical Mechanics of Classical and Disordered Systems* (2018), Springer Proceedings in Mathematics & Statistics, pp. 63-93.

[4] E. Bolthausen, S. Nakajima, N. Sun, C. Xu. Gardner Formula for Ising Perceptron Models at Small Densities. Preprint: arXiv:2111.02855.

[5] C. Brennecke, H.-T. Yau. The Replica Symmetric Formula for the SK Model Revisited. *J. Math. Phys.* **63**, 073302 (2022).

[6] J. Ding, N. Sun. Capacity lower bound for the Ising perceptron. Preprint: arXiv:1809.07742.

[7] M. Talagrand. Mean Field Models for Spin Glasses. Volume I: Basic Examples. A Series of Modern Surveys in Mathematics, **Vol. 54** (2011), *Springer Verlag Berlin–Heidelberg.*

[8] M. Talagrand. Mean Field Models for Spin Glasses. Volume II: Advanced Replica-Symmetry and Low Temperature. A Series of Modern Surveys in Mathematics, **Vol. 54** (2011), *Springer Verlag Berlin–Heidelberg.*