Introduction to Constructive Quantum Field Theory

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Abstract

In these notes, we discuss basic mathematical results in constructive quantum field theory. We start with a short motivation and recall basic ideas from classical mechanics, quantum mechanics and quantum field theory, concluding with a precise notion of a quantum field theory based on Wightman's axioms. In the first main part we then focus on the explicit construction of several important free field theories (scalar, vector and Dirac spinor fields) which presupposes a detailed discussion of distributions and basic aspects of the representation theory of the Lorentz and Poincaré groups. In the second part we discuss several topics related to general properties of quantum field theories and to the construction of interacting theories. This includes basic results on spin and statistics, locality, the reconstruction from Wightman functions, aspects of the interaction picture and Haag's theorem as well as the Euclidean approach. The last part of the notes concludes with basic, recent results on non-Abelian (lattice Yang-Mills) gauge theories.

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1 Introduction

In this section, we introduce basic notions and discuss standard examples from classical and quantum physics. The main purpose of this section is to motivate a certain notion of a quantum field theory which is based on the so called Wightman axioms. This is the central notion of a quantum field discussed in these notes. The main references on which these notes are based are [4, 22, 21], which contain extensive reference lists to background, advanced and the original research literature. For detailed physics background on classical mechanics, special relativity, quantum mechanics and quantum field theory, we refer the interested reader to [7, 8, 9, 24].

1.1 Classical Mechanics

A basic starting point for the discussion of classical mechanics is Newton's law. Consider a massive point particle of mass $m > 0$ which moves in some Euclidean space $\mathbb{R}^d$. In classical mechanics, our goal is to describe the trajectory of the particle at every given instance of time. This can, in principle, be determined based on Newton's law. If the position of the particle is described by the curve $t \mapsto x(t) \in \mathbb{R}^d$ and its momentum is $t \mapsto p(t) = m(dx/dt)(t) \in \mathbb{R}^d$, then Newton's law states that the change of momentum in time is equal to the force that acts on the particle, that is

$$\frac{dp}{dt} = m\,\frac{d^2x}{dt^2} = F(x). \tag{1.1}$$

Here, $F : \mathbb{R}^d \to \mathbb{R}^d$ denotes the force that acts on the particle. Note that (1.1) describes a second order ODE and admits, under suitable assumptions on F, a unique regular (e.g. twice continuously differentiable) solution $t \mapsto x(t)$ with initial data $x_0, p_0 \in \mathbb{R}^d$ such that $x(0) = x_0, m(dx/dt)(0) = p_0$. In typical situations of interest, the force F is conservative which means that $F = -\nabla V$ for some potential $V : \mathbb{R}^d \to \mathbb{R}$. For instance, the gravitational force between a particle of mass $m_1 > 0$ at $x \in \mathbb{R}^3$ and a particle of mass $m_2 > 0$ fixed at the origin $0 \in \mathbb{R}^3$ is described by $F_{\text{gravity}}(x) = -Gm_1m_2x/|x|^3$ so that $F_{\text{gravity}} = -\nabla V_{\text{gravity}}$ for $V_{\text{gravity}}(x) = -Gm_1m_2/|x|$ (G denotes the gravitational constant). In the following we restrict our attention to conservative forces.

The dynamical law (1.1) implies that when no force ($F = 0$) acts on the particle, the particle moves at constant speed (and the trace of its position is a straight line). This is called, more specifically, Newton's first law or the law of inertia. Classical mechanics assumes this law to be correct under the assumption that the coordinate system that we use to describe our particle is a so called inertial frame. Such frames are rather vaguely described (see e.g. [7, Chapter 1]) as coordinate frames in which space is homogeneous and isotropic (no point and no direction play a mechanically distinct role) and time is homogeneous (no instance of time plays a mechanically distinct role). The existence of such frames is assumed (an example of a frame which is not an inertial system is one which accelerates with regards to a particle at rest that does not interact with anything else) and the Galilean relativity principle states that the mechanical laws can not distinguish one inertial system from another. Based on these considerations,

it follows that different inertial systems move with constant speed with regards to each other (consider a massive particle at rest upon which no force acts, centered at the origin of $\mathbb{R}^d$ with regards to one coordinate system, and apply the Galilean principle so that $x(t) = vt + x_0$ for all $t \in \mathbb{R}$ in the transformed system). From a more practical point of view, what this discussion implies is that if K and K' denote two inertial systems moving with relative speed $v \in \mathbb{R}^d$ with regards to one another and the position of our particle in K is described by the curve $t \mapsto x(t)$, then its position with regards to the coordinate system in K' is described by

$$x'(t) = x(t) - vt,$$

assuming that there is a universal time, parametrized by $t \in \mathbb{R}$, that is used in both systems K and K' (that is $t' = t$). In other words, within the framework of classical mechanics, we assume that time is measured independently of the (inertial) reference frame that we choose to describe mechanical phenomena.

Given a potential $V : \mathbb{R}^d \to \mathbb{R}$ and a solution $t \mapsto x(t)$ of (1.1), one observes that

$$H(x, p) = \frac{|p|^2}{2m} + V(x)$$

is preserved in time. Indeed, we readily find that

$$\frac{d}{dt} H(x, p) = p(t) \cdot \frac{d^2 x}{dt^2} + \nabla V(x) \cdot \frac{dx}{dt} = 0.$$

The function $H : \mathbb{R}^{2d} \to \mathbb{R}$ is called Hamiltonian and it is identified with the energy of the system (the sum of kinetic and potential energies). One can reformulate the dynamics described by (1.1) as a system of first order ODE that involve H. This is the starting point for the Hamiltonian formulation of classical mechanics in which the possible mechanical states are described by points $(x, p) \in U \times \mathbb{R}^n$ for an open set $U \subset \mathbb{R}^n$ and some $n \in \mathbb{N}$ (more generally, the states are described by points in a $2n$-dimensional manifold). The dynamics of the states is determined by Hamilton's equations

$$\frac{dx}{dt} = \nabla_p H, \quad \frac{dp}{dt} = -\nabla_x H, \tag{1.2}$$

where $\nabla_x = (\partial_{x_1}, \ldots, \partial_{x_n})$ and $\nabla_p = (\partial_{p_1}, \ldots, \partial_{p_n})$ denote conventionally the gradients in $U \times \mathbb{R}^n$ with regards to the first and, respectively, last n coordinates.

Example 1.1 (Single Particle in External Field)**.** The Hamiltonian of a single particle of mass $m > 0$ that moves in $U \subset \mathbb{R}^d$ under the influence of an external potential $V : U \to \mathbb{R}$, whose position is $x \in U$ and whose momentum is $p \in \mathbb{R}^d$ is given by

$$H(x, p) = \frac{|p|^2}{2m} + V(x).$$

The Hamiltonian equations read

$$\frac{dx}{dt} = \nabla_p H = \frac{p}{m}, \qquad \frac{dp}{dt} = -\nabla_x H = -\nabla V = F,$$

which is equivalent to our previous definition $p = m(dx/dt)$ and Newton's law (1.1).

Example 1.2 (System of Interacting Particles)**.** Consider a collection of $N \in \mathbb{N}$ particles of mass $m > 0$ that move in $\mathbb{R}^d$ and interact with each other through some interaction potential $V : \mathbb{R}^d \to \mathbb{R}$. Their positions and momenta can be described collectively by $(x_1, \dots, x_N, p_1, \dots, p_N) \in \mathbb{R}^{dN} \times \mathbb{R}^{dN}$. The many-body energy is described by

$$H(x_1, \dots, x_N, p_1, \dots, p_N) = \sum_{i=1}^{N} \frac{|p_i|^2}{2m} + \sum_{1 \le i < j \le N} V(x_i - x_j).$$

The first contribution is called the kinetic energy while the second term on the right hand side describes the interaction energy among the particles. The dynamics reads

$$\frac{dx_i}{dt} = \frac{p_i}{m}, \qquad \frac{dp_i}{dt} = \sum_{1 \le j \le N : j \neq i} F(x_i - x_j) \qquad (\forall\, i = 1, \dots, N).$$

Example 1.3 (Particle in Static Magnetic and Electric Fields)**.** In classical physics, the electromagnetic phenomena are described by Maxwell's equations. If we denote the (time-dependent) electric and magnetic fields by $E : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ and, respectively, $B : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, then in the presence of charge and current densities $\rho : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ and, respectively, $j : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, Maxwell's equations read

$$\begin{aligned} \mathrm{div} E = \rho, \qquad & \nabla \times E = -\frac{1}{c} \partial_t B, \\ \mathrm{div} B = 0, \qquad & \nabla \times B = \frac{1}{c} (\partial_t E + j). \end{aligned} \tag{1.3}$$

Here, c denotes the speed with which electromagnetic waves travel in the vacuum. The charge density ρ (which describes the electric charge per unit volume) describes the distribution of all electric charges so that the total charge Q in $U \subset \mathbb{R}^d$ at time t equals

$$Q(t) = \int_U dx\, \rho(t, x).$$

By (1.3), the densities ρ and j are necessarily related by the continuity equation

$$\partial_t \rho + \mathrm{div} j = 0. \tag{1.4}$$

In particular, if the current density has compact support, say in $B_R(0) \subset \mathbb{R}^d$ for some $R > 0$, this implies that the total charge in this region

$$\frac{dQ}{dt}\,(\cdot) = \int_{B_R(0)} dx\, \partial_t \rho(\cdot, x) = -\int_{\partial B_R(0)} \sigma(dx)\, j(\cdot, x) = 0$$

is constant. Here, the second step follows from integration by parts. In other words, Maxwell's equations explain the conservation of the total electric charge. In case of a single particle of charge e moving in $\mathbb{R}^3$ whose position is described by $t \mapsto x(t)$, the charge and current densities are equal to the distributions (cf. Section 2.1 below)

$$t \mapsto \rho(t, x) = e\delta_{x(t)} \in \mathcal{D}'(\mathbb{R}^3), \qquad t \mapsto j(x, t) = e\delta_{x(t)} \frac{dx}{dt}(t) \in \mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3), \tag{1.5}$$

where here and in the following, $\delta_y \in \mathcal{D}'(U)$ describes the Dirac distribution centered at $y \in U \subset \mathbb{R}^d$. On the other hand, the force exerted by electric and magnetic fields E and B on a particle of charge e is equal to the Lorentz force so that by Newton's law

$$m\frac{d^2x}{dt^2} = eE + \frac{e}{c}\frac{dx}{dt} \times B. \tag{1.6}$$

The non-trivial system of equations (1.3), (1.5), (1.6) and generalizations thereof to a system of interacting charged particles (including possibly not only electromagnetic, but also other interactions such as gravity) models an enormous range of phenomena.

To relate the dynamics (1.6) to a Hamiltonian system, we need to relate the electric and magnetic fields to certain potentials. In typical situations of interest, this can be done as follows. Consider a particle moving in $\mathbb{R}^3$ whose influence on the charge and current densities ρ and j is negligible. Moreover, assume that the particle travels through a smooth magnetic field which is time-independent, that is $\partial_t B = 0$. Then (1.3) implies

$$\big(\partial_i E_j - \partial_j E_i\big) = 0$$

for all $i, j \in \{1, 2, 3\}, i \neq j$. Assuming E to be smooth and identifying the rotation $\nabla \times E$ with the exterior differential $d\mathrm{E}$ of the one form $\mathrm{E} = \sum_{i=1}^3 E_i dx_i \in \Omega^1(\mathbb{R}^3)$, note that

$$d\mathrm{E} = \sum_{1 \leq i < j \leq 3} \big(\partial_i E_j - \partial_j E_i\big)\, dx_i \wedge dx_j = 0.$$

In other words, $\partial_t B = 0$ and (1.3) imply that E is closed. By the Poincaré Lemma (see e.g. [10, Prop. 6.30 & Theorem 15.14]), we conclude that $\mathrm{E} = d\Phi$ for some potential $\Phi : \mathbb{R}^3 \to \mathbb{R}$ or, equivalently by slight abuse of notation, $E = \nabla\Phi$. Identifying similarly the magnetic field B with the differential two form

$$\mathrm{B} = B_1 dx_2 \wedge dx_3 - B_2 dx_1 \wedge dx_3 + B_3 dx_1 \wedge dx_2 \in \Omega^2(\mathbb{R}^3),$$

the second identity in (1.3) is equivalent to $d\mathrm{B} = 0$, so that $\mathrm{B} = dA$ for $\mathrm{A} = \sum_{i=1}^3 A_i dx_i$ or, equivalently, $B = \nabla \times A$ for a magnetic vector potential $A = (A_1, A_2, A_3) : \mathbb{R}^3 \to \mathbb{R}^3$.

In terms of the electric and magnetic potentials (Φ, A), the identity (1.6) reads

$$m\frac{d^2x}{dt^2} = e\nabla\Phi + \frac{e}{c}\frac{dx}{dt} \times \nabla \times A$$

and setting $p = mdx/dt + eA/c$, this dynamics can be reformulated as

$$\frac{dx}{dt} = \frac{1}{m}\Big(p - \frac{e}{c}A\Big), \qquad \frac{dp_i}{dt} = \frac{e}{mc}\Big(p - \frac{e}{c}A\Big) \cdot \partial_i A - e\partial_i\Phi \quad (i = 1, 2, 3).$$

Notice that this corresponds to the Hamiltonian dynamics for the energy function

$$(x, p) \mapsto H(x, p) = \frac{1}{2m}\big|p - \frac{e}{c}A(x)\big|^2 + e\Phi(x).$$

1.1.1 Canonical Transformations, Flows and Symmetries

Consider now a general Hamiltonian system with smooth energy $H \in C^\infty(\mathcal{P})$ and (flat) phase space $\mathcal{P} = U \times \mathbb{R}^n$. As already remarked in the previous section, the Hamiltonian H is preserved in time under the Hamiltonian dynamics (1.2). Indeed, this follows from

$$\frac{dH}{dt} = \nabla_x H \cdot \frac{dx}{dt} + \nabla_p H \cdot \frac{dp}{dt} = \sum_{i=1}^{n} \big(\partial_{x_i} H \partial_{p_i} H - \partial_{p_i} H \partial_{x_i} H\big) = 0.$$

In typical examples the Hamiltonian H is not the only quantity that is preserved in time, but also other quantities such as the momentum or angular momentum may be preserved. This is related to specific symmetries of the Hamiltonian. In order to make this more precise, and in foresight of the following sections, let us introduce some additional machinery. Let $F \in C^\infty(\mathcal{P})$ be a smooth function and denote by $t \mapsto (x(t), p(t)) \in \mathcal{P}$ a (local) solution of Hamilton's equations (1.2). Then a straightforward calculation shows

$$\frac{d}{dt} F \circ (x,p) = \Big(\nabla_x F \cdot \frac{dx}{dt} + \nabla_p F \cdot \frac{dp}{dt}\Big) \circ (x,p) = \{F, H\} \circ (x,p), \tag{1.7}$$

where for $F, G \in C^\infty(\mathcal{P})$, we introduced the Poisson bracket

$$\{F, G\} = \sum_{i=1}^{n} \big(\partial_{x_i} F \, \partial_{p_i} G - \partial_{p_i} F \, \partial_{x_i} G\big).$$

It is bilinear as a map $\{\cdot,\cdot\} : C^\infty(\mathcal{P}) \times C^\infty(\mathcal{P}) \to C^\infty(\mathcal{P})$, it satisfies

$$\{F, G\} + \{G, F\} = 0, \quad \{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0$$

and an explicit calculation shows that for all $i, j \in \{1, \dots, n\}$, we have

$$\{x_i, x_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{x_i, p_j\} = \delta_{ij}. \tag{1.8}$$

By (1.7), the vanishing of $\{F, H\} = 0$ implies that F is a conserved quantity under the Hamiltonian dynamics (1.2). It is thus natural to look for criteria that ensure the vanishing $\{F, G\} = 0$ of the Poisson bracket. Below, we relate this to certain observable-associated flows, for which we need to introduce a few additional tools.

Denote by $\mathbf{J} \in \mathbb{R}^{2n \times 2n}$ the matrix

$$\mathbf{J} = \begin{pmatrix} 0 & \mathbf{1}_{\mathbb{R}^n} \\ -\mathbf{1}_{\mathbb{R}^n} & 0 \end{pmatrix}$$

and recall that the symplectic group $\mathrm{Sp}(2n) = \mathrm{Sp}(2n, \mathbb{R})$ is defined by

$$\mathrm{Sp}(2n) = \big\{M \in \mathbb{R}^{2n \times 2n} : M^T \mathbf{J} M = \mathbf{J}\big\}.$$

Matrices $M \in \mathrm{Sp}(2n)$ are called symplectic and they leave the symplectic bilinear form $\sigma : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$, defined by $\sigma(\zeta_1, \zeta_2) = \zeta_1 \cdot \mathbf{J}\zeta_2$, invariant. That is

$$\sigma(M\zeta_1, M\zeta_2) = \sigma(\zeta_1, \zeta_2)$$

for all $\zeta_1, \zeta_2 \in \mathbb{R}^{2n}$. Symplectic matrices are invertible, because $1 = \det(M^T \mathbf{J} M) = |\det M|^2$ and we have that $M^{-1} = -\mathbf{J} M^T \mathbf{J}$, because $\mathbf{J}^2 = -\mathbf{1}_{\mathbb{R}^{2n}}$.

Symplectic matrices occur naturally in the question for which coordinate transformations $\phi \in C^\infty(\mathcal{P}, \mathbb{R}^{2n})$ the form of the Hamiltonian equations (1.2) is preserved. For a precise statement in the following lemma, let us say that $\phi \in C^\infty(\mathcal{P}, \mathbb{R}^{2n})$ is a canonical transformation if its differential satisfies $(D\phi)(\zeta) \in \mathrm{Sp}(2n)$, for every $\zeta \in \mathcal{P}$. Note that this implies in particular that ϕ is a local diffeomorphism, by the inverse function theorem. In other words, we can think of ϕ as a (local) coordinate transformation.

Lemma 1.1. *Let $t \mapsto \zeta(t)$ be a solution to Hamiltonian's equations so that*

$$\frac{d\zeta}{dt} = \boldsymbol{J} \nabla H(\zeta).$$

If $\zeta' = \phi \circ \zeta$ for a canonical transformation $\phi \in C^\infty(\mathcal{P})$, then

$$\frac{d\zeta'}{dt} = \boldsymbol{J} \nabla (H \circ \phi^{-1})(\zeta').$$

Moreover, setting $\phi^* F = F \circ \phi$, ϕ is canonical if and only if for all $F, G \in C^\infty(\mathbb{R}^{2n})$

$$\phi^* \{F, G\} = \{\phi^* F, \phi^* G\}.$$

Remark 1.1. *Note that if H describes the energy of a particle system described within a given reference frame whose coordinates are $\zeta = (x, p) \in \mathcal{P}$, then if $\phi \in C^\infty(\mathcal{P}, \mathbb{R}^{2n})$ is a (possibly local) coordinate transformation (a local diffeomorphism), then $H \circ \phi^{-1}$ describes the energy in terms of the new coordinates $\zeta' = \phi \circ \zeta$.*

Remark 1.2. *Geometrically, $\phi^* F$ is the pullback of the form $F \in \Omega^0(\mathbb{R}^{2n}) = C^\infty(\mathbb{R}^{2n})$.*

Proof. Using the chain rule, we verify that

$$\begin{aligned} \frac{d\zeta'}{dt} = D\phi(\zeta) \frac{d\zeta}{dt} = D\phi(\zeta) \mathbf{J} \nabla H(\zeta) &= D\phi(\zeta) \mathbf{J} \nabla \big(H \circ \phi^{-1}\big)(\zeta') \\ &= D\phi(\zeta) \mathbf{J} D\phi^T(\zeta) \nabla (H \circ \phi^{-1})(\zeta') \\ &= \mathbf{J} \nabla (H \circ \phi^{-1})(\zeta'), \end{aligned}$$

where we used that $\nabla H = (DH)^T$ and that $M^T \in \mathrm{Sp}(2n)$ if $M \in \mathrm{Sp}(2n)$ (*exercise*).

For the second statement, notice that $\{F, G\} = \nabla F \cdot \mathbf{J} \nabla G$ as well as $\nabla(\phi^* F) = (D\phi)^T (\nabla F) \circ \phi$, so that the pullback identity is equivalent to the statement that

$$\big(\nabla F \cdot \mathbf{J} \nabla G\big) \circ \phi = \big((\nabla F) \circ \phi\big) \cdot D\phi \, \mathbf{J} (D\phi)^T (\nabla G) \circ \phi.$$

Now, on the one hand, if ϕ is canonical, then $D\phi \, \mathbf{J} (D\phi)^T = \mathbf{J}$ and we conclude the invariance of the Poisson bracket under taking the pullback by ϕ. On the other hand, assuming that the invariance of the Poisson bracket, we can choose the canonical coordinate functions x_i, p_j (for F, G), for $i, j = 1, \ldots, n$, in $\mathbb{R}^{2n}$ to deduce that $D\phi \mathbf{J} (D\phi)^T = \mathbf{J}$. $\square$

Next, let us recall the notion of the flow and the Lie derivative with regards to a vector field. Let $X \in C^\infty(\mathcal{P}, \mathbb{R}^{2n})$ be a smooth vector field (identifying $\mathcal{P} = U \times \mathbb{R}^n$ with a flat, smooth manifold and $\mathbb{R}^{2n} \simeq \mathrm{T}\mathcal{P} = \bigsqcup_{(x,p)\in\mathcal{P}} \mathbb{R}^{2n}$ with its tangent space). By basic existence and uniqueness theory for ordinary differential equations, recall that for every $\zeta \in \mathcal{P}$ there exists a (non-empty) time interval $(-t_\zeta, t_\zeta) \subset \mathbb{R}$, an open set $U_\zeta \subset \mathcal{P}$ that contains ζ and a smooth map $\Phi_X : (-t_\zeta, t_\zeta) \times U_\zeta \mapsto \mathcal{P}$ such that for every $\xi \in U_\zeta$, the map $t \mapsto \Phi_X(t, \xi)$ is equal to the unique, smooth solution of the initial value problem

$$\begin{cases} \dfrac{df}{dt} = X(f), \\ f(0) = \xi. \end{cases}$$

In particular, $\Phi_X(t, \cdot)$ is a local diffeomorphism with inverse $\Phi_X(-t, \cdot)$, because

$$\Phi_X(s, \cdot) \circ \Phi_X(t, \cdot) = \Phi_X(t, \cdot) \circ \Phi_X(s, \cdot) = \Phi_X(s + t, \cdot).$$

Given $F \in C^\infty(\mathcal{P})$, a vector field X and its flow Φ_X, the Lie derivate $\mathcal{L}_X(F) \in C^\infty(\mathcal{P})$ of F in the direction of X is defined by

$$\mathcal{L}_X(F)(\zeta) = \frac{d}{dt} F(\Phi_X(t, \zeta))_{|t=0} = \lim_{t\to 0} \frac{1}{t}\big(F(\Phi_X(t, \zeta)) - F(\zeta)\big).$$

Recall that this is a natural geometric definition of the directional derivative of F (or, suitably extended, of a tensor field) in direction X. In standard coordinates $X = (X_1, \ldots, X_{2n})$, we find from the chain rule that

$$\mathcal{L}_X(F)(\zeta) = \sum_{i=1}^{2n} (X_i \partial_{\zeta_i} F)(\zeta) = (\nabla F \cdot X)(\zeta) = (XF)(\zeta)$$

and, based on the group property of the flow, we find that

$$\begin{aligned} \mathcal{L}_X(\Phi_X(t, \cdot)^* F) &= \lim_{s\to 0} \frac{1}{s}\big(F(\Phi_X(t, \cdot) \circ \Phi_X(s, \cdot)) - F(\Phi_X(t, \cdot))\big) \\ &= \lim_{s\to 0} \frac{1}{s}\big(F(\Phi_X(s, \Phi_X(t, \cdot))) - F(\Phi_X(t, \cdot))\big) = \Phi_X(t, \cdot)^*\big(\mathcal{L}_X(F)\big), \end{aligned}$$

as well as

$$\frac{d}{dt}(\Phi_X(t, \cdot)^* F) = (\nabla F \cdot X) \circ \Phi_X(t, \cdot) = \Phi_X(t, \cdot)^*\big(\mathcal{L}_X(F)\big) = \mathcal{L}_X\big(\Phi_X(t, \cdot)^* F\big).$$

Given these notions, observe that Hamilton's equations (1.2) are equivalent to the flow dynamics with regards to the Hamiltonian vector field $X_H = \mathbf{J}\nabla H \in C^\infty(\mathcal{P}, \mathbb{R}^{2n})$. Consequently, the Hamiltonian dynamics (1.7) of observables can be rewritten as

$$\frac{d}{dt} \Phi_{X_H}(t, \cdot)^* F = \Phi_{X_H}(t, \cdot)^* \{F, H\} = \big\{\Phi_{X_H}(t, \cdot)^* F, H\big\}.$$

Here, first step is equality is identical to (1.7). For the second equality, on the other hand, we used that $\mathcal{L}_{X_H}(F) = \{F, H\}$, which follows from evaluating the first equality at $t = 0$, and that $\frac{d}{dt}\Phi_{X_H}(t, \cdot)^* F = \mathcal{L}_{X_H}\big(\Phi_{X_H}(t, \cdot)^* F\big)$, proved above. The connection of the previous observations to canonical coordinates is the content of the next lemma.

Lemma 1.2. *Denote by $(t,\zeta) \mapsto \Phi_t(\zeta) = \Phi_{X_H}(t,\zeta)$ the flow of the Hamiltonian vector field $X_H = \boldsymbol{J}\nabla H$. Then, for every $t \in \mathbb{R}$ for which Φ_t exists, $\Phi_t(\cdot)$ is canonical.*

Proof. By Lemma 1.1, it suffices to show that for all $F, G \in C^\infty(\mathcal{P})$, we have that

$$\Phi_t^*\{F,G\} = \{\Phi_t^* F, \Phi_t^* G\}. \tag{1.9}$$

To this end, let us set $F_t = \Phi_t^* F$ and $G_t = \Phi_t^* G$ so that $dF_t/dt = \{F_t, H\}$ and thus

$$\frac{d}{dt}\{F_t, G_t\} = \big\{\{F_t, H\}, G_t\big\} + \big\{F_t, \{G_t, H\}\big\} = \big\{\{F_t, G_t\}, H\big\}$$

with $\{F_t, G_t\}_{|t=0} = \{F, G\}$ (all identities holding true pointwise in $\mathcal{P}$). Since

$$\frac{d}{dt}\Phi_t^*\{F,G\} = \big\{\Phi_t^*\{F,G\}, H\big\}$$

with $(\Phi_t^*\{F,G\})_{|t=0} = \{F,G\}$, the map $t \mapsto \Phi_t^*\{F,G\}$ solves the same initial value problem. By standard results for ordinary differential equations, we conclude (1.9). □

Let us now get back to the question of characterizing quantities that are conserved under (1.2). We noted earlier that an observable $F \in C^\infty(\mathcal{P})$ is conserved if $\{F, H\} = 0$. From the previous discussion, we conclude that this is equivalent to the statement that $\Phi_{X_H}(t,\cdot)^* F = F$, that is, F is invariant under the flow associated to $X_H = \mathbf{J}\nabla H$.

Observe that we can also turn this picture around: F is conserved under (1.2) if the Hamiltonian $H = \Phi_{X_F}(t,\cdot)^* H$ is invariant under the flow $\Phi_{X_F}(t,\cdot)$ generated by F, where $X_F = \mathbf{J}\nabla F$. This naturally suggests to look at the symmetries of H in order to find conserved quantities. Indeed, in typical examples, symmetries of H can be described by suitable Lie group actions. Recall that a Lie group G is a smooth manifold that is also a group and such that the group multiplication and inverse maps are smooth. Typical examples are matrix groups such as linear, orthogonal transformations

$$\mathrm{O}(n) = \big\{\ R \in \mathbb{R}^{n\times n} : R \text{ linear}, R^* R = \mathbf{1}_{\mathbb{R}^n}\big\}.$$

If H is invariant under the action of a Lie group G, it is invariant under the action of all its one-parameter subgroups (that is, continuous group homomorphisms) $\gamma : \mathbb{R} \to G$. The latter can be identified with the integral curves (that is, the flow) of left-invariant vector fields on G (its Lie algebra), s.t. the connection between symmetries and conserved quantities becomes obvious. Let's illustrate this strategy through some explicit examples. For a general geometric discussion, see e.g. [10, Chapters 18 & 20].

Example 1.4 (Momentum Conservation)**.** Consider a single particle moving in $U \subset \mathbb{R}^d$ in an external field $V : \mathbb{R}^d \to \mathbb{R}$. The Hamiltonian is of the form

$$H(x,p) = \frac{|p|^2}{2m} + V(x),$$

as in Example 1.1. Consider the translation group $\mathcal{G} = (\mathbb{R}^d, +)$ acting on $\mathcal{P} = U \times \mathbb{R}^d$ as $a(x,p) = (x+a, p)$, for $a \in \mathcal{G}$, and consider its one-parameter subgroups $\mathcal{G}_n =$

$\big(\{nt : t \in \mathbb{R}\}, +\big)$, for directions $n \in \mathbb{R}^d$ with $|n| = 1$. Then, the flow Φ_{X_n} associated to the (constant, Hamiltonian) vector field $X_n = (n, 0) = \mathbf{J}\nabla P_n \in C^\infty(\mathcal{P}, \mathbb{R}^{2d})$, for $P_n \in C^\infty(\mathcal{P})$ denoting the momentum $P_n(x, p) = p \cdot n$ in direction n, is equal to

$$\Phi_{X_n}(t, x, p) = (x + tn, p).$$

Indeed, $\Phi_{X_n}(0, x, p) = (x, p)$ and $d\Phi_{X_n}(\cdot, x, p)/dt = (n, 0) = X_n \circ \Phi_{X_n}(\cdot, x, p)$.

Thus, P_n is conserved under (1.2) if

$$\Phi_{X_n}(t, \cdot)^* H = H \quad \Longleftrightarrow \quad V(x) = V(x + tn),\ \forall\ (x, p) \in \mathcal{P}, t \in \mathbb{R}.$$

Generalized to the N-body setting as in Example 1.2, consider the Hamiltonian vector field $\mathbf{X}_n = (n, \dots, n, 0, \dots, 0) = \mathbf{J}\nabla \mathbf{P}_n \in C^\infty(\mathcal{P}^N, \mathbb{R}^{2dN})$, for $\mathbf{P}_n \in C^\infty(\mathcal{P}^N)$ denoting the momentum $\mathbf{P}_n(x, p) = \sum_{i=1}^N p_i \cdot n$. Arguing as above, the total momentum $\mathbf{P} = \sum_{i=1}^N p_i$ is a conserved quantity, because of the invariance

$$\Phi_{\mathbf{X}_n}(t, \cdot)^* H_N = H_N \iff \Phi_{\mathbf{X}_n}(t, \cdot)^* V(x_i - x_j) = V(x_i - x_j),\ \forall\ i, j,\ n \in \mathbb{R}^d, |n| = 1,$$

where we recall that $H_N(x_1, \dots, x_N, p_1, \dots, p_N) = \sum_{i=1}^N \frac{|p_i|^2}{2m} + \sum_{1 \le i < j \le N} V(x_i - x_j)$.

Example 1.5 (Angular Momentum Conservation)**.** Consider a single particle moving in $\mathbb{R}^3$ with Hamiltonian $H : \mathcal{P} \to \mathbb{R}$ defined by

$$H(x, p) = \frac{|p|^2}{2m} + V(x),$$

as in Example 1.4, for $\mathcal{P} = \mathbb{R}^3 \times \mathbb{R}^3$. Recall that the special orthogonal group

$$\mathrm{SO}(3) = \big\{R \in \mathbb{R}^{3\times 3} : R^T R = \mathbf{1}_{\mathbb{R}^3}, \det R = 1\big\}$$

describes rotations in $\mathbb{R}^3$ (for more on SO(3), see Section 3.2). It is a well-known fact from linear algebra that a general rotation matrix $R \in SO(3)$ can be written as

$$R = e^{\omega\, n \cdot X}, \quad \text{where} \quad n \cdot X = \sum_{i=1}^3 n_i X_i,$$

for an angle $\omega \in [0, 2\pi)$, a rotation axis $n \in \mathbb{R}^3$, $|n| = 1$ and where the matrices

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.10}$$

form a basis of the linear space of skew-symmetric matrices

$$\mathfrak{so}(3) = \big\{X \in \mathbb{R}^{3\times 3} : X = -X^*\big\}.$$

Equivalently, one can show that every $R \in \mathrm{SO}(3)$ can be decomposed into a product

$$R = e^{\omega_1 X_1} e^{\omega_2 X_2} e^{\omega_3 X_3}$$

of three rotations around the standard coordinate axes. We assume these basic facts in the sequel without proof; for detailed explanations, see e.g. [11, Sections 2.1 and 7.1]

In geometric terms, $\mathfrak{so}(3) \simeq T_{\mathbf{1}_{\mathbb{R}^3}}\mathrm{SO}(3)$, equipped with the usual matrix commutator as the Lie bracket $[\cdot,\cdot] : \mathfrak{so}(3) \times \mathfrak{so}(3) \to \mathfrak{so}(3)$, represents the Lie algebra of the Lie group SO(3) and the corresponding exponential map $\exp : \mathfrak{so}(3) \to \mathrm{SO}(3)$, defined by

$$\exp(X) = e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!},$$

is surjective. Matrices $R \in \mathrm{SO}(3)$ act naturally on points in $\mathcal{P}$ as $R(x,p) = (Rx, Rp)$. Consider then e.g. the one-parameter subgroup $\{R_{e_3,\omega} : \omega \in [0,2\pi)\} \subset \mathrm{SO}(3)$ of rotations around the coordinate axis $e_3 = (0,0,1)$, where an explicit calculation yields

$$R_{e_3,\omega} = \exp(\omega X_3) = \begin{pmatrix} \cos(\omega) & -\sin(\omega) & 0 \\ \sin(\omega) & \cos(\omega) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, $\omega \mapsto R_{e_3,\omega}(x,p) = \left(e^{X_3\omega}x, e^{X_3\omega}p\right) = \Phi_{X_{e_3}}(\omega,x,p)$ is equal to the flow of

$$(x,p) \mapsto X_{e_3}(x,p) = (X_3x, X_3p) = \mathbf{J}\nabla L_3(x,p) \in C^\infty(\mathcal{P},\mathcal{P}) \text{ for } L_3(x,p) = x_1p_2 - x_2p_1.$$

In particular, if $V(R_{e_3,\omega}\cdot) = V(\cdot)$ is invariant under rotations around e_3, the angular momentum L_3 around e_3 is a conserved quantity. If, more generally, $V(R\,\cdot) = V(\cdot)$ for all $R \in SO(3)$, then the angular momentum $(x,p) \mapsto L(x,p) = x \times p$ is conserved. We leave the detailed verification of the last two statements as a basic *exercise*.

Problem 1.1. *Prove that* O(3) *and* SO(3) *are Lie groups and that* $\mathfrak{so}(3) \simeq T_{\mathbf{1}_{\mathbb{R}^3}}\mathrm{SO}(3)$.

1.2 Quantum Mechanics

Despite describing a vast range of phenomena, there are various physical observations that can not be explained based on the principles of classical mechanics. This includes e.g. the discreteness of atomic spectra or internal particle properties such as spin. Quantum theory generalizes classical mechanics in order to describe such phenomena. The mathematical setting is quite different, so let us summarize a few of the basic axioms.

First of all, possible states of a quantum mechanical system are normalized vectors, the so called wave functions, $\psi \in \mathcal{H}, \|\psi\| = 1$, in a complex, separable Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. In accordance with standard physics notation, we assume throughout these notes that inner products are conjugate linear in the first slot and linear in the second slot. Wave functions $\psi_1, \psi_2 \in \mathcal{H}$ which only differ by a complex phase so that $\psi_1 = e^{i\omega}\psi_2$ for some $\omega \in [0,2\pi)$, are identified as describing the same physics (in other words, one may identify the state space more precisely with the space of unit rays $\mathcal{H}/\sim$, where $\psi_1 \sim \psi_2$ if and only if $\psi_1 = e^{i\omega}\psi_2$ for some $\omega \in [0,2\pi)$).

Physical observables, such as the position or momentum of a massive particle, correspond to suitable self-adjoint operators $A : D_A \to \mathcal{H}$, D_A denoting the linear, dense domain of A. Recall that the adjoint A^* of such an operator $A : D_A \to \mathcal{H}$ has domain

$$D(A^*) = \left\{\varphi \in \mathcal{H} : D(A) \ni \psi \mapsto \langle\varphi, A\psi\rangle \in \mathbb{C} \text{ extends to a linear functional in } \mathcal{H}^*\right\}.$$

Riesz' lemma shows that if $\varphi \in D(A^*)$, then there exists a unique $\xi_\varphi \in \mathcal{H}$ such that $\langle \varphi, A\psi\rangle = \langle \xi, \psi\rangle$ for every $\psi \in D(A)$ and one thus defines $A^*\varphi = \xi_\varphi$. An operator is self-adjoint if $D(A) = D(A^*)$ and $A = A^*$. Such operators have a canonical form which is the content of the spectral theorem. For its proof, see e.g. [14, Chapters VII & VIII].

Theorem 1.1 (Spectral Theorem)**.** *Let $A : D_A \to \mathcal{H}$ be self-adjoint. Then, there is a measure space $\mathcal{M} = (\mathcal{X}, \mathcal{B}(\mathcal{X}), \mu)$ with a finite Borel measure μ, a unitary map $U : \mathcal{H} \to L^2(\mathcal{M})$ and a real-valued, μ-measurable function $f : \mathcal{X} \to \mathbb{R}$ such that*

$$UD_A = \big\{\varphi \in L^2(\mathcal{M}) : f\varphi \in L^2(\mathcal{M})\big\} \quad \text{and} \quad UAU^*\varphi = f\varphi \in L^2(\mathcal{M}),\ \forall\, \varphi \in UD_A.$$

Define $g(A) = U^*(g \circ f)U$ for every bounded, measurable function $g : \mathbb{R} \to \mathbb{R}$, where $g \circ f$ is interpreted as multiplication operator in $L^2(\mathcal{M})$, and denote by $\chi_\Omega : \mathbb{R} \to \mathbb{R}$ the characteristic function of $\Omega \in \mathcal{B}(\mathbb{R})$. Then, the family of bounded, self-adjoint operators $(\chi_\Omega(A))_{\Omega\in\mathcal{B}(\mathbb{R})}$ forms a projection valued measure, which means that

i) $\chi_\emptyset(A) = 0$ *and* $\chi_{\mathbb{R}}(A) = \mathbf{1}_{\mathcal{H}}$,

ii) $\chi_{\Omega_1}(A)\chi_{\Omega_2}(A) = \chi_{\Omega_1\cap\Omega_2}(A)$ *for every* $\Omega_1, \Omega_2 \in \mathcal{B}(\mathbb{R})$ *and*

iii) $\chi_\Omega(A) = \sum_{i=1}^\infty \chi_{\Omega_i}(A)$ *strongly in* $\mathcal{H}$, *if* $\Omega = \cup_{i=1}^\infty \Omega_i$ *with* $\Omega_i \cap \Omega_j = \emptyset$, $\forall\, i \neq j$.

In terms of this projection-valued measure, $A : D_A \to \mathcal{H}$ has the spectral decomposition

$$A = \int_{\sigma(A)} \lambda\, \chi_{d\lambda}(A), \tag{1.11}$$

where $\sigma(A) \subset \mathbb{R}$ denotes the spectrum of A.

Remark 1.3. *Recall that $\sigma(A) = \mathbb{C} \setminus \rho(A)$ where the resolvent set $\rho(A) \subset \mathbb{C}$ is given by*

$$\rho = \big\{z \in \mathbb{C} : (A - z) \text{ admits a bounded inverse } (A - z)^{-1} : \mathcal{H} \to D_A\big\}.$$

We split $\sigma(A) = \sigma_{\rm d}(A) \cup \sigma_{\rm ess}(A)$ into a discrete part $\sigma_{\rm d}(A)$, the set of isolated eigenvalues of A of finite multiplicity, and its complement $\sigma_{\rm ess}(A)$, the essential spectrum.

Remark 1.4. *The theorem generalizes the well-known fact from linear algebra that every Hermitian matrix $H = H^* \in \mathbb{C}^{n\times n}$ can be diagonalized and admits an orthonormal eigenbasis $(\varphi_i)_{i=1}^n$ so that $H\varphi_i = \lambda_i\varphi_i$, for suitable (real) eigenvalues $\lambda_i \in \mathbb{R}$. In this case, the spectral projection valued measure represention of H is simply given by*

$$H = \sum_{i=1}^n \lambda_i |\varphi_i\rangle\langle\varphi_i|.$$

The map U can be defined by linearly extending $\mathbb{C}^n \ni \varphi_i \mapsto \chi_{\{\lambda_i\}} \in L^2(\Omega, \mathcal{B}(\Omega), \mu)$, where $\Omega = \{\lambda_i : i = 1, \dots, n\}$ and where μ denotes the counting measure on Ω.

Based on the spectral theorem, let us point out how, in the context of quantum mechanics, observables like the position or the momentum of a particle are connected with self-adjoint operators. Suppose that $A : D_A \to \mathcal{H}$ represents some observable $\mathcal{O}$ and that the system is in state $\psi \in \mathcal{H}$. Then, based on the normalization $\|\psi\| = 1$ and on the spectral decomposition (1.11) of A, one identifies $\mathcal{O}$ with a real-valued random variable (ranging almost surely in the spectrum $\sigma(A) \subset \mathbb{R}$ of A) and the probability $\mathbb{P}$ that $\mathcal{O}$ takes a specific value in some measurable set $\Omega \in \mathcal{B}(\mathbb{R})$ is defined as

$$\mathbb{P}(\mathcal{O} \in \Omega) = \int_\Omega \langle \psi, \chi_{d\lambda}(A)\psi \rangle = \langle \psi, \chi_\Omega(A)\psi \rangle. \tag{1.12}$$

Notice that the law $\Omega \mapsto \mathcal{O}_*(\mathbb{P})(\Omega) = \mathbb{P}(\mathcal{O} \in \Omega)$ defines indeed a Borel probability measure on $\mathbb{R}$. In other words, quantum mechanics only provides probabilistic predictions for the outcomes of physical measurements. Within this probabilistic interpretation, the expectation value $\mathbb{E}\mathcal{O}$ of the observable $\mathcal{O}$ represented by A is then equal to

$$\mathbb{E}\mathcal{O} = \int d\mathbb{P}\, \mathcal{O} = \int_{\mathbb{R}} \mathcal{O}_*(\mathbb{P})(d\lambda)\, \lambda = \int_{\sigma(A)} \lambda \, \langle \psi, \chi_{d\lambda}(A)\psi \rangle = \langle \psi, A\psi \rangle \tag{1.13}$$

and for this reason, we refer in the sequel to inner products like that on the r.h.s. of the previous equation as expectation values. Other basic statistical quantities from probability theory can be similarly related to suitable inner products, e.g. the variance

$$\mathbb{E}(\mathcal{O} - \mathbb{E}\mathcal{O})^2 = \mathbb{E}\mathcal{O}^2 - (\mathbb{E}\mathcal{O})^2 = \langle \psi, A^2\psi \rangle - \langle \psi, A\psi \rangle^2. \tag{1.14}$$

Observe that while (1.12) is well-defined for every $\psi \in \mathcal{H}$ (as it should be if every state $\psi \in \mathcal{H}$ is a possible state of the system), this need not be the case for (1.13) and (1.14).

Before switching to some concrete examples, let us explain how one describes the dynamics of quantum systems. Here, one postulates the existence of a strongly continuous one-parameter unitary group $(U_t)_{t\in\mathbb{R}}$ acting on $\mathcal{H}$. If the system is in state $\psi \in \mathcal{H}$ at time $t = 0$, then the system is in state $U_t\psi$ at every other time $t \in \mathbb{R}$. Note that $U_t\psi$ is indeed a valid state, because $\|U_t\psi\| = \|\psi\| = 1$, for all $t \in \mathbb{R}$. By (1.2), the classical dynamics is determined by the energy (the Hamiltonian). By analogy, one defines the Hamilton operator $H : D_H \to \mathbb{R}$ in quantum mechanics as the generator of the quantum dynamics $(U_t)_{t\in\mathbb{R}}$. This relies on the following fundamental result.

Theorem 1.2 (Stone's Theorem)**.** *Let $(U_t)_{t\in\mathbb{R}}$ be a strongly continuous one-parameter unitary group on $\mathcal{H}$. Then, there exists a self-adjoint operator $H : D_H \to \mathcal{H}$ such that*

$$U_t = e^{-itH}, \ \forall\, t \in \mathbb{R}.$$

Proof. We follow [14]. Before defining our candidate for H, we first need to find a suitable dense domain on which we can differentiate $t \mapsto U_t(\cdot)$. Using that, heuristically, $\phi \approx e^{-itH}\phi$ for small t (assuming we knew the existence of H already), it is useful to consider for $f \in C_c^\infty(\mathbb{R})$ and $\phi \in \mathcal{H}$ the vector space generated by vectors of the form

$$\phi_f = \int_{\mathbb{R}} dt\, f(t) U_t\phi \in \mathcal{H}.$$

Here, the integral on the r.h.s. can be defined as a vector-valued Riemann integral (and coincides with the usual Bochner integral). Set

$$D = \mathrm{span}\big\{\phi_f : f \in C_c^\infty(\mathbb{R}), \phi \in \mathcal{H}\big\}.$$

Then $D \subset \mathcal{H}$ is dense, because for a standard approximation of the identity $(f_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R})$ (choose e.g. $f = nf(n.)$ for some $f \in C_c^\infty(\mathbb{R})$ such that $0 \leq f \leq 1$, $\int_{\mathbb{R}} f = 1$ so that $\mathrm{supp}(f_n) \subset (-1/n; 1/n)$, $\forall\, n \in \mathbb{N}$), we have that

$$\limsup_{n\to\infty} \|\phi_{f_n} - \phi\| = \limsup_{n\to\infty} \left\| \int_{\mathbb{R}} dt\, f_n(t)(U_t\phi - \phi) \right\| \leq \limsup_{n\to\infty} \sup_{|t|\leq 1/n} \|U_t\phi - \phi\| = 0.$$

Next, we define H (initially on D) by differentiating $t \mapsto U_t$. For $\phi_f \in D$, we compute

$$\lim_{t\to 0} \frac{1}{t}(U_t\phi_f - \phi_f) = \lim_{t\to 0} \frac{1}{t} \int_{\mathbb{R}} ds\, f(s)(U_{t+s} - U_s)\phi = -\int_{\mathbb{R}} f'(s)U(s)\phi = -\phi_{f'},$$

where in the last step we applied the dominated convergence theorem. This suggests to define the operator $H : D \to D$ by

$$H\phi_f = i \lim_{t\to 0} \frac{1}{t}(U_t\phi_f - \phi_f) = -i\phi_{f'}.$$

By definition of $\phi_f \in D$, note that $U_t : D \to D$ for each $t \in \mathbb{R}$ ($U_t\phi_f = \phi_{f(.-t)}$), $H : D \to D$, $[U_t, H] = 0$ in D and that H is a symmetric operator, because

$$\begin{aligned}
\langle H\phi_f, \psi_g\rangle &= \langle -i\phi_{f'}, \psi_g\rangle \\
&= i \int_{\mathbb{R}^2} dsdt\, f'(t)g(s)\, \langle \phi, U_{-t+s}\psi\rangle \\
&= i \int_{\mathbb{R}^2} dsdt\, f'(t)g(s+t)\langle \phi, U_s\psi\rangle \\
&= -i \int_{\mathbb{R}^2} dsdt\, f(t)g'(s+t)\langle \phi, U_s\psi\rangle \\
&= \langle \phi_f, H\psi_g\rangle.
\end{aligned}$$

To conclude the theorem, it suffices to show that H is essentially self-adjoint and that the exponential of its (self-adjoint) closure is equal to U_t. For the first part, suppose that $\psi \in D(H^*)$ with $H^*\psi = i\psi$. Then, for each $\phi \in D$, we compute

$$\partial_t\langle U_t\phi, \psi\rangle = \langle -iHU_t\phi, \psi\rangle = -\langle U_t\phi, \psi\rangle.$$

This implies that $\langle U_t\phi, \psi\rangle = \langle \phi, \psi\rangle e^{-t}$ so that $\langle \phi, \psi\rangle = 0$, because $e^{-t} \to \infty$ as $t \to -\infty$ while $|\langle -U_t\phi, \psi\rangle| \leq \|\phi\|\|\psi\|$. Since $\phi \in D$ was arbitrary and $\overline{D} = \mathcal{H}$, this implies that $\psi = 0$. Repeating an analogous argument for the case $H^*\psi = -i\psi$, we deduce that $H : D \to D$ is essentially self-adjoint. Now, denote by $\overline{H} : D(\overline{H}) \to \mathcal{H}$ the self-adjoint closure of H and set $V_t = e^{-it\overline{H}}$. Given $\phi \in D$, we compute that

$$\partial_t\big(U_t\phi - V_t\phi\big) = -iHU_t\phi - i\overline{H}V_t\phi = -i\overline{H}(U_t - V_t)\phi,$$

which implies

$$\partial_t \|U_t\phi - V_t\phi\|^2 = 2\,\mathrm{Im}\big\langle \overline{H}(U_t\phi - V_t\phi), U_t\phi - V_t\phi\big\rangle = 0.$$

Thus, $U_t\phi = V_t\phi$ for all $t \in \mathbb{R}$ and $\phi \in D$, so that $U_t = V_t$, using $\overline{D} = \mathcal{H}$. □

In quantum mechanics, one identifies the dynamics conventionally with $(U_t)_{t\in\mathbb{R}} = (e^{-iHt/\hbar})_{t\in\mathbb{R}}$, where $\hbar$ is a fundamental small constant, called Planck's constant. The unitary dynamics is then related to the Schrödinger equation: if H has domain $D_H \subset \mathcal{H}$, then an application of the spectral theorem shows that for every $\varphi \in D_H$, the map

$$t \mapsto \psi(t) = e^{-itH/\hbar}\varphi \in C(\mathbb{R}, D_H) \cap C^1(\mathbb{R}, \mathcal{H}),$$

is the unique $C(\mathbb{R}, D_H) \cap C^1(\mathbb{R}, \mathcal{H})$ solution of the initial value problem

$$\begin{cases} i\hbar \dfrac{d\psi}{dt} = H\psi, \\ \psi(0) = \varphi. \end{cases} \tag{1.15}$$

For general initial data $\varphi \in \mathcal{H}$, not necessarily in D_H, the dynamics $(e^{-itH/\hbar}\varphi)_{t\in\mathbb{R}}$ solves (1.15) weakly (e.g. in form sense by testing the equation with elements from D_H).

Similarly as in (1.7) in the context of classical mechanics, one can study the dynamics of observables. Suppose that a physical observable $\mathcal{O}$ is represented by a self-adjoint operator $A : D_A \to \mathcal{H}$ and that, at time $t = 0$, the system is in state $\psi \in \mathcal{H}$, $\|\psi\| = 1$. As mentioned earlier in (1.12), the probability that $\mathcal{O} \in \Omega$ is defined as

$$\mathbb{P}(\mathcal{O} \in \Omega) = \langle \psi, \chi_\Omega(A)\psi\rangle.$$

If the system evolves in time until $t \in \mathbb{R}$ via $(U_t)_{t\in\mathbb{R}}$, this probability changes to

$$\mathbb{P}(\mathcal{O}_t \in \Omega) = \langle \psi, U_{-t}\chi_\Omega(A)U_t\psi\rangle = \langle \psi, \chi_\Omega(U_{-t}AU_t)\psi\rangle.$$

This clearly motivates to view $A_t = U_{-t}AU_t : U_{-t}D_A \to \mathcal{H}$ as representing the dynamical observable $(\mathcal{O}_t)_{t\in\mathbb{R}}$ at time $t \in \mathbb{R}$. Note that $A_t = A_t^*$, because unitary conjugation preserves self-adjointness. The formal quantum analogue of (1.7) thus becomes

$$\frac{d}{dt}A = \frac{1}{i\hbar}[A, H]. \tag{1.16}$$

Due to potential domain constraints, the rigorous interpretation of (1.16) requires some care. However, assuming e.g. that $A \in \mathcal{L}(\mathcal{H})$ extends in fact to a self-adjoint, bounded operator on $\mathcal{H}$, the operator dynamics (1.16) always admits a rigorous, weak formulation in quadratic form sense (by taking the expectation of (1.16) w.r.t. vectors in D_H).

1.2.1 Basic Examples via Canonical Quantization

In Section 1.1, we discussed several basic examples of classical mechanical systems. In this section, we describe the quantum mechanical versions of these models.

In order to generalize Examples 1.1, 1.2 and 1.3, we need to understand which self-adjoint operators should represent the position and the momentum of a particle. In our examples, the corresponding operators can be found based on a recipe called canonical quantization. This method is an important tool to guess and to formulate quantum mechanical (or quantum field theoretical) models with desirable properties that generalize their classical counterparts. It should be kept in mind, however, that this method has certain limitations (some of which are discussed below) and that, after all, quantum mechanics (or quantum field theory) is a more accurate theory to describe interacting particle systems than classical mechanics. For a quick introductory discussion of quantization schemes and their limitations, see e.g. [5, Chapters 13, 22 & 23].

Keeping the previous remarks in mind, let us explain the canonical quantization scheme. Comparing (1.7) with (1.16) motivates to replace the Poisson bracket $\{\cdot,\cdot\}$ of two classical observables by the commutator $\frac{1}{i\hbar}[\cdot,\cdot]$ of the corresponding quantum operators when passing from a classical to a quantum mechanical description. The constant of proportionality $i\hbar$ ensures consistency of the dynamics (1.7) and (1.16).

Let us apply the quantization scheme by reconsidering Examples 1.1, 1.2 and 1.3. Consider first a single, massive particle moving in $\mathbb{R}^d$, similarly as in Example 1.1. Its classical dynamics is determined by its position and momentum $(x,p) \in \mathcal{P} = \mathbb{R}^d \times \mathbb{R}^d$. Recalling (1.8), these satisfy as observables (that is, as coordinate functions in $C^\infty(\mathcal{P})$)

$$\{x_i, x_j\} = 0, \quad \{p_i, p_j\} = 0, \quad \{x_i, p_j\} = \delta_{ij}.$$

By the canonical quantization scheme, we now want to find a Hilbert space $\mathcal{H}$ and self-adjoint operators $X = (X_1, \dots, X_d), P = (P_1, \dots, P_d)$ that represent the position and, respectively, momentum of the particle so that (on a suitable domain $D \subset \mathcal{H}$)

$$[X_i, X_j] = 0, \quad [P_i, P_j] = 0, \quad [X_i, P_j] = i\hbar\delta_{ij}. \tag{1.17}$$

Let us start to look for a suitable position operator X. In classical probability, the position $\mathcal{X}$ of an object can be modelled as a random variable with distribution $\mathbb{P} \in \mathrm{P}(\mathbb{R}^d)$ (the set of Borel probability measures on $\mathbb{R}^d$) through the map $x \mapsto \mathcal{X}(x) = x$, defined on the probability space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mathbb{P})$. If $\mathbb{P}$ is absolutely continuous with regards to the Lebesgue measure dx, it admits a density $\rho : \mathbb{R}^d \to [0,\infty)$ and we have that

$$\mathbb{P}(\mathcal{X} \in \Omega) = \int_\Omega dx\, \rho(x).$$

Comparing this with (1.12), it is natural to define $\mathcal{H} = L^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), dx) = L^2(\mathbb{R}^d)$ and to define, for $i \in \{1, \dots, d\}$, the position coordinate operators $X_i : D_{X_i} \to L^2(\mathbb{R}^d)$ by

$$(X_i\psi)(x) = x_i\psi(x) \text{ for } a.e.\ x \in \mathbb{R}^d,\ \psi \in D_{X_i} = \big\{\psi \in L^2(\mathbb{R}^d) : x \mapsto x_i\psi(x) \in L^2(\mathbb{R}^d)\big\}.$$

Then each X_i is self-adjoint by Theorem 1.1 and there exists a dense domain $D \subset L^2(\mathbb{R}^d)$, e.g. $D = \mathcal{S}(\mathbb{R}^d)$, such that for all $i, j \in \{1, \dots, d\}$, we have $D \subset D_{X_j}, X_i D \subset D_{X_j}$ and

$$[X_i, X_j]_{|D} = 0.$$

In this so called Schrödinger representation, the modulus square $|\psi|^2$ of an element $\psi \in L^2(\mathbb{R}^d)$ has the interpretation of a probability density. This means that $|\psi(x)|^2 dx$ describes the probability to find the particle near the point $x \in \mathbb{R}^d$.

The momentum operators P_j can certainly be no functions of $X = (X_1, \dots, X_d)$ and one may attempt instead to look for differential operators that satisfy (1.17). A quick computation shows that we can define $P_j = -i\hbar\partial_{x_j} : D_{P_j} \to L^2(\mathbb{R}^d)$ on the domain

$$D_{P_j} = \big\{\psi \in L^2(\mathbb{R}^d) : \partial_{x_i}\psi \in L^2(\mathbb{R}^d)\ \big\} = \big\{\psi \in L^2(\mathbb{R}^d) : p \mapsto p_j\widehat{\psi}(p) \in L^2(\mathbb{R}^d)\big\},$$

where ∂_{x_j} denotes the j-th distributional derivate in $\mathbb{R}^d$ and where $\widehat{\psi}$ denotes the $L^2(\mathbb{R}^d)$-Fourier transform of ψ. Recall that for $\psi, \widehat{\psi} \in L^1 \cap L^2(\mathbb{R}^d)$, we have $\mathbb{R}^d$-$a.s.$ that

$$\widehat{\psi}(p) = \int_{\mathbb{R}^d} dx\, e^{-2\pi i p x}\psi(x), \qquad \psi(x) = \int_{\mathbb{R}^d} dp\, e^{2\pi i p x}\widehat{\psi}(p). \tag{1.18}$$

Moreover, the map $L^2(\mathbb{R}^d) \ni \psi \mapsto \widehat{\psi} \in L^2(\mathbb{R}^d)$ is unitary and it maps $\mathcal{S}(\mathbb{R}^d)$ to itself.

It is straightforward to verify that the above choices imply that, on a suitable dense domain $D \subset L^2(\mathbb{R}^d)$ that is preserved by all the X_i, P_j (e.g. $D = \mathcal{S}(\mathbb{R}^d)$), we have that

$$[X_i, X_j] = 0, \qquad [P_i, P_j] = 0, \qquad [X_i, P_j] = i\hbar\delta_{ij}.$$

The relations (1.17) are called canonical commutation relations (CCR) and they have a fundamental physical consequence, called the Heisenberg uncertainty principle. For the following bounds, fix e.g. some normalized state $\psi \in \mathcal{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$. Recalling the identity (1.14), notice that (1.17) and Cauchy-Schwarz imply that

$$\begin{aligned}
\frac{\hbar}{2} &= \frac{1}{2}|\langle\psi, [X_j - \langle\psi, X_j\psi\rangle, P_j - \langle\psi, P_j\psi\rangle]\psi\rangle| \\
&\leq \big\langle\psi, (X_j - \langle\psi, X_j\psi\rangle)^2\psi\big\rangle^{\frac{1}{2}}\ \big\langle\psi, (P_j - \langle\psi, P_j\psi\rangle)^2\psi\big\rangle^{\frac{1}{2}} \\
&= \big(\langle\psi, X_j^2\psi\rangle - \langle\psi, X_j\psi\rangle^2\big)^{\frac{1}{2}}\ \big(\langle\psi, P_j^2\psi\rangle - \langle\psi, P_j\psi\rangle^2\big)^{\frac{1}{2}}.
\end{aligned}$$

In other words, the variances of the position and momentum distributions of a particle cannot simultaneously be small, at the scale $\hbar \ll 1$. This means that if the position distribution of a particle is localized (the particle position is known), its momentum distribution is necessarily rather wide (the particle momentum is completely uncertain, because all momenta are equally likely to be measured). Obviously, this is in strong contrast to classical mechanics which does not imply such constraints. The smaller $\hbar$, the weaker the uncertainty constraints and in the semiclassical limit $\hbar \to 0$ one recovers classical behavior in the sense that the position and momentum operators commute. For an introductory discussion of basic, semiclassical results, see e.g. [5, Chapter 15].

Having set up a model for the position and momentum operators of a particle moving in $\mathbb{R}^d$, the Hamilton operator H that describes the energy of a particle of mass $m > 0$ in an external field $V : \mathbb{R}^d \to \mathbb{R}$ is, by analogy to (1.1), given by

$$H = \frac{|i\hbar\nabla|^2}{2m} + V(x) = \frac{\hbar^2}{2m}(-\Delta) + V(x).$$

Here and in the following, we write $V(x)$ for the multiplication operator $\psi \mapsto V\psi$ in $L^2(\mathbb{R}^d)$. Note that H is a well-defined, symmetric operator on $\mathcal{S}(\mathbb{R}^d)$. Under suitable conditions on V, it is essentially self-adjoint and thus admits a unique self-adjoint extension. See e.g. [15] for a thorough discussion of self-adjointness of Schrödinger operators.

It should be obvious how the above considerations can be generalized to formulate a quantum model of a particle moving in some subregion $U \subset \mathbb{R}^d$. Similarly, we can generalize Examples 1.2 and 1.3. If we want to describe a system of N particles of masses $m_j > 0$ that move in $\mathbb{R}^d$ and interact through a potential $V : \mathbb{R}^d \to \mathbb{R}$, a possible state space is $\mathcal{H} = L^2(\mathbb{R}^{dN}, \mathcal{B}(\mathbb{R}^{dN}), dx) = \bigotimes_{j=1}^N L^2(\mathbb{R}^d)$ and the Hamiltonian takes the form

$$H = \sum_{j=1}^N \frac{\hbar^2}{2m_j}(-\Delta_{x_j}) + \sum_{1\le i<j\le N} V(x_i - x_j).$$

Similarly, if a particle moves in $\mathbb{R}^3$ in a static electromagnetic field that is generated by the electric and magnetic potentials $\Phi : \mathbb{R}^3 \to \mathbb{R}$ and, respectively, $A : \mathbb{R}^3 \to \mathbb{R}^3$, its energy is described by a suitable self-adjoint extension of the operator

$$H = \frac{1}{2m}\left|i\hbar\nabla + \frac{e}{c}A(x)\right|^2 + e\Phi(x).$$

The previous examples were obtained via canonical quantization, described at the beginning of this section. Although one can not hope for an intuitive derivation of the quantization scheme, it is quite natural to wonder whether it allows for other choices of the position and momentum operators. The following fundamental result implies that the choice is essentially unique for systems of finitely many degrees of freedom.

Theorem 1.3 (Stone-von Neumann)**.** *Let $(U_s)_{s\in\mathbb{R}^d}, (V_t)_{t\in\mathbb{R}^d}$ be two strongly continuous families of unitary operators on a separable Hilbert space $\mathcal{H}$ that satisfy the integrated canonical commutation relations (ICCR)*

$$U_sU_t = U_{s+t}, \;\; V_sV_t = V_{s+t}, \;\; V_tU_s = e^{is\cdot t}U_sV_t, \; \forall\, s,t \in \mathbb{R}^d. \tag{1.19}$$

Assume, moreover, that $\mathcal{H}$ has no non-trivial closed subspace D (i.e., $D \neq \{0\}$ and $D \neq \mathcal{H}$) that is invariant under $(U_s)_{s\in\mathbb{R}^d}$ and $(V_t)_{t\in\mathbb{R}^d}$ and call such a pair an irreducible realization of the ICCR. Then, any two non-trivial irreducible realizations of the ICCR are unitarily equivalent.

Problem 1.2. *Suppose that $A = A^*, B = B^* \in \mathcal{L}(\mathcal{H})$. Show that A and B satisfy the commutation relation $[A, B] = i$ if and only if $e^{isB}e^{itA} = e^{its}e^{itA}e^{isB}$ for every $s, t \in \mathbb{R}$.*

Remark 1.5. *The previous problem motivates why we can view the relations* (1.19) *as an integrated version of* (1.17) *for the strongly continuous families of unitary maps*

$$U_s = e^{is\cdot X},\ V_t = e^{it\cdot P/\hbar},\ \forall\ s,t \in \mathbb{R}^d.$$

The integrated variant (1.19) *of the CCR is easier to analyze, since all involved operators are bounded so that no domain issues (due to the unboundedness of X and P) arise.*

Proof of Theorem 1.3. We prove the theorem for $d = 1$. The general case follows along the same lines and is left as an *exercise* for the interested reader.

In order to prove the theorem, it suffices to consider the case $\mathcal{H} = L^2(\mathbb{R})$ (any two separable Hilbert spaces are unitarily equivalent) and to show that every irreducible realization $(U'_s)_{s\in\mathbb{R}}, (V'_t)_{t\in\mathbb{R}}$ of (1.19) is equivalent to the canonical one, defined by

$$U_s = e^{isX},\ V_t = e^{itP/\hbar},\ \forall\ s,t \in \mathbb{R}.$$

To see that this defines indeed an irreducible realization of the integrated canonical commutation relations (1.19), note first of all that for every $\varphi, \psi \in L^2(\mathbb{R})$, we have that

$$\langle \varphi, V_t\psi\rangle = \int_{\mathbb{R}} dp\, \overline{\widehat{\varphi}}(p) e^{2\pi i t p}\widehat{\psi}(p) = \int_{\mathbb{R}} dx\, \overline{\varphi}(x)\psi(x+t) = \langle \varphi, \psi(\cdot + t)\rangle,$$

so that $(V_t\psi)(\cdot) = \psi(\cdot + t)$, for every $t \in \mathbb{R}$. Based on this observation, one readily verifies the identities (1.19), the non-trivial case being

$$(V_tU_s\psi)(\cdot) = V_t\big(e^{is\cdot}\psi(\cdot)\big) = e^{ist}e^{is\cdot}\psi(\cdot + t) = e^{ist}(U_sV_t\psi)(\cdot),\ \forall\,\psi \in L^2(\mathbb{R}).$$

Furthermore, the pair $(U_s)_{s\in\mathbb{R}^d}$ and $(V_t)_{t\in\mathbb{R}^d}$ is irreducible. Indeed, suppose there is a non-trivial closed subspace $D \subset L^2(\mathbb{R})$ that is both invariant under all maps U_t and all maps V_s, for $s, t \in \mathbb{R}$. Then we can pick a normalized element $\varphi \in D$ and consider a vector $\psi \in D^\perp$ in the orthogonal complement of D. By the invariance of D under all maps U_s and V_t, we find that for every $\zeta \in \mathcal{S}(\mathbb{R})$, it holds true that

$$\int_{\mathbb{R}} dx\, \widehat{\zeta}(x)\overline{\psi}(x)\varphi(x) = \int_{\mathbb{R}} dp\, \zeta(p)\bigg(\int_{\mathbb{R}} dx\, \overline{\psi}(x)e^{-2\pi i x p}\varphi(x)\bigg) = 0.$$

A standard density argument implies that $|\psi(x)\varphi(x)| = 0$ for *a.e.* $x \in \mathbb{R}$. Since $V_t\varphi \in D$, the same argument implies $|\psi(x)\varphi(x+t)| = 0$ *a.s.* in $\mathbb{R}$, for every $t \in \mathbb{R}$. But then

$$0 = \int_{\mathbb{R}^2} dxdy\, |\psi(x)\varphi(x-y)|\zeta(y) = \int_{\mathbb{R}} dx\, |\psi(x)|\big(|\varphi| * \zeta\big)(x)$$

for every $\zeta \in \mathcal{S}(\mathbb{R})$. Choosing a strictly positive $\zeta \in \mathcal{S}(\mathbb{R})$, so that $\zeta * |\varphi| > 0$ everywhere since $\|\varphi\| = 1$, we conclude $|\psi(x)| = 0$ for *a.e.* $x \in \mathbb{R}$ and thus $\psi = 0 \in L^2(\mathbb{R})$. This means that $D = \overline{D} = \mathcal{H}$ and it proves that $(U_s)_{s\in\mathbb{R}}, (V_t)_{t\in\mathbb{R}}$ is irreducible. We leave it as an *exercise* to show that a simple generalization of the previous analysis implies that the realization of the ICCR from Remark 1.5 is irreducible, for every $d \in \mathbb{N}$.

In order to proceed with the proof, it turns out convenient to set

$$W_{s,t} = e^{\frac{ist}{2}} U_t V_s = e^{-\frac{its}{2}} V_s U_t, \quad W'_{s,t} = e^{\frac{ist}{2}} U'_t V'_s = e^{-\frac{its}{2}} V'_s U'_t,$$

so that our goal is equivalent to finding a unitary map $\mathcal{U} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ that satisfies

$$\mathcal{U} W_{s,t} \mathcal{U}^* = W'_{s,t}, \ \forall\, s,t \in \mathbb{R}. \tag{1.20}$$

Now, observe that for every normalized $\psi, \psi' \in L^2(\mathbb{R})$, the linear spaces

$$D_\psi = \mathrm{span}\big\{W_{s,t}\psi : s,t \in \mathbb{R}\big\}, \qquad D'_{\psi'} = \mathrm{span}\big\{W'_{s,t}\psi' : s,t \in \mathbb{R}\big\}$$

are dense in $L^2(\mathbb{R})$, because they are invariant under the irreducible realizations of the ICCR. To construct $\mathcal{U}$, we look for two normalized elements $\psi, \psi' \in L^2(\mathbb{R})$ that satisfy

$$\langle \psi, W_{s,t}\psi\rangle = \langle \psi', W'_{s,t}\psi'\rangle, \ \forall\, s,t \in \mathbb{R}. \tag{1.21}$$

Once such elements ψ, ψ' are found, we conclude from (1.19) that

$$\begin{aligned}
\Big\langle \sum_{j=1}^n \lambda_j W_{s_j,t_j}\psi, \sum_{k=1}^m \mu_k W_{s_k,t_k}\psi \Big\rangle &= \sum_{j=1}^n \sum_{k=1}^m \overline{\lambda}_k \mu_k \langle \psi, W^*_{s_j,t_j} W_{s_k,t_k}\psi\rangle \\
&= \sum_{j=1}^n \sum_{k=1}^m \overline{\lambda}_k \mu_k \langle \psi, W_{-s_j,-t_j} W_{s_k,t_k}\psi\rangle \\
&= \sum_{j=1}^n \sum_{k=1}^m \overline{\lambda}_k \mu_k e^{-\frac{i}{2}(s_j t_k - s_k t_j)} \langle \psi, W_{s_k-s_j,t_k-t_j}\psi\rangle \\
&= \Big\langle \sum_{j=1}^n \lambda_j W'_{s_j,t_j}\psi', \sum_{k=1}^m \mu_k W'_{s_k,t_k}\psi' \Big\rangle.
\end{aligned}$$

This and the density of $D_\psi, D_{\psi'}$ in $L^2(\mathbb{R})$ imply that $\mathcal{U} : D_\psi \mapsto D_{\psi'}$, defined by

$$\mathcal{U}\Big(\sum_{k=1}^n \lambda_k W_{s_k,t_k}\psi\Big) = \sum_{k=1}^n \lambda_k W'_{s_k,t_k}\psi', \ \forall\, \lambda_k \in \mathbb{C}, s_k, t_k \in \mathbb{R}, k \in \mathbb{N},$$

is an isometry and extends to a unitary map that satisfies, by definition, (1.20).

It thus remains to find two vectors $\psi, \psi' \in L^2(\mathbb{R})$, $\|\psi\| = \|\psi'\| = 1$, that satisfy (1.21). Finding such elements is not obvious and to explain the key idea, let us rewrite

$$\langle \psi, W_{s,t}\psi\rangle = \mathrm{tr}\, |\psi\rangle\langle\psi| W_{s,t}.$$

To find suitable ψ and ψ', it would be useful if the trace on the r.h.s. would only involve the operators $(W_{s,t})_{s,t\in\mathbb{R}}$ and this suggests to look for averages of the form

$$|\psi\rangle\langle\psi| = \int_{\mathbb{R}^2} ds dt\, \varphi(s,t) W_{s,t} \tag{1.22}$$

for some $\varphi : \mathbb{R}^2 \to \mathbb{R}$, say $\varphi \in \mathcal{S}(\mathbb{R}^2)$. In particular, this would imply that

$$\psi(x) \int_{\mathbb{R}} ds\, \overline{\psi}(s) f(s) = \int_{\mathbb{R}^2} dsdt\, e^{\frac{i}{2}st + itx} \varphi(s,t) f(x+s) = \int_{\mathbb{R}^2} dsdt\, e^{\frac{i}{2}(s+x)t} \varphi(s-x,t) f(s)$$

for every $f \in \mathcal{S}(\mathbb{R})$ and for *a.e.* $x \in \mathbb{R}$. Recalling the well-known fact that the Fourier transform of a Gaussian density remains a Gaussian, that is

$$\int_{\mathbb{R}} dx\, e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}},\ \forall\, a \in (0,\infty), b \in \mathbb{C},$$

it is left as a *exercise* to verify that (1.22) is indeed satisfied for the explicit choice

$$x \mapsto \psi(x) = \pi^{-\frac{1}{4}} e^{-\frac{1}{2}x^2} \in \mathcal{S}(\mathbb{R}), \quad (s,t) \mapsto \varphi(s,t) = \frac{1}{2\pi} e^{-\frac{1}{2}(s^2+t^2)} \in \mathcal{S}(\mathbb{R}^2). \tag{1.23}$$

Similarly, it is left as an elementary *exercise* to verify that for the choice (1.23), we have

$$\langle \psi, W_{s,t}\psi \rangle = e^{-\frac{1}{4}(s^2+t^2)},\ \forall\, s,t \in \mathbb{R}.$$

Now, to find a corresponding state $\psi' \in L^2(\mathbb{R})$, consider by analogy to (1.22) the operator

$$P' = \int_{\mathbb{R}^2} dsdt\, \varphi(s,t) W'_{s,t} \in \mathcal{L}(L^2(\mathbb{R}))$$

whose boundedness is a direct consequence of the integrability of φ and the unitarity of U'_s, V'_t. Recalling that $(W'_{s,t})^* = W'_{-s,-t}$ and that $\varphi(s,t) = \varphi(-s,-t)$, we find that $P' = (P')^*$ is self-adjoint. Motivated by (1.22), we may expect that $P' \neq 0$ is a non-trivial projection. To see this, assume first by contradiction that $P' = 0$. Then, using

$$W'_{-s_1,-t_1} W'_{s_2,t_2} W'_{s_1,t_1} = e^{i(s_1 t_2 - s_2 t_1)} W'_{s_2,t_2},$$

we conclude that for every $\zeta \in \mathcal{S}(\mathbb{R}^2)$, we also have that

$$\begin{aligned} \int_{\mathbb{R}^2} dsdt\, \zeta(s,t)\varphi(s,t) W'_{s,t} &= \frac{1}{4\pi^2} \int_{\mathbb{R}^4} ds_1 dt_1 ds_2 dt_2\, \widehat{\zeta}(s_1,t_1) e^{is_1 s_2 + it_1 t_2} \varphi(s_2,t_2) W'_{s_2,t_2} \\ &= -\frac{1}{4\pi^2} \int_{\mathbb{R}^2} dsdt\, \widehat{\zeta}(-t,s) W'_{-s,-t} P' W'_{s,t} = 0. \end{aligned}$$

But $\varphi > 0$ in $\mathbb{R}^2$, so pick e.g. a non-negative bump function $\zeta_\epsilon \in C_c^\infty(B_\epsilon(0))$ which satisfies $\int_{\mathbb{R}^2} dsdt\, \zeta_\epsilon(s,t)\varphi(s,t) = 1$. Moreover, pick a normalized $\tau' \in L^2(\mathbb{R}^2)$. Then, by the strong continuity of the map $(s,t) \mapsto W_{s,t}$, we find for $\epsilon > 0$ small enough that

$$\Big\| \int_{\mathbb{R}^2} dsdt\, \zeta_\epsilon(s,t)\varphi(s,t) W_{s,t}\tau' - \tau' \Big\| \leq \sup_{|s'|,|t'| \leq \epsilon} \|W_{s,t}\tau' - \tau'\| \int_{\mathbb{R}^2} dsdt\, \zeta_\epsilon(s,t)\varphi(s,t) < 1$$

and thus $\int_{\mathbb{R}^2} dsdt\, \zeta_\epsilon(s,t)\varphi(s,t)W_{s,t} \neq 0$, a contradiction. This shows that $P' \neq 0$. To see that $(P')^2 = P'$ is a projection, we compute explicitly that

$$\begin{aligned}
(P')^2 &= \int_{\mathbb{R}^4} ds_1dt_1ds_2dt_2\, \varphi(s_1,t_1)\varphi(s_2,t_2)W'_{s_1,t_1}W'_{s_2,t_2} \\
&= \int_{\mathbb{R}^4} ds_1dt_1ds_2dt_2\, \varphi(s_1,t_1)\varphi(s_2,t_2)e^{\frac{i}{2}(s_1t_2-s_2t_1)}W'_{s_1+s_2,t_1+t_2} \\
&= \int_{\mathbb{R}^4} dsdtds'dt'\, \varphi(s-s',t-t')\varphi(s',t')e^{\frac{i}{2}((s-s')t'-s'(t-t'))}W'_{s,t} \\
&= \int_{\mathbb{R}^2} dsdt\, \varphi(s,t)W'_{s,t}\bigg(\int_{\mathbb{R}^2} \frac{ds'dt'}{2\pi}e^{-(s')^2-(t')^2+\frac{1}{2}(s-it)s'+\frac{i}{2}(s-it)t'}\bigg) = P'.
\end{aligned}$$

More generally, motivated once again by (1.22), a similar computation (*exercise*) yields

$$P'W'_{s,t}P' = P'e^{-\frac{1}{4}(s^2+t^2)}P' = e^{-\frac{1}{4}(s^2+t^2)}P'.$$

Given these observations, we conclude the proof by finding some normalized vector $\psi' = P(\varphi')$, $\|\psi'\| = 1$, for some $0 \neq \varphi' \in L^2(\mathbb{R})$, so that $P'\psi' = \psi'$ and thus

$$\langle \psi', W'_{s,t}\psi'\rangle = \langle \psi', P'W'_{s,t}P'\psi'\rangle = e^{-\frac{1}{4}(s^2+t^2)} = \langle \psi, W_{s,t}\psi\rangle,\ \forall\, s,t \in \mathbb{R}.$$

This proves (1.21) which, as explained earlier, implies the unitary equivalence (1.20). □

The canonical quantization scheme has been quite important historically, because it leads to natural candidates for quantum variants of well-known classical models in a relatively straightforward way. This applies in particular to the development of quantum field theories, which describe quantum versions of classical fields. A classical field Φ, such as the electromagnetic field, can be understood as a collection of observables $(\Phi_x)_{x\in U\subset\mathbb{R}^3}$ labelled by the space coordinates (Φ is typically tensor-valued, e.g. scalar- or vector-valued). In this sense, it consists of an infinite number of degrees of freedom. When applying the canonical quantization scheme to it, one associates, loosely speaking, to each observable Φ_x a pair of classical canonical variables that satisfy (1.8) (like position and momentum in case of a single particle) and replaces it with a corresponding pair of operators that satisfy the canonical commutation relations (1.17). Theorem 1.3 shows that in case of finitely many degrees of freedom (including e.g. all interacting many-body systems of finitely many particles), this procedure yields a quantum model which is essentially unique. In the remaining part of this section we explain that, in contrast to the finite particle case, for an infinite number of degrees of freedom, uniqueness is lost.

For a simple motivation that suggests the loss of uniqueness for infinitely many variables, consider the pair $(U_s)_{s\in\mathbb{R}^d}, (V_t)_{t\in\mathbb{R}^d}$ as in Remark 1.5 and set for $\sigma > 0$

$$U'_s = e^{is\cdot X/\sigma} = U_{s/\sigma},\ V'_t = e^{i\sigma t\cdot P/\hbar} = V_{\sigma t},\ \forall\, s,t \in \mathbb{R}^d.$$

With analogous notation as in the proof of Theorem 1.3, a basic change of variables shows that $\mathcal{U}W_{s,t}\mathcal{U}^* = W'_{s,t}$ for $\mathcal{U} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, defined by

$$(\mathcal{U}\varphi)(\cdot) = \sigma^{-\frac{d}{2}}\varphi(\cdot/\sigma).$$

In other words, rescaling by $\sigma > 0$ yields a unitarily equivalent, irreducible realization of the ICCR (1.19), for every $d \in \mathbb{N}$. Heuristically, we may expect the loss of unitary equivalence if $d = \infty$, in which case the above expression for $\mathcal{U}$ does no longer make sense. To set up this observation rigorously, we first need to define a reasonable version of $L^2\big(\prod_{j\in\mathbb{N}} \mathbb{R}\big)$ that generalizes in a suitable sense the Lebesgue spaces $L^2(\mathbb{R}^d)$, for $d \in \mathbb{N}$. To this end, observe that for every $d \in \mathbb{N}$, the map

$$\mathcal{V}_d : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu^{\otimes d}),\ \mu(dx) = \frac{1}{\sqrt{\pi}} e^{-x^2} dx,\ (\mathcal{V}_d\varphi)(\cdot) = \pi^{\frac{1}{4}} e^{\frac{1}{2}|\cdot|^2} \varphi(\cdot)$$

is a unitary map and that $\mu^{\otimes d}$ is a Gaussian probability measure on $\mathbb{R}^d$.

Problem 1.3. *Let $d \in \mathbb{N}$ and set $U'_s = \mathcal{V}_d U_s \mathcal{V}_d^*$, $V'_t = \mathcal{V}_d V_t \mathcal{V}_d^*$ and $W'_{s,t} = \mathcal{V}_d e^{\frac{i}{2} s\cdot t} U_t V_s \mathcal{V}_d^*$ as well as $x \mapsto \psi'(x) = 1 \in L^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu^{\otimes d})$. Prove that*

$$\begin{aligned} (U'_s\varphi)(x) &= e^{is\cdot x}\varphi(x), \\ (V'_t\varphi)(x) &= e^{-t\cdot x - \frac{1}{2}|t|^2}\varphi(x+t), \\ \langle \psi', W'_{s,t}\psi'\rangle &= e^{-\frac{1}{4}(|s|^2+|t|^2)}, \end{aligned}$$

for every $s, t \in \mathbb{R}^d$, $\varphi \in L^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu^{\otimes d})$ and a.e. $x \in \mathbb{R}^d$.

Problem 1.3 suggests that instead of considering the canonical models in the different spaces $L^2(\mathbb{R}^d)$ for each fixed $d \in \mathbb{N}$, we can set up a model with an infinite number of degrees of freedom into which one can naturally embed all finite particle models. More precisely, applying the Kolmogorov extension theorem, see e.g. [3, Chapter 10], we define μ_∞ as the unique (regular Borel) probability measure on $(\Omega, \mathcal{B}(\Omega))$ for $\Omega = \prod_{j=1}^\infty \mathbb{R}$ s.t.

$$\int_\Omega \mu_\infty(dx) f(x) = \int_\Omega \mu_\infty(dx) f(x_1, x_2, \ldots, x_d) = \int_{\mathbb{R}^d} \mu^{\otimes d}(dx) f(x_1, x_2, \ldots, x_d)$$

for every integrable $f : \Omega \to \mathbb{C}$ that only depends on the first $d \in \mathbb{N}$ coordinates in Ω. Then, identifying for $d \in \mathbb{N}$ an element $\varphi \in L^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with the function $x \mapsto (\varphi \otimes 1)(x) = \varphi(x_1, \ldots, x_d) \in L^2(\Omega, \mathcal{B}(\Omega), \mu_\infty)$, we have the isometric embeddings

$$L^2(\mu^{\otimes d}) \hookrightarrow L^2(\mu^{\otimes d+1}) \hookrightarrow L^2(\mu^{\otimes d+2}) \hookrightarrow \ldots \hookrightarrow L^2(\mu_\infty)$$

for $L^2(\mu^{\otimes d}) = L^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu^{\otimes d})$ and $L^2(\mu_\infty) = L^2(\Omega, \mathcal{B}(\Omega), \mu_\infty)$. Moreover, since integrable functions in $(\Omega, \mathcal{B}(\Omega), \mu_\infty)$ can be approximated up to arbitrarily small errors by linear combinations of characteristic functions of open sets in Ω (equipped with the product topology admitting a countable basis for its topology), we verify that

$$L^2_{<\infty} = \bigcup_{d=1}^\infty L^2(\mu^{\otimes d}) \subset L^2(\mu_\infty)$$

is dense in $L^2(\mu_\infty)$. As a consequence, the canonical projection $\Pi_d : L^2(\mu_\infty) \to L^2(\mu^{\otimes d})$ satisfies $\lim_{d\to\infty} \|\psi - \Pi_d\psi\| = 0$, for every $\psi \in L^2(\mu_\infty)$. We leave the careful proof of the last two statements as an *exercise*. In view of Problem 1.3, we then define

$$(U_s\varphi)(x) = e^{is\cdot x}\varphi(x),\ (V_t\varphi)(x) = e^{-t\cdot x - \frac{1}{2}|t|^2}\varphi(x+t),\ (W_{s,t}\varphi)(x) = e^{\frac{is\cdot t}{2}}(U_tV_s\varphi)(x)$$

for every $s,t \in \mathbb{R}^{<\infty} = \bigcup_{d\in\mathbb{N}} \mathbb{R}^d \hookrightarrow \Omega$ and $\varphi \in L^2(\mu_\infty)$. Notice that, heuristically, the pair $(U_s)_{s\in\mathbb{R}^{<\infty}}, (V_t)_{t\in\mathbb{R}^{<\infty}}$ corresponds to a countably infinite sequence of canonical position and momentum operators $(X_j, P_j)_{j\in\mathbb{N}}$ that satisfy the CCR (1.17).

As in the proof of Theorem 1.3, we obtain for every $s,t \in \mathbb{R}^{<\infty}$ that

$$U_sU_t = U_{s+t},\ \ V_sV_t = V_{s+t},\ V_tU_s = e^{is\cdot t}U_sV_t \tag{1.24}$$

and that

$$\langle \psi, W_{s,t}\psi\rangle = e^{-\frac{1}{4}(|s|^2+|t|^2)}.$$

Here, $x \mapsto \psi(x) = 1 \in L^2(\mu_\infty)$ with $\|\psi\| = 1$. Since $(U_s)_{s\in\mathbb{R}^{<\infty}}$, $(V_t)_{t\in\mathbb{R}^{<\infty}}$ define isometries on $L^2_{<\infty}$, they must be unitary on $L^2(\mu_\infty)$ (alternatively, unitarity can be checked explicitly based on (1.24)). The pair $(U_s)_{s\in\mathbb{R}^{<\infty}}, (V_t)_{t\in\mathbb{R}^{<\infty}}$ is also irreducible. Indeed, assuming $D \subset L^2(\mu_\infty)$ to be a closed, non-empty invariant subspace, we can pick $\varphi \in D, \|\varphi\| = 1$ and $\psi \in D^\perp$, as in the proof of Theorem 1.3. As before

$$0 = \int \mu_\infty(dx)\, \zeta(x)\overline{\psi}(x)V_t\varphi(x)$$

for every $\zeta \in \mathcal{S}(\mathbb{R}^d) \hookrightarrow L^2_{<\infty}$ and every $t \in \mathbb{R}^{<\infty}$ so that $|\psi(x)e^{-t\cdot x-|t|^2/2}\varphi(x+t)| = 0$ and hence $|\psi(x)\varphi(x+t)| = 0$ *a.s.* in Ω. Now, note that for $d \in \mathbb{N}$ sufficiently large and $0 \neq \xi \in L^2(d\mu_\infty)$ we have explicitly

$$(\Pi_d\xi)(x,X) = \frac{\int_{\Omega_{>d}} \mu^{\otimes>d}(dX)\, \xi(x,X)}{\int_{\mathbb{R}^d} \mu^{\otimes d}(dx)\big|\int_{\Omega_{>d}} \mu^{\otimes>d}(dX)\, \xi(x,X)\big|^2}$$

for *a.e.* $x \in \mathbb{R}^d$, $\Omega_{>d} = \prod_{j=d+1}^\infty \mathbb{R}$ and $\mu^{\otimes>d}$ defined such that $\mu_\infty = \mu^{\otimes d} \otimes \mu^{\otimes>d}$ (based on another application of Kolmogorov's theorem). If we assume by contradiction that $\psi \neq 0$, Cauchy-Schwarz implies for suitable $C > 0$ (by normalization) that

$$\begin{aligned}
|\langle \Pi_d\psi, \Pi_dU_sV_t\varphi\rangle| &\leq C\int_{\mathbb{R}^d} \mu^{\otimes d}(dx)\Big|\int_{\Omega_{>d}} \mu^{\otimes>d}(dX)\, \psi(x,X)\int_{\Omega_{>d}} \mu^{\otimes>d}(dY)\, V_t\varphi(x,Y)\Big| \\
&\leq C\int_{\mathbb{R}^d} \mu^{\otimes d}(dx)\, \langle|\psi|(x,\cdot),1\rangle_{L^2(\mu^{\otimes>d})}\big\langle 1, |U_sV_t\varphi|(x)\big\rangle_{L^2(\mu^{\otimes>d})} \\
&\leq C\int_\Omega \mu_\infty(dx)\, e^{-t\cdot x-\frac{1}{2}|t|^2}|\psi(x)\varphi(x+t)| = 0
\end{aligned}$$

for every $t \in \mathbb{R}^{<\infty}$. But now arguing as in the finite dimensional case (using that U_s and V_t commute with Π_d if $s,t \in \mathbb{R}^d$), we conclude that $\Pi_d\psi = 0 \in L^2(\mu^{\otimes d})$ for every $d \in \mathbb{N}$. This shows $\psi = 0 \in L^2(\mu_\infty)$, a contradiction, and the irreducibility follows.

Now, note that for every other irreducible pair $(U'_s)_{s\in\mathbb{R}^{<\infty}}, (V'_t)_{s\in\mathbb{R}^{<\infty}}$ that satisfies (1.24) and that is unitarily equivalent to $(U_s)_{s\in\mathbb{R}^{<\infty}}, (V_t)_{t\in\mathbb{R}^{<\infty}}$ (via some map $\mathcal{U}$, say), there must exist a normalized state $\psi'(=\mathcal{U}\psi) \in L^2(\mu_\infty)$ such that

$$\langle \psi', W'_{s,t}\psi'\rangle = e^{-\frac{1}{4}(|s|^2+|t|^2)}$$

for every $s, t \in \mathbb{R}^{<\infty}$. Our previous considerations then suggest to fix $\sigma > 0$ and to set

$$U'_s = U_{s/\sigma},\ V'_t = V_{\sigma t}.$$

Clearly, $(U'_s)_{s\in\mathbb{R}^{<\infty}}, (V'_t)_{s\in\mathbb{R}^{<\infty}}$ satisfies (1.24) and is irreducible. But this pair can not be unitarily equivalent to $(U_s)_{s\in\mathbb{R}^{<\infty}}, (V_t)_{t\in\mathbb{R}^{<\infty}}$. Otherwise, there exists some $\psi' \in L^2(\mu_\infty)$, $\|\psi'\| = 1$, such that $\langle \psi', V'_{t_d}\psi'\rangle = e^{-\frac{1}{4}}$ for every $t_d = (0, \ldots, 0, 1, 0, \ldots)$ (the non-zero entry being at the d-th slot) and every $d \in \mathbb{N}$. But if $\psi'_d = \Pi_d\psi'$ is the projection of ψ' onto $L^2(\mu^{\otimes d}) \hookrightarrow L^2(\mu_\infty)$, we infer from a small variation (*exercise*) of Problem 1.3 that

$$\langle \psi'_d, V'_{t_{d+1}}\psi'_d\rangle = e^{-\frac{\sigma^2}{4}}\|\psi'_d\| \to e^{-\frac{\sigma^2}{4}}$$

as $d \to \infty$. On the other hand, $\langle \psi', V'_{t_{d+1}}\psi'\rangle = e^{-\frac{1}{4}}$ and strong continuity implies

$$|\langle \psi', V'_{t_{d+1}}\psi'\rangle - \langle \psi'_d, V'_{t_{d+1}}\psi'_d\rangle\ | \to 0$$

as $d \to \infty$. Thus, we obtain for every $\sigma \neq 1$ the contradiction that $e^{-\frac{\sigma^2}{4}} = e^{-\frac{1}{4}}$. In these cases, the pair $(U'_s)_{s\in\mathbb{R}^{<\infty}}, (V'_t)_{s\in\mathbb{R}^{<\infty}}$ is not unitarily equivalent to $(U_s)_{s\in\mathbb{R}^{<\infty}}, (V_t)_{t\in\mathbb{R}^{<\infty}}$.

The previous counter examples to uniqueness rely on a simple scaling argument so that, heuristically, also the corresponding position and momentum operators are merely rescaled versions of the canonical ones. For a more detailed discussion including a characterization of unitary equivalence in the setting of $L^2(\mu_\infty)$, see [22, Section C.4].

1.2.2 Symmetries and Their Representations

Following [24, Section 2.2], a symmetry transformation is a transformation that does not change the physics of a system, but rather our point of view of how to describe it. Basic examples include Euclidean transformations such as rotations that describe the change of the coordinate system used to measure e.g. the position of a particle.

Suppose now that we consider a quantum system whose state is described by a normalized wave function $\psi \in \mathcal{H}$ in a Hilbert space $\mathcal{H}$. The probabilistic interpretation of quantum mechanics, summarized in (1.12), implies that all probabilities one may potentially be interested in reduce to the computation of squares of inner products

$$|\langle \varphi, \psi\rangle|^2 \quad \text{for} \quad \varphi, \psi \in \mathcal{H},\ \|\varphi\| = \|\psi\| = 1. \tag{1.25}$$

Assume that T describes a symmetry transformation. What are reasonable conditions to impose on T? Since ψ and $e^{i\omega}\psi$, for $\omega \in [0, 2\pi)$, describe the same physical state, it is natural to assume that $T : \mathcal{R}_1 \to \mathcal{R}_1$ is a map on the space of unit rays. Here, we set

$$\mathcal{R}_1 = \bigcup_{\psi\in\mathcal{H},\|\psi\|=1} \{\varphi \in \mathcal{H} : \varphi \sim \psi\} = \bigcup_{\psi\in\mathcal{H},\|\psi\|=1} [\psi],$$

recalling that $\psi_1 \sim \psi_2$ if and only if $\psi_1 = e^{i\omega}\psi_2$ for some $\omega \in [0, 2\pi)$. This means that T maps one physical state to another physical state. Furthermore, T must not influence measurement outcomes which means that

$$T[\varphi] \cdot T[\psi] = [\varphi] \cdot [\psi] \tag{1.26}$$

for every $[\varphi], [\psi] \in \mathcal{R}_1$, where we set $[\varphi] \cdot [\psi] = |\langle \varphi, \psi \rangle|$. It is left as a basic *exercise* to show that (1.26) implies that T is injective. Since we can usually switch from one coordinate system to another and back, it is also reasonable to assume that T is onto.

In the sequel, we call a map $T : \mathcal{R}_1 \to \mathcal{R}_1$ with the previous properties a symmetry transformation. Notice in particular that every linear, unitary and every antilinear, antiunitary operator on $\mathcal{H}$ induces a symmetry transformation. It is a fundamental result that these operators constitute all possible symmetry transformations.

Theorem 1.4 (Wigner)**.** *Let $T : \mathcal{R}_1 \to \mathcal{R}_1$ be a symmetry transformation. Then there exists a linear, unitary or an antilinear, antiunitary operator $U : \mathcal{H} \to \mathcal{H}$ such that $[U\psi] = T[\psi]$ for every $\psi \in \mathcal{H}$, $\|\psi\| = 1$. In particular, T is equal to the symmetry transformation that is induced by U. If $\dim \mathcal{H} \geq 2$, the map $U : \mathcal{H} \to \mathcal{H}$ is unique up to multiplication by a constant of modulus one.*

Proof. We follow [2] and begin with a few preliminary remarks.

We assume throughout the proof that $\dim \mathcal{H} \geq 3$ (see Problem 1.4 for the remaining, simpler cases). Since the equivalence $\varphi \sim \psi$ for two non-zero vectors implies that $[\varphi/\|\varphi\|] = [\psi/\|\psi\|] \in \mathcal{R}_1$, we can extend $T : \mathcal{R} \to \mathcal{R}$ via $T[\psi] = \|\psi\| T[\psi/\|\psi\|]$ for every non-zero ray $\psi \in \mathcal{R} = \mathcal{H}/\sim$. Then (1.26) remains valid for every $\varphi, \psi \in \mathcal{R}$.

Next, observe that if $([\psi]_i)_{i=1}^n$ is a sequence of orthonormal rays so that $[\psi]_i \cdot [\psi]_j = \delta_{ij}$, then we have by (1.26) for every pair of representatives $\psi_i \in [\psi]_i, \psi_j \in [\psi]_j$ and $\psi_i' \in T[\psi]_i, \psi_j' \in T[\psi]_j$ that

$$|\langle \psi_i, \psi_j \rangle| \ = |\langle \psi_i', \psi_j' \rangle| = \delta_{ij} = \langle \psi_i, \psi_j \rangle = \langle \psi_i', \psi_j' \rangle.$$

As a consequence, if $\varphi = \sum_{i=1}^n \lambda_i \psi_i \, (= \sum_{i=1}^n \langle \psi_i, \varphi \rangle \psi_i)$, then we have for every $\varphi' \in T[\varphi]$

$$\begin{aligned} \Big\| \varphi' - \sum_{i=1}^n \langle \psi_i', \varphi' \rangle \psi_i' \Big\|^2 = \|\varphi'\|^2 - \sum_{i=1}^n |\langle \psi_i', \varphi' \rangle|^2 = (T[\varphi] \cdot T[\varphi]) - \sum_{i=1}^n (T[\psi_i] \cdot T[\varphi])^2 \\ = \Big\| \varphi - \sum_{i=1}^n \langle \psi_i, \varphi \rangle \psi_i \Big\|^2 = 0. \end{aligned}$$

In other words, $\varphi' = \sum_{i=1}^n \lambda_i' \psi_i'$ for constants $|\lambda_i'| = |\lambda_i|$ for each $i \in \{1, \ldots, n\}$. Notice that this looks already close to linearity. Loosely speaking, our goal is to make sure that the transformed phases λ_i' can be chosen in a consistent (linear or antilinear) way.

Now, let's begin to construct the map U. This is done in three main steps. In the first step, we have some freedom how to set the direction of U on a fixed unit vector. To be more precise, let $[e] \in \mathcal{R}_1$ and choose some $e \in [e]$, $e' \in T[e] \in \mathcal{R}_1$. Then we may set

$$Ue = e'. \tag{1.27}$$

In the second step, we extend U to $\{e\}^\perp$ in a way that is consistent with (1.27) and with the statement of the theorem. To this end, consider $\psi = e + \varphi$ for some $0 \neq \varphi \in \{e\}^\perp$. Then, we can write equivalently $\psi = \lambda_1 e + \lambda_2 e_\varphi$ for the normalized

vector $e_\varphi = \varphi/\|\varphi\| \in \{e\}^\perp$, $\lambda_1 = 1$ and $\lambda_2 = \|\varphi\|$. Fixing some $e'_\varphi \in T[e_\varphi]$, we know based on the preliminary remarks that for every $\psi' \in T[\psi]$, we have that

$$\psi' = \lambda'_1 e' + \lambda'_2 e'_\varphi$$

for some $\lambda'_1, \lambda'_2 \in \mathbb{C}$ with $|\lambda'_1| = 1$, $|\lambda'_2| = \|\varphi\|$. In particular, by orthogonality of e' and e'_φ, there is one and only one element $\psi' = e' + \lambda'_2 e'_\varphi \in T[\psi]$ such that $\lambda'_1 = 1$, $|\lambda'_2| = \|\varphi\|$. Based on this specific choice (which determines λ'_2), we define

$$U_\perp \varphi = \lambda'_2 e'_\varphi \in T[e_\varphi], \;\; U(e+\varphi) = e' + U_\perp \varphi \in T[e+\varphi], \;\; U\varphi = U_\perp \varphi \in T[e_\varphi]. \quad (1.28)$$

This implies in particular the additivity $U(e+\varphi) = Ue + U\varphi$ for every $\varphi \in \{e\}^\perp$.

Next, we analyze the map $U_\perp$ in detail. Our goal is to verify that $U_\perp$ defines a linear or an antilinear map on $\{e\}^\perp$. To this end, pick $\varphi_1, \varphi_2 \in \{e\}^\perp$. By (1.26), the preliminary remarks and (1.28), we have that $|\langle U_\perp \varphi_1, U_\perp \varphi_2\rangle|^2 = |\langle \varphi_1, \varphi_2\rangle|^2$ as well as

$$|1 + \langle U_\perp \varphi_1, U_\perp \varphi_2\rangle|^2 = |\langle e' + U_\perp \varphi_1, e' + U_\perp \varphi_2\rangle|^2 = |\langle e + \varphi_1, e + \varphi_2\rangle|^2 = |1 + \langle \varphi_1, \varphi_2\rangle|^2.$$

By expanding the squares, this implies that $\operatorname{Re}\langle U_\perp \varphi_1, U_\perp \varphi_2\rangle = \operatorname{Re}\langle \varphi_1, \varphi_2\rangle$. In particular, if $\langle \varphi_1, \varphi_2\rangle = \operatorname{Re}\langle \varphi_1, \varphi_2\rangle$ is real, we must have $\operatorname{Im}\langle U_\perp \varphi_1, U_\perp \varphi_2\rangle = 0$ (recalling that $|\langle U_\perp \varphi_1, U_\perp \varphi_2\rangle| = |\langle \varphi_1, \varphi_2\rangle|$) so that in this case

$$\langle U_\perp \varphi_1, U_\perp \varphi_2\rangle = \langle \varphi_1, \varphi_2\rangle. \quad (1.29)$$

Next, let us pick some vector $\psi_2 \in \{e\}^\perp$ (at this point, we make use of the assumption that $\dim \mathcal{H} \geq 3$) that is orthonormal to $\psi_1 = e_{\varphi_1} = \varphi_1/\|\varphi_1\| \in \{e\}^\perp$ (assuming in the sequel without loss of generality that $\varphi_1 \neq 0$) and such that

$$\varphi_1 = \lambda_1 \psi_1, \quad \varphi_2 = \mu_1 \psi_1 + \mu_2 \psi_2. \quad (1.30)$$

From the preliminary remarks, we recall that if $\psi'_i = U_\perp \psi_i \in T[\psi_i]$, then ψ'_1 and ψ'_2 are still orthonormal. By definition of $U_\perp$, we have for every $\psi \in \{e\}^\perp, \nu \in \mathbb{C}$ that

$$U_\perp(\nu\psi) = \chi_\psi(\nu)\psi' \;\text{ with }\; |\chi_\psi(\nu)| = |\nu|.$$

In the sequel, let's abbreviate $\chi_i(\nu) = \chi_{\psi_i}(\nu)$. Note in particular that $\chi_i(1) = 1$, by definition of ψ'_i. Moreover, by the previous observations, we find for $\nu_1, \nu_2 \in \mathbb{C}$ that

$$\operatorname{Re}\overline{\chi_i(\nu_1)}\chi_i(\nu_2) = \operatorname{Re}\langle U_\perp(\nu_1\psi_i), U_\perp(\nu_2\psi_i)\rangle = \operatorname{Re}\langle \nu_1\psi_i, \nu_2\psi_i\rangle = \operatorname{Re}\overline{\nu}_1\nu_2$$

so that

$$\operatorname{Re}\chi_i(\nu) = \operatorname{Re}\overline{\chi_i(1)}\chi_i(\nu) = \operatorname{Re}\nu \;\text{ and }\; \chi_i(\nu) = \nu \;\text{ if } \nu \in \mathbb{R} \text{ (since } |\chi_i(\nu)| = |\nu|).$$

Now, pick an arbitrary linear combination $\psi = \nu_1\psi_1 + \nu_2\psi_2$. Then, by the preliminary remarks, we know that $U_\perp \psi = \nu'_1\psi'_1 + \nu'_2\psi'_2$ for suitable coefficients $|\nu'_i| = |\nu_i|$, $i \in \{1,2\}$. If $\nu_i = 0$, then $\nu'_i = 0 = \chi_i(\nu_i)$. Otherwise, if $\nu_i \neq 0$, then $\langle(\overline{\nu}_i)^{-1}\psi_i, \nu_i\psi_i\rangle = 1$ so that

$$\langle(\overline{\nu}_i)^{-1}\psi_i, \nu_i\psi_i\rangle = \operatorname{Re}\langle(\overline{\nu}_i)^{-1}\psi_i, \nu_i\psi_i\rangle = \langle U_\perp(\overline{\nu}_i)^{-1}\psi_i, U_\perp\nu_i\psi_i\rangle = \overline{\chi_i((\overline{\nu}_i)^{-1})}\chi_i(\nu_i).$$

At the same time, $\langle(\overline{\nu}_i)^{-1}\psi_i, \nu_i\psi_i\rangle = \langle(\overline{\nu}_i)^{-1}\psi_i, \psi\rangle$, so that by the same argument

$$\langle(\overline{\nu}_i)^{-1}\psi_i, \psi\rangle = \langle U_\perp(\overline{\nu}_i)^{-1}\psi_i, U_\perp\psi\rangle = \langle U_\perp(\overline{\nu}_i)^{-1}\psi_i, \nu_1'\psi_1' + \nu_2'\psi_2'\rangle = \overline{\chi_i((\overline{\nu}_i)^{-1})}\,\nu_i'.$$

Combining this, we find that $\nu_i' = \chi_i(\nu_i)$. In other words, for every $\nu_1, \nu_2 \in \mathbb{C}$, we have

$$U_\perp(\nu_1\psi_1 + \nu_2\psi_2) = \chi_1(\nu_1)U_\perp\psi_1 + \chi_2(\nu_2)U_\perp\psi_2.$$

Since this implies

$$\chi_{\psi_1+\psi_2}(\nu)(\psi_1' + \psi_2') = \chi_{\psi_1+\psi_2}(\nu)U_\perp(\psi_1 + \psi_2) = U_\perp(\nu\psi_1 + \nu\psi_2) = \chi_1(\nu)\psi_1' + \chi_2(\nu)\psi_2',$$

we have by orthogonality in fact that $\chi_1(\nu) = \chi_2(\nu)$ for all $\nu \in \mathbb{C}$, that is

$$U_\perp(\nu_1\psi_1 + \nu_2\psi_2) = \chi(\nu_1)U_\perp\psi_1 + \chi(\nu_2)U_\perp\psi_2.$$

where from now on we set $\chi(\nu) = \chi_1(\nu)$ for all $\nu \in \mathbb{C}$. This leads us to asking what possibilities we have for the function χ? By the previous observations, $|\chi(i)| = 1$ and $\operatorname{Re}\chi(i) = \operatorname{Re} i = 0$ so that $\chi(i) = \sigma i$ for $\sigma \in \{-1, 1\}$. For a general $\nu \in \mathbb{C}$, on the other hand, we know that $\operatorname{Re}\chi(\nu) = \operatorname{Re}\nu$ and, similarly, that

$$\operatorname{Im}\chi(\nu) = \operatorname{Re}(-i\chi(\nu)) = \sigma\operatorname{Re}(\overline{\chi(i)}\chi_1(\nu)) = \sigma\operatorname{Re}(-i\nu) = \sigma\operatorname{Im}\nu$$

so that, in conclusion, $\chi(\nu) = \nu$ for all $\nu \in \mathbb{C}$ or $\chi(\nu) = \overline{\nu}$ for all $\nu \in \mathbb{C}$. In particular, $\chi : \mathbb{C} \to \mathbb{C}$ is either a linear or an antilinear map.

Let us collect the previous observations and draw some conclusions on $U_\perp$. Recalling that $\varphi_1 = \lambda_1\psi_1$ and $\varphi_2 = \mu_1\psi_1 + \mu_2\psi_2$, the previous arguments imply that

$$\begin{aligned} U_\perp(\varphi_1 + \varphi_2) = U_\perp((\lambda_1 + \mu_1)\psi_1 + \mu_2\psi_2) &= \chi_1(\lambda_1 + \mu_1)U_\perp\psi_1 + \chi_1(\mu_2)U_\perp\psi_2 \\ &= \chi_1(\lambda_1)U_\perp\psi_1 + \chi_1(\mu_1)U_\perp\psi_1 + \chi_1(\mu_2)U_\perp\psi_2 \\ &= U_\perp\varphi_1 + U_\perp\varphi_2, \end{aligned}$$

that $U_\perp(\nu\varphi_i) = \chi(\nu)\varphi_i,\ i \in \{1, 2\}$ for $\chi(\nu) = \nu$ or $\chi(\nu) = \overline{\nu}$ and, finally, that

$$\langle U_\perp\varphi_1, U_\perp\varphi_2\rangle = \overline{\chi(\lambda_1)}\chi(\mu_1) = \chi(\overline{\lambda}_1)\chi(\mu_1) = \chi(\overline{\lambda}_1\mu) = \chi(\langle\varphi_1, \varphi_2\rangle).$$

Since $\varphi_1(\neq 0)$ and φ_2 were arbitrary vectors in $\{e\}^\perp$, this proves that $U_\perp$ extends either to a linear, isometric or to an antilinear, anti-isometric map on $\{e\}^\perp$ that is consistent with the symmetry transformation $T : \mathcal{R}_1 \to \mathcal{R}_1$, by construction.

We are now ready to finish the construction of the map U. To this end, it only remains to define its action on vectors of the form $\alpha e + \varphi$ for $\varphi \in \{e\}^\perp$ and for some $\alpha \in \mathbb{C} \setminus \{0, 1\}$. Here, we finally set

$$U(\alpha e+\varphi) = \chi(\alpha)U(e+\alpha^{-1}\varphi) = \chi(\alpha)e'+\chi(\alpha)U_\perp(\alpha^{-1}\varphi) = \chi(\alpha)e'+U_\perp\varphi = \chi(\alpha)Ue+U\varphi$$

and it is readily verified that this yields a map $U : \mathcal{H} \to \mathcal{H}$ that is either linear and unitary or antilinear and antiunitary (depending on the function $\chi : \mathbb{C} \to \mathbb{C}$) with the consistency property that $U\psi \in T[\psi]$ for every $\psi \in \mathcal{H}$.

We conclude the proof by showing the uniqueness of U, up to multiplication by a constant phase. Suppose that $U_1, U_2 : \mathcal{H} \to \mathcal{H}$ are compatible with the symmetry transformation and that both are either linear, unitary or antilinear, antiunitary (we leave it as an *exercise* that T cannot be induced by both a linear, unitary and an antilinear, antiunitary map). By compatibility with T, we have $U_1\psi \sim U_2\psi$ for all $\psi \in \mathcal{H}$. This means that for every $\psi \in \mathcal{H}$, there exists $\omega_\psi \in [0, 2\pi)$ so that

$$U_2\psi = e^{i\omega_\psi} U_1\psi.$$

If $\varphi, \psi \in \mathcal{H}$ are linearly independent, we infer from

$$e^{i\omega_\varphi} U_1\varphi + e^{i\omega_\psi} U_1\psi = U_2(\varphi + \psi) = e^{i\omega_{\varphi+\psi}}(U_1\varphi + U_1\psi) = e^{i\omega_{\varphi+\psi}} U_1\varphi + e^{i\omega_{\varphi+\psi}} U_1\psi$$

that $e^{i\omega_\varphi} = e^{i\omega_{\varphi+\psi}} = e^{i\omega_\psi}$. Note here that linear independence of φ and ψ is preserved by U_1, by (anti-)unitarity and the fact that φ and ψ are linearly independent if and only if $|\langle\varphi, \psi\rangle| < \|\varphi\|\|\psi\|$. On the other hand, if φ, ψ are linearly dependent, we can find some $\zeta \in \mathcal{H}$ independent of both φ and ψ so that the previous argument implies

$$e^{i\omega_\varphi} = e^{i\omega_{\varphi+\zeta}} = e^{i\omega_\zeta} = e^{i\omega_{\psi+\zeta}} = e^{i\omega_\psi}.$$

In conclusion, this shows that $U_2 = e^{i\omega} U_1$ for some $\omega \in [0, 2\pi)$. □

Problem 1.4. *Verify Theorem 1.4 for* $\dim \mathcal{H} = 2$. *For* $\dim \mathcal{H} = 1$, *prove the existence of a unitary map as in Theorem 1.4, but explain why uniqueness as in Theorem 1.4 fails.*

Theorem 1.4 shows that unitary (or antiunitary) operators are appropriate to model symmetries in quantum mechanics where states are described by suitable vectors in a Hilbert space. We conclude this section by presenting some basic examples.

Example 1.6 (Translations)**.** Consider a particle of mass $m > 0$ moving in $\mathbb{R}^d$ in an external field $V : \mathbb{R}^d \to \mathbb{R}$. As discussed in Section 1.2.1, we can model this by choosing the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$ and the Hamiltonian

$$H = \frac{|i\hbar\nabla|^2}{2m} + V(x).$$

As usual, we assume that V satisfies suitable constraints so that H can be realized as a self-adjoint operator on a dense domain in $L^2(\mathbb{R}^d)$. A basic symmetry transformation is to translate the coordinate system that we use to measure positions. If the state of the system is described in the original coordinate system by $\psi \in L^2(\mathbb{R}^d)$ and the system is translated by $a \in \mathbb{R}^d$, the state from the viewpoint of the transformed system reads

$$x \mapsto (U_a\psi)(x) = \psi(x - a) \in L^2(\mathbb{R}^d) = U_a L^2(\mathbb{R}^d).$$

Clearly, the maps $(U_a)_{a\in\mathbb{R}^d}$ form a family of unitary transformations.

In analogy to Example 1.4, consider a direction $n \in \mathbb{R}^d, |n| = 1$. In the classical setting, the group of translations $(tn)_{t\in\mathbb{R}}$ leads to the conservation of the momentum in

direction n if V is translation invariant in direction n, i.e. $V(\cdot + tn) = V(\cdot)$. An analogue remains valid in the quantum mechanical setting. To see this, note that Theorem 1.2 applied to $(U_{tn})_{t\in\mathbb{R}}$ implies the existence of a self-adjoint operator $p \cdot n$ that satisfies

$$U_{tn} = e^{-itp\cdot n/\hbar}, \ \forall t \in \mathbb{R}.$$

A direct calculation shows that

$$(p \cdot n\, \psi)(x) = i\hbar \lim_{t\to 0} \frac{1}{t}(U_{tn} - U_0\psi)(x) = i\hbar \lim_{t\to 0} \frac{\psi(x - tn) - \psi(x)}{t} = \big(- i\hbar\nabla \cdot n\, \psi\big)(x)$$

for all $\psi \in L^2(\mathbb{R}^d)$ that satisfy $\nabla \cdot n\, \psi \in L^2(\mathbb{R}^d)$. Choosing for $n \in \mathbb{R}^d$ the canonical base vectors e_j in $\mathbb{R}^d$, for $j = 1, \ldots, d$, the corresponding translations are generated by

$$p_j : D_{p_j} = \big\{\psi \in L^2(\mathbb{R}^d) : \partial_{x_j}\psi \in L^2(\mathbb{R}^d)\big\} \to L^2(\mathbb{R}^d), \ p_j\psi = -i\hbar\partial_{x_j}\psi, \ j \in \{1, \ldots, d\}.$$

In other words, the generators of translations in $\mathbb{R}^d$ are the momentum operators that we introduced earlier in Section 1.2.1, based on the canonical quantization method.

Now, let us explain why translation invariance $V(\cdot + tn) = V(\cdot)$ of V implies the conservation of $p \cdot n$ under the Schrödinger dynamics (1.15). In fact, a direct calculation shows the invariance $U_{sn}HU_{sn}^* = H$, for all $s \in \mathbb{R}$. The functional calculus thus implies

$$U_{sn}e^{-itH/\hbar}\, U_{sn}^* = e^{-itH/\hbar} \iff e^{-is(p\cdot n)_t/\hbar}U_{sn}^* = \mathbf{1}_{L^2(\mathbb{R}^d)} \iff e^{-is(p\cdot n)_t/\hbar} = e^{-isp\cdot n/\hbar}$$

for every $s, t \in \mathbb{R}$ and this implies by (the proof of) Theorem 1.2 that

$$(p \cdot n)_t = e^{itH/\hbar}\, p \cdot n\, e^{-itH/\hbar} = p \cdot n, \ \forall\, t \in \mathbb{R}.$$

Example 1.7 (Rotations)**.** Consider a particle in $\mathbb{R}^3$, in the same setting as in the previous Example 1.6. Symmetry transformations that describe rotations of the coordinate system can be implemented through the unitary operators

$$U_R : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3), \ (U_R\psi)(x) = \psi(R^{-1}x), \ \text{ for } R \in SO(3).$$

In analogy to Example 1.5, the quantum angular momentum $L \cdot n$ in direction n, $|n|$, is the generator of the one-parameter unitary group $(U_{e^{\omega\, n\cdot X}})_{\omega\in\mathbb{R}}$, where the matrices X_1, X_2, X_3 denote the generators of $\mathfrak{so}(3)$, defined in (1.10). For example, the angular momentum in direction e_3 takes the explicit form

$$\begin{aligned}(L_3\psi)(x) &= i\hbar \lim_{\omega\to 0} \frac{\psi(e^{-\omega X_3}x) - \psi(x)}{\omega} \\ &= i\hbar \lim_{\omega\to 0} \frac{1}{\omega}\big(\psi\big((\cos(\omega)x_1 + \sin(\omega)x_2, -\sin(\omega)x_1 + \cos(\omega)x_2, x_3\big)\big) - \psi(x)\big) \\ &= \big(x_2(i\hbar\partial_{x_1}) - x_1(i\hbar\partial_{x_2})\psi\big)(x) = \big((x_1p_2 - x_2p_1)\psi\big)(x).\end{aligned}$$

More generally, one readily verifies that $L = (L_1, L_2, L_3) = x \times p$ on a suitable, dense domain, where x and p denote the quantum mechanical position and momentum operators. Once again, this is consistent with the definition of quantum angular momentum via canonical quantization of the classical angular momentum. Recall that the latter corresponds to the classical observable $(x, p) \mapsto L(x, p) = x \times p \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$.

The two previous examples illustrate that a symmetry that corresponds to a Lie group may give rise to a strongly continuous unitary representation of the group on the state space. At this point, let us introduce some basic definitions. Let G be a group. Then, a group representation $\rho : G \to \mathrm{GL}(V) = \left\{ A : V \to V : A \text{ linear }, A^{-1} \text{ exists} \right\}$ of G on a vector space V is a group homomorphism to $\mathrm{GL}(V)$, the vector space of invertible linear maps on V. That is, ρ satisfies

$$\rho(g_1 g_2) = \rho(g_1)\rho(g_2), \ \forall\, g_1, g_2 \in G.$$

Unless mentioned otherwise, all considered vector spaces V are assumed to be vector spaces over $\mathbb{C}$. We call $\dim(V)$ the dimension of ρ. If G is a topological group and V is a normed space, we call ρ strongly continuous if the map $G \ni g \mapsto \rho(g)v \in V$ is continuous, for every $v \in V$. We call ρ irreducible if there does not exist a non-trivial, closed subspace $S \subset V$, so that $S \neq \{0\}$ and $S \neq V$, which is invariant under ρ, that is, $\rho(g)S \subset S$ for all $g \in G$. Otherwise, ρ is called reducible. From a physical point of view, according to the ideas of E. Wigner [25, 26], if a quantum system comes with a symmetry group that is described by an irreducible representation, then the system can be considered elementary in the sense that the symmetry can not be used to resolve some finer structure of the model. If it is reducible, on the other hand, one can factorize the state space accordingly into finer, elementary subspaces.

In view of Theorem 1.4, in the context of quantum mechanics typical symmetry groups, like e.g. SO(3) in Example 1.7, lead to representations that take values in the space of unitary operators $\mathcal{U}(\mathcal{H}) = \{U : \mathcal{H} \to \mathcal{H} : U \text{ is unitary}\}$, if $\mathcal{H}$ denotes the Hilbert space describing the possible states. Unitary operators (as opposed to antiunitary operators) naturally occur if the symmetry is described by a connected Lie group that contains the identity (which is unitary). A representation $\rho : G \to \mathcal{U}(\mathcal{H})$ that takes its values in $\mathcal{U}(\mathcal{H})$ is called a unitary representation of G.

Lemma 1.3. *A unitary representation $\rho : G \to \mathcal{U}(\mathcal{H})$ is irreducible if and only if for every $\psi \in \mathcal{H}, \psi \neq 0$, we have that*

$$\mathcal{H} = \overline{\mathrm{span}\{\rho(g)\psi : g \in G\}} \iff \{0\} = \{\varphi \in \mathcal{H} : \langle U(g)\psi, \varphi\rangle = 0, \ \forall\, g \in G\}.$$

Proof. The space $\overline{\mathrm{span}\{U(g)\psi : g \in G\}}$ is clearly invariant under ρ. If ρ is irreducible, this implies it must be equal to $\{0\}$ ($\psi = 0$) or $\mathcal{H}$ ($\psi \neq 0$). On the other hand, if $\mathcal{H} = \overline{\mathrm{span}\{\rho(g)\psi : g \in G\}}$ for every non-zero $\psi \in \mathcal{H}$ and if there exists a non-empty, closed linear subspace $S \subset \mathcal{H}$ invariant under ρ, we pick $\varphi \in S$, $\varphi \neq 0$ and find $\mathcal{H} = \overline{\mathrm{span}\{\rho(g)\varphi : g \in G\}} \subset S \subset \mathcal{H}$. In other words, $S = \mathcal{H}$ or equivalently $S^\perp = \{0\}$. □

Lemma 1.4 (Schur)**.** *Consider a finite-dimensional, unitary representation $\rho : G \to \mathcal{H}$. Then ρ is irreducible if and only if every linear map A that commutes with every operator $\rho(g)$, for $g \in G$, is a multiple of the identity $\mathbf{1}_{\mathcal{H}}$.*

Proof. Recall that every linear map A has at least one eigenvalue $\lambda \in \mathbb{C}$ in a complex vector space $\mathcal{H}$. If $\varphi \neq 0$ is a normalized eigenvector and ρ is irreducible, we know that $\overline{\mathrm{span}\{\rho(g)\varphi : g \in G\}} = \mathcal{H}$. In particular, if $\psi = U(g)\varphi$, we see that

$$A\psi = A\rho(g)\varphi = \rho(g)A\varphi = \lambda\rho(g)\varphi = \lambda\psi,$$

which implies by continuity that $A = \lambda \mathbf{1}_{\mathcal{H}}$. On the other hand, if ρ reducible, pick a non-trivial subspace $S \subset \mathcal{H}$, $S \neq \mathcal{H}$, that is invariant under ρ. This induces a unitary representation $\rho' : G \to \mathcal{U}(S)$, because $\rho(g)S \subset S$. Without loss of generality, we may assume that ρ' is irreducible, because otherwise we can repeat the previous step until it is (recalling that $\dim(\mathcal{H}) < \infty$). By the previous step, we may thus assume that $S = \overline{\text{span}\{\rho(g)\varphi : g \in G\}}$ for some $0 \neq \varphi \in S$. Notice that the invariance of S under ρ also implies the invariance of $S^\perp$ under ρ, which readily follows from unitarity and

$$\langle \rho(g)\psi, \zeta \rangle = \langle \psi, \rho(g)^* \zeta \rangle = \langle \psi, \rho(g^{-1})\zeta \rangle.$$

Now, choose $P : \mathcal{H} \to S$ to be the orthogonal projection onto S and let $\psi \in \mathcal{H}, \psi \neq 0$. By a simple density argument, we may assume w.l.o.g. that $\psi = \psi_1 + \psi_2 = \rho(g)\varphi + \psi_2$ for $\psi_1 \in S, \psi_2 \in S^\perp$. Then, using that $P\rho(h)\varphi = \rho(h)\varphi \in S$ as well as $P\rho(h)\psi_2 = 0$ for every $h \in G$, we obtain

$$\begin{aligned} P\rho(h)\psi = P\rho(h)\rho(g)\varphi + P\rho(h)\psi_2 = P\rho(hg)\varphi = \rho(hg)\varphi = \rho(h)(P\rho(g)\varphi + \rho(h)P\psi_2) \\ = \rho(h)P\psi. \end{aligned}$$

This implies that $[P, \rho(g)] = 0$ for every $g \in G$, but $P \neq \lambda \mathbf{1}_{\mathcal{H}}$, for every $\lambda \in \mathbb{C}$. □

In Example 1.7, we analysed the unitary representation $\rho : \text{SO}(3) \to \mathcal{U}(\mathcal{H})$, given by

$$\rho(R)\psi = \psi(R^{-1}.)$$

for $\psi \in \mathcal{H} = L^2(\mathbb{R}^3)$, to describe the rotation symmetry of a system that describes a massive particle in $\mathbb{R}^3$. This representation is induced by the group action

$$\text{SO}(3) \times \mathbb{R}^3 \ni (R, x) \mapsto R^{-1}x \in \mathbb{R}^3$$

of SO(3) on $\mathbb{R}^3$ and the SO(3)-invariance of the Lebesgue measure. In the same way, every other finite-dimensional, unitary representation $T : \text{SO}(3) \to \mathcal{U}(\mathbb{C}^n)$ induces a unitary representation $\rho : \text{SO}(3) \to \mathcal{U}(L^2(\mathbb{R}^3, \mathbb{C}^n))$ through

$$\rho(R)\psi = T(R)\psi(R^{-1}.), \ \forall \ \psi \in L^2(\mathbb{R}^3, \mathbb{C}^n). \tag{1.31}$$

In fact, in view of quantum mechanics it is more appropriate to allow for projective unitary representations which are maps $\rho : G \to \mathcal{U}(\mathcal{H})$ so that for every $g_1, g_2 \in G$, there exists a phase $\lambda(g_1, g_2) \in \mathbb{C}, |\lambda(g_1, g_2)| = 1$, such that

$$\rho(g_1 g_2) = \lambda(g_1, g_2)\rho(g_1)\rho(g_2). \tag{1.32}$$

This naturally suggests to study all the possible projective representations of a given symmetry group. Interestingly, in important cases including SO(3), the projective representations can be understood in terms of ordinary representations of a larger, so called covering group. This is what we want to explain next in the context of SO(3). To this end, we use the following result that relates SO(3) to SU(2), which is defined by

$$\begin{aligned} \text{SU}(2) &= \big\{A \in \mathbb{C}^{2\times 2} : A^*A = \mathbf{1}_{\mathbb{C}^2}, \det A = 1\big\} \\ &= \big\{A = (a_{ij})_{i,j=1}^2 \in \mathbb{C}^{2\times 2} : a_{11} = \overline{a}_{22}, a_{21} = -\overline{a}_{12}, |a_{11}|^2 + |a_{12}|^2 = 1\big\}. \end{aligned} \tag{1.33}$$

Observe that the second equality implies that SU(2) is a (locally path) connected, compact and simply connected Lie group, for it is diffeomorphic to the unit sphere $S^3 \subset \mathbb{R}^4$.

Proposition 1.1. *There exists a two-to-one group homomorphism $R : \mathrm{SU}(2) \to \mathrm{SO}(3)$, which is in particular a local homeomorphism.*

Proof. Denote by $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{C}^{2\times 2}$ the Pauli spin matrices, defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.34}$$

Note that $i\sigma_1, i\sigma_2, i\sigma_3 \in \mathbb{C}^{2\times 2}$ form a basis of the linear space of traceless, antisymmetric matrices in $\mathbb{C}^{2\times 2}$, which, as a side remark, equals the Lie algebra $\mathfrak{su}(2)$ of SU(2).

Now, consider the map

$$\mathbb{R}^3 \ni x \mapsto \sigma \cdot x = \sum_{j=1}^{3} \sigma_j x_j = (\sigma \cdot x)^*$$

so that $\operatorname{tr} \sigma \cdot x = 0$ for every $x \in \mathbb{R}^3$ and observe that

$$\det(\sigma \cdot x) = -x_3^2 - |x_1 - ix_2|^2 = -|x|^2.$$

Given that $A(\sigma \cdot x)A^*$ is self-adjoint and traceless, and since the $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{C}^{2\times 2}$ form a basis of the traceless, self-adjoint matrices in $\mathbb{C}^{2\times 2}$, we find for every $A \in \mathrm{SU}(2)$ a unique $y_A(x) \in \mathbb{R}^3$ such that

$$A(\sigma \cdot x)A^* = \sigma \cdot y_A(x), \quad \text{with} \quad (y_A(x))_j = \operatorname{tr} \sigma_j A(\sigma \cdot x)A^*, j \in \{1, 2, 3\}.$$

The map $x \to y_A(x)$ is clearly linear, continuous and can be written as $y_A(x) = R(A)x$ for some $R(A) \in \mathbb{R}^{3\times 3}$. We claim that the map

$$\mathrm{SU}(2) \ni A \mapsto R(A) \in \mathbb{R}^{3\times 3}$$

is a two-to-one homomorphism from SU(2) to SO(3). To see this, note first that

$$|R(A)x|^2 = -\det \sigma \cdot (R(A)x) \ = -\det A(\sigma \cdot x)A^* = -\det \sigma \cdot x = |x|^2.$$

Combined with the fact that $R(\mathbf{1}_{\mathbb{C}^2}) = \mathbf{1}_{\mathbb{C}^2}$ with $\det \mathbf{1}_{\mathbb{C}^2} = 1$, that $\mathrm{SU}(2) \ni A \mapsto \det R(A) \in \{-1, 1\}$ is continuous and the fact that SU(2) is connected, we conclude that $R(A) \in \mathrm{SO}(3)$. The group homomorphism property follows from

$$\sigma \cdot (R(AB)x) = AB(\sigma \cdot x)B^*A^* = A(\sigma \cdot (R(B)x))A^* = \sigma \cdot (R(A)R(B)x)$$

so that $R(AB) = R(A)R(B)$. That $\mathrm{SU}(2) \ni A \mapsto R(A) \in \mathrm{SO}(3)$ is surjective, follows from Problem 1.5. Finally, suppose that $R(A) = \mathbf{1}_{\mathbb{R}^3}$. This is the case if and only if

$$A\,\sigma \cdot x = \sigma \cdot x\,A$$

for every $x \in \mathbb{R}^3$. In particular, this implies that $[A, \sigma_i] = 0$ for all $i \in \{1, 2, 3\}$. Choosing $i = 3$, an explicit computation then verifies that $\beta = 0$ if

$$A = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}.$$

Combining this with the condition $[A, \sigma_1] = 0$ implies that $\mathrm{Im}(\alpha) = 0$ and since $A^*A = \mathbf{1}_{\mathbb{C}^2}$, we conclude $A = \mathbf{1}_{\mathbb{C}^2}$ or $A = -\mathbf{1}_{\mathbb{C}^2}$. Combined with the homomorphism property, we see that $R(A_1) = R(A_2)$ if and only if $A_1 = A_2$ or $A_1 = -A_2$ so that the map $\mathrm{SU}(2) \ni A \mapsto R(A) \in \mathrm{SO}(3)$ is a continuous two-to-one homomorphism. In fact, the two-to-one property readily implies that R is locally bijective (*exercise*). Since $\mathrm{SU}(2)$ is compact, this and the continuity of R imply that R is a local homeomorphism. □

Problem 1.5. *Recall the generators $X_1, X_2, X_3 \in \mathbb{R}^{3\times 3}$ from (1.10). Prove explicitly that*

$$e^{-\frac{i}{2}\omega\sigma_3}\big(\sigma \cdot x\big)e^{\frac{i}{2}\omega\sigma_3} = \sigma \cdot (e^{\omega X_3}x) \tag{1.35}$$

for every $x \in \mathbb{R}^3, \omega \in [0, 2\pi)$. Prove analogous formulas for rotation matrices around the axes $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$. Finally, use these results to argue that the map $SU(2) \ni A \mapsto R(A) \in SO(3)$ constructed in Proposition 1.1 is surjective.

Our next goal is to show that every finite-dimensional projective unitary representation of SO(3) corresponds to an ordinary unitary representation of SU(2). In other words, by enlarging the symmetry group, we can work with a much simpler mathematical object (an ordinary representation compared to a potentially complicated projective representation). To make this more precise, we say that a projective representation ρ can be lifted, or deprojectivized, to an ordinary representation ρ' if there exists a function $\lambda : G \to \{z \in \mathbb{C} : |z| = 1\}$ such that $\rho(g) = \lambda(g)\rho'(g)$, for every $g \in G$.

Proposition 1.2. *Let G be a compact, connected and simply connected Lie group. Then, every finite dimensional strongly continuous projective unitary representation of G can be lifted to a strongly continuous unitary representation of G.*

Remark 1.6. *The reason we assume G to be compact is that under suitable additional assumptions, it is a general fact that non-compact Lie groups do not admit non-trivial finite-dimensional unitary representations. Details on this can be found e.g. in [6].*

Remark 1.7. *Prop. 1.2 is a simplified version of a much more general result from [1].*

Before proving Prop. 1.2, we record its consequences for SO(3)-representations.

Corollary 1.1. *Let $\rho : SO(3) \to \mathcal{U}(\mathcal{H})$ be a finite-dimensional strongly continuous projective unitary representation of $SO(3)$. Then, there exists an ordinary representation τ of $SU(2)$ and a continuous map $\lambda : SU(2) \to \{z \in \mathbb{C} : |z| = 1\}$ such that*

$$\rho(R(A)) = \lambda(A)\tau(A), \ \forall\ A \in SU(2).$$

Here, $R : SO(3) \to SU(2)$ denotes the homomorphism constructed in Proposition 1.1.

Proof. As noted earlier, SU(2) is diffeomorphic to the unit sphere $S^3 \subset \mathbb{R}^4$ and is thus compact, connected and simply connected. Applying Prop. 1.2 to

$$\mathrm{SU}(2) \ni A \mapsto \rho(R(A)) \in \mathcal{U}(\mathcal{H}),$$

which defines a projective representation of SU(2), implies the claim. □

Proof of Prop. 1.2. Let $\rho : G \to \mathrm{GL}(\mathbb{C}^n)$ be a projective unitary representation so that

$$\rho(g_1 g_2) = \mu(g_1, g_2)\rho(g_1)\rho(g_2) \quad \text{with} \quad |\mu(g_1, g_2)| = 1, \ \ \forall\, g_1, g_2 \in G.$$

Since ρ is assumed to be strongly continuous and finite-dimensional, it is continuous in the usual sense if we equip $\mathcal{U}(\mathbb{C}^n) \subset \mathrm{GL}(\mathbb{C}^n)$ with the standard Euclidean topology. Notice that this, combined with the continuity of the maps $G^2 \ni (g_1, g_2) \mapsto g_1 g_2 \in G$ as well as $G^2 \ni (g_1, g_2) \mapsto \rho(g_1)\rho(g_2) \in \mathcal{U}(\mathbb{C}^n)$, implies also the continuity of

$$G^2 \ni (g_1, g_2) \mapsto \frac{\rho(g_1 g_2)}{\rho(g_1)\rho(g_2)} = \mu(g_1, g_2) \in \{z \in \mathbb{C} : |z| = 1\}.$$

In the sequel, we assume furthermore that $\rho(\mathbf{1}_G) = \mathbf{1}_{\mathbb{C}^n}$ (if this is not the case, multiply ρ by a suitable constant of modulus one, which yields again a projective representation).

The proof consists of two main steps. In the first step, we prove the claim locally at the identity $\mathbf{1}_G$ of G. In the second step we bootstrap the construction based on the connectedness properties of G and the local homomorphism from the first step, to obtain a global representation of G with the desired properties.

Let's start with the local step. Among the key observations is the simple fact that

$$G \ni g \mapsto \det \rho(g) \in \{z \in \mathbb{C} : |z| = 1\}$$

is continuous. Since $\det \rho(\mathbf{1}_G) = 1$, continuity implies the existence of a small open neighborhood $V \subset G$, $\mathbf{1}_G \in V$, in which $|\det \rho(g) - 1|$ is small enough, so that $V \ni g \mapsto \lambda(g) = (\det \rho(g))^{-1/n}$ is well-defined and continuous. We then simply set

$$\tau(g) = \lambda(g)\rho(g) \in \mathcal{U}(\mathcal{H}) \tag{1.36}$$

for every $g \in V$ so that by the properties of ρ, we have that

$$\tau(g_1 g_2) = \frac{\lambda(g_1 g_2)\mu(g_1 g_2)}{\lambda(g_1)\lambda(g_2)}\tau(g_1)\tau(g_2).$$

Using that $\det \tau(g) = 1$, by definition of λ, this implies that

$$\left(\frac{\lambda(g_1 g_2)\mu(g_1 g_2)}{\lambda(g_1)\lambda(g_2)}\right)^n = 1.$$

Assuming $V \subset G$ to so small such that the last step implies $\frac{\lambda(g_1 g_2)\mu(g_1 g_2)}{\lambda(g_1)\lambda(g_2)} = 1$, we find

$$\tau(g_1 g_2) = \tau(g_1)\tau(g_2), \ \ \forall\, g_1, g_2 \in V. \tag{1.37}$$

Loosely speaking, this means ρ can be lifted locally to a homomorphism $\tau : V \to \mathcal{U}(\mathcal{H})$.

Our next goal is to show that $\tau : V \to \mathcal{U}(\mathcal{H})$ can be extended to a global homomorphism $\tau : G \to \mathcal{U}(\mathcal{H})$ in such a way that ρ is induced by τ. Here, we use the connectedness properties of G. In particular, without loss of generality, we assume in the following that $V \subset G$ is path connected.

The key step is how to extend τ to G. Afterwards, straightforward arguments imply that the extension corresponds to a de-projectification of ρ. So, let's first extend τ from V to all of G. Since $\tau(\mathbf{1}_G) = \mathbf{1}_{\mathbb{C}^n}$, a natural strategy is to choose for general $g \in G$ a continuous path $\gamma : [0,1] \to g$ with $\gamma(0) = \mathbf{1}_G$, $\gamma(1) = g$ and to define $\tau(g)$ as

$$\tau(g,\gamma) = \tau(\gamma_{t_k}\gamma_{t_{k-1}}^{-1})\tau(\gamma_{t_{k-1}}\gamma_{t_{k-2}}^{-1})\dots\tau(\gamma_{t_1}\gamma_{t_0}^{-1}), \tag{1.38}$$

where $0 = t_0 < t_1 < t_2 < \ldots < t_k = 1$ is a partition that is fine enough to ensure that $\gamma_{t_i}\gamma_{t_{i-1}}^{-1} \in V$ for every $i \in \{0,\dots,k\}$. We leave it as a simple *exercise* to show that (1.37) implies that a refinement of the partition does not change the r.h.s. in (1.38). Thus, $\tau(g,\gamma)$ only depends on γ and not on the specific partition that we choose. In the next step, we show that the simple connectedness of G implies that

$$\tau(g,\gamma_1) = \tau(g,\gamma_2) \tag{1.39}$$

for every pair of continuous paths $\gamma_i : [0,1] \to G$ s.t. $\gamma_i(0) = \mathbf{1}_G$, $\gamma_i(1) = g$, $i \in \{1,2\}$. Assuming for the moment the validity of (1.39), we can then define

$$\tau(g) = \tau(g,\gamma_1)$$

for some (and hence all) $\gamma : [0,1] \to G$ s.t. $\gamma(0) = \mathbf{1}_G$, $\gamma(1) = g$. Since V is path connected, note that this coincides with our earlier definition $\tau : V \to \mathcal{U}(\mathcal{H})$. Furthermore, note that for two paths $\gamma_1 : [0,1] \to G$ with $\gamma_1(0) = \mathbf{1}_G$, $\gamma_1(1) = g_1 \in G$ and $\gamma_2 : [0,1] \to G$ with $\gamma_1(0) = \mathbf{1}_G$, $\gamma_2(1) = g_2 \in G$, we can form a path $\gamma : [0,1] \to G$ with $\gamma(0) = \mathbf{1}_G$, $\gamma(1) = g_1g_2 \in G$ as the concatenation $\gamma = \gamma_1 \circ (-\gamma_2) \circ \gamma_2$. Then (1.38) and (1.39) imply (*exercise*) that

$$\tau(g_1g_2) = \tau(g_1g_2,\gamma) = \tau(g_1,\gamma_1)\tau(g_2,\gamma_2) = \tau(g_1)\tau(g_2),$$

so that $\tau : G \to \mathcal{U}(\mathcal{H})$ is a representation of G. Similarly, (1.36) and (1.38) imply that for every $g \in G$, there exists $\lambda'(g) \in \mathbb{C}$ with $|\lambda'(g)| = 1$ such that

$$\begin{aligned}\tau(g) &= \tau(\gamma_{t_k}\gamma_{t_{k-1}}^{-1})\tau(\gamma_{t_{k-1}}\gamma_{t_{k-2}}^{-1})\dots\tau(\gamma_{t_1}\gamma_{t_0}^{-1}) \\ &= \lambda^{-1}(\gamma_{t_k}\gamma_{t_{k-1}}^{-1})\rho(\gamma_{t_k}\gamma_{t_{k-1}}^{-1})\dots\lambda^{-1}(\gamma_{t_1}\gamma_{t_0}^{-1})\rho(\gamma_{t_1}\gamma_{t_0}^{-1}) \\ &= \lambda'(g)\rho(g).\end{aligned}$$

In other words, $\tau : G \to \mathcal{U}(\mathcal{H})$ is a de-projectification of $\rho : G \to \mathcal{U}(\mathcal{H})$. Observe, moreover, that the map $G \ni g \mapsto \lambda'(g)$ is continuous.

To finish the proof, it remains to verify that $\tau(g,\gamma)$ is independent of $\gamma : [0,1] \to G$ with $\gamma(0) = \mathbf{1}_G$, $\gamma(1) = g$. By standard concatenation, this is equivalent to showing

that $\tau(\mathbf{1}_G, \gamma) = \mathbf{1}_{\mathbb{C}^n}$ for every loop $\gamma : [0,1] \to G$ starting and ending at $\mathbf{1}_G$. So consider such a loop and choose a homotopy $\psi : [0,1] \times [0,1] \to G$ between γ and the constant loop $[0,1] \ni t \mapsto \gamma_0(t) = \mathbf{1}_G$, by the simple connectedness of G. Note that trivially $\tau(\mathbf{1}_G, \gamma_0) = \mathbf{1}_{\mathbb{C}^n}$. We now show that for $|u_1 - u_2|$ sufficiently small, we also have that

$$\tau(g, \psi_{u_1}) = \tau(g, \psi_{u_2}), \tag{1.40}$$

where $\psi_u(t) = \psi(u,t)$ such that $\psi_u(0) = \psi_u(1) = \mathbf{1}_G$, for every $u \in [0,1]$, and such that $\psi_0 = \gamma_0$, $\psi_1 = \gamma$. Once (1.40) is proved, the path independence of $\tau(g, \gamma)$ follows.

To prove (1.40), we choose a partition $0 = t_0 < t_1 < t_2 < \ldots < t_k = 1$ so that

$$\tau(g, \psi_{u_1}) = \tau(\psi_{u_1}(t_k)\psi_{u_1}(t_{k-1})^{-1}) \ldots \tau(\psi_{u_1}(t_1)\psi_{u_1}(t_0)^{-1}),\ \psi_{u_1}(t_i)\psi_{u_1}(t_{i-1})^{-1} \in V$$

and

$$\tau(g, \psi_{u_2}) = \tau(\psi_{u_2}(t_k)\psi_{u_2}(t_{k-1})^{-1}) \ldots \tau(\psi_{u_2}(t_1)\psi_{u_2}(t_0)^{-1}),\ \psi_{u_2}(t_i)\psi_{u_1}(t_{i-1})^{-1} \in V$$

for every $i \in \{1 \ldots, k\}$. Continuity and (1.37) then also imply that

$$\psi_{u_1}(s)\psi_{u_1}(t)^{-1},\ \psi_{u_2}(s)\psi_{u_2}(t)^{-1} \in V,\ \forall\ t_i \leq s, t \leq t_{i+1}, i \in \{0, 1, \ldots, k-1\}.$$

Now, for every fixed $i \in \{1, \ldots, k\}$, write

$$\psi_{u_1}(t_i)\psi_{u_1}(t_{i-1})^{-1} = \big(\psi_{u_1}(t_i)\psi_{u_2}(t_i)^{-1}\big)\big(\psi_{u_2}(t_i)\psi_{u_2}(t_{i-1})^{-1}\big)\big(\psi_{u_2}(t_{i-1})\psi_{u_1}(t_{i-1})^{-1}\big)$$

and notice that for $|u_1 - u_2|$ sufficiently small, the continuity of the homotopy ψ implies that all three elements in the brackets on the r.h.s. belong to V. Applying (1.37) yields

$$\begin{aligned}
&\tau\big(\psi_{u_1}(t_i)\psi_{u_1}(t_{i-1})^{-1}\big) \\
&= \tau\big(\psi_{u_1}(t_i)\psi_{u_2}(t_i)^{-1}\big)\tau\big(\psi_{u_2}(t_i)\psi_{u_2}(t_{i-1})^{-1}\big)\tau\big(\psi_{u_2}(t_{i-1})\psi_{u_1}(t_{i-1})^{-1}\big).
\end{aligned}$$

Similarly, continuity of ψ implies that

$$\big(\psi_{u_2}(t_{i-1})\psi_{u_1}(t_{i-1})^{-1}\big)^{-1} = \psi_{u_1}(t_{i-1})\psi_{u_2}(t_{i-1})^{-1} \in V$$

whenever $|u_1 - u_2|$ is small so that another application of (1.37) shows that

$$\begin{aligned}
&\tau(\psi_{u_1}(t_k)\psi_{u_1}(t_{k-1})^{-1}) \ldots \tau(\psi_{u_1}(t_1)\psi_{u_1}(t_0)^{-1}) \\
&= \tau\big(\mathbf{1}_G\big)\tau\big(\psi_{u_2}(t_k)\psi_{u_2}(t_{k-1})^{-1}\big)\tau\big(\psi_{u_2}(t_{k-1})\psi_{u_1}(t_{k-1})^{-1}\big) \\
&\quad \times \tau\big(\psi_{u_1}(t_{k-1})\psi_{u_2}(t_{k-1})^{-1}\big)\tau\big(\psi_{u_2}(t_{k-1})\psi_{u_2}(t_{k-2})^{-1}\big)\tau\big(\psi_{u_2}(t_{k-2})\psi_{u_1}(t_{k-2})^{-1}\big) \\
&\quad \times \ldots \times \tau\big(\psi_{u_1}(t_1)\psi_{u_2}(t_1)^{-1}\big)\tau\big(\psi_{u_2}(t_1)\psi_{u_2}(t_0)^{-1}\big)\tau\big(\mathbf{1}_G\big) \\
&= \tau(\psi_{u_2}(t_k)\psi_{u_2}(t_{k-1})^{-1}) \ldots \tau(\psi_{u_2}(t_1)\psi_{u_2}(t_0)^{-1}).
\end{aligned}$$

This shows that $\tau(g, \psi_{u_1}) = \tau(g, \psi_{u_2})$ whenever $|u_1 - u_2|$ is sufficiently small. □

Consider now once more the transformation behavior of a rotation invariant quantum system whose symmetry is described by a (possibly projective) unitary representation

$$\rho(R)\psi = T(R)\psi(R^{-1}.),\ \forall\ \psi \in L^2(\mathbb{R}^3, \mathbb{C}^n) \tag{1.41}$$

as in (1.31), where $T : \mathrm{SO}(3) \to \mathcal{U}(\mathbb{C}^n)$ is some finite-dimensional (possibly projective) unitary representation of SO(3). As discussed earlier, the transformation behavior (1.41) is interpreted as the change of the physical states under a rotation of the coordinate system that is used to describe the system. An important physical insight is that not only spatial symmetries describing changes of coordinates may be relevant for the correct description of a quantum system. Instead, one may also allow for other, so called internal symmetries that have no relation to classical coordinate transformations. This idea is central for the formulation of modern gauge theories which are quantizations of classical field theories equipped with an internal symmetry group G (fields are modelled as connection forms on a principal G-bundle).

In view of Corollary 1.1, a simple generalization of (1.41) is to consider instead of SO(3) the symmetry group SU(2). The latter does not act directly on $\mathbb{R}^3$. However, it is a double cover of SO(3), so we can define a unitary representation by

$$\rho(A)\psi = T(A)\psi(R(A)^{-1}.),\ \forall\ \psi \in L^2(\mathbb{R}^3, \mathbb{C}^n) \tag{1.42}$$

for $A \in \mathrm{SU}(2)$, where $T : \mathrm{SU}(2) \to \mathcal{U}(\mathbb{C}^n)$ is some finite-dimensional unitary representation of SU(2) and where $R : \mathrm{SU}(2) \mapsto \mathrm{SO}(3)$ is the covering map from Prop. 1.1.

The previous discussion motivates to study the unitary representations of SU(2). We defer this to Section 3.2 and conclude this section with some further examples instead.

Example 1.8 (Spin-$\frac{1}{2}$ Particles)**.** Consider the state space $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^2)$, the defining representation $\mathrm{SU}(2) \ni A \mapsto A \in \mathbb{C}^{2\times 2}$ of SU(2) and the induced representation ρ on $\mathcal{H}$, defined through (1.42) for $n = 2$. This model is used to describe elementary particles such as electrons, positrons or neutrons (non-relativistically). Based on ρ, we obtain a generalized angular momentum, say in the direction of $e_3 = (0, 0, 1) \in \mathbb{R}^3$, by

$$(J_3\psi)(x) = i \lim_{\omega\to 0} \frac{1}{\omega}\Big(e^{-\frac{i}{2}\sigma_3\omega}\psi(e^{\omega X_3}x) - \psi(x)\Big) = \Big(L_3 + \frac{\sigma_3}{2}\Big)\psi(x), \quad \text{for } a.e.\ x \in \mathbb{R}^3.$$

Here, we used Problem 1.5 which shows that $R(e^{-\frac{i}{2}\sigma_3\omega}) = e^{\omega X_3}$. The operator $S_3 = \frac{1}{2}\sigma_3$ is called the spin operator in direction e_3. Similarly, one defines spin operators S_1 and S_2 and collects them into the spin vector $S = (S_1, S_2, S_3)$ which measures a quantum mechanical property that has no classical counterpart. In analogy to spatial angular momentum, one interprets spin as some internal angular momentum. It is readily verified that the spectrum $\mathrm{spec}(S_j) = \{-s, s\}$ for $s = \frac{1}{2}$ so that the spin in each coordinate direction is quantized. Particles as above are therefore considered to be of spin $s = \frac{1}{2}$.

Spin is an important property used to explain e.g. the Stern-Gerlach experiment or the fine structure of atomic spectra. To describe energetic effects in the non-relativistic setting, a suitable Hamiltonian that takes spin into account is the Pauli-Hamiltonian

$$H = \frac{1}{2m}\big|i\hbar\nabla + \frac{e}{c}A(x)\big|^2 + e\Phi(x) - \frac{e}{mc}S \cdot B(x),$$

where $B(x) = \nabla \times A(x)$ denotes the magnetic field generated by A. It is worth to note that for $A(x) = (B \times x)/2$, so that $B(x) = B = const.$ for all $x \in \mathbb{R}^3$, one has that

$$\frac{1}{2m}\big|i\hbar\nabla + \frac{e}{c}A(x)\big|^2 = \frac{\hbar^2}{2m}(-\Delta) - \frac{\hbar e}{2mc}L \cdot B + O(B^2).$$

What this shows is that the spin S couples in the Pauli-Hamiltonian to the magnetic field B like a small constant magnetic field couples to the spatial angular momentum in the Hamiltonian of a massive, charged particle that moves in an electromagnetic field.

Example 1.9 (Indistinguishable Particles)**.** Despite the spin, which is related to the internal symmetry group SU(2), there exist also other symmetries that have no classical counterpart. Consider for instance a system of $N \in \mathbb{N}$ particles moving in $\mathbb{R}^d$. We saw earlier that a natural state space is $\mathcal{H} = \bigotimes_{i=1}^N L^2(\mathbb{R}^d) = L^2(\mathbb{R}^{dN})$. If $\psi \in \mathcal{H}$ describes the state of the system, it enables us via (1.12) to predict the probabilities for certain measurement outcomes. But what if the N particles are indistinguishable, based on their basic properties such as their mass, spin or charge?

In this case, it seems reasonable to postulate that permuting the particles is a symmetry of the system. Since the permutation group $\mathcal{S}_N$ of N elements acts on $\mathbb{R}^{dN}$ via

$$\pi(x_1, \ldots, x_N) = \big(x_{\pi(1)}, \ldots, x_{\pi(N)}\big),\ \forall\, \pi \in \mathcal{S}_N, (x_1, \ldots, x_N) \in \mathbb{R}^{dN},$$

we can define the associated unitary permutation operators $U_\pi : \mathcal{H} \to \mathcal{H}$ by

$$(U_\pi\psi)(x_1, \ldots, x_N) = \psi\big(x_{\pi(1)}, \ldots, x_{\pi(N)}\big).$$

Now, in general, we clearly have $\psi \not\sim U_\pi\psi$. So, in the setting of indistinguishable particles, it is natural to restrict the state space to those wave functions ψ that satisfy

$$U_\pi\psi \sim \psi \quad \Longleftrightarrow \quad U_\pi\psi = \lambda_\pi\psi$$

for every $\pi \in \mathcal{S}_N$, where $\lambda_\pi \in \{z \in \mathbb{C} : |z| = 1\}$. Since $U_{\pi_1 \circ \pi_2} = U_{\pi_1}U_{\pi_2}$, this implies

$$\lambda_{\pi_1\circ\pi_2} = \lambda_{\pi_1}\lambda_{\pi_2},\ \forall\, \pi_1, \pi_2 \in \mathcal{S}_N,$$

so that $\lambda : \mathcal{S}_N \to \{z \in \mathbb{C} : |z| = 1\}$ defines a one dimensional representation of $\mathcal{S}_N$. Now, it is well-known that every permutation $\pi = \tau_1 \circ \ldots \circ \tau_k$ is equal to a product of transpositions $(\tau_j)_{j=1}^k$, $k \leq N$, swapping exactly two elements. For a transposition τ, on the other hand, the homomorphism property implies $\lambda_{\tau\circ\tau} = \lambda_\tau^2 = \lambda(\mathbf{1}_{\mathcal{S}_N}) = 1$, so that $\lambda_\tau \in \{-1, 1\}$ and thus $\lambda_\pi \in \{-1, 1\}$ for every $\pi \in \mathcal{S}_N$. Furthermore, based on the simple observation $(j, k)\circ(j, l) = (j, l)\circ(l, k)$ for every $j, k, l \in \{1, \ldots, N\}$, it follows that $\lambda_\tau = 1$ for all transpositions or $\lambda_\tau = -1$ for all transpositions $\tau \in \mathcal{S}_N$. This determines the one dimensional representations of $\mathcal{S}_N$. According to each representation, we can now build the appropriate physical state spaces. One calls the particles described by states in

$$L^2_s(\mathbb{R}^{dN}) = \big\{\psi \in L^2(\mathbb{R}^{dN}) : U_\pi\psi = \psi,\ \forall\, \pi \in \mathcal{S}_N\big\} = \bigotimes_{\text{sym}}^N L^2(\mathbb{R}^d)$$

bosons while fermions are particles that are described by states in

$$L^2_{as}(\mathbb{R}^{dN}) = \big\{\psi \in L^2(\mathbb{R}^{dN}) : U_\pi \psi = (-1)^{\deg(\pi)}\psi,\ \forall\, \pi \in \mathcal{S}_N\big\} = \bigwedge^N L^2(\mathbb{R}^d).$$

For more on indistinguishability, see also the discussion in [24, Sections 4.1 & 9.7].

1.3 Special Relativity

In Section 1.1, we described some of the basic ideas of classical mechanics. In particular, Examples 1.1, 1.2 and 1.3 describe the standard Hamiltonians for charged, massive particles interacting with electromagnetic fields. Although a vast range of mechanical and electromagnetic phenomena is well-described by these models, towards the end of the 19th century, fundamental difficulties emerged in the unification of Newton's and Maxwell's theories. These problems were related to the understanding of the propagation of light (e.g. what is the medium in which electromagnetic waves propagate, interpreting electromagnetic waves as a mechanical phenomenon?) and, consequently, of the classical notions of space and time. This lead in particular to a reconsideration of the concept of inertial systems (for which Newton's law is supposed to be valid), briefly described in Section 1.1, and the relativity principle which states that the form of physical laws does not depend on the choice of the inertial frame used to describe the physical phenomena.

Anticipating the formalism of special relativity, it is useful to make the assumptions of Newtonian mechanics on space and time more precise using a geometric formulation; we follow [13, Chapter 6]. According to Newtonian mechanics, absolute space is a three dimensional Riemannian manifold (E, g) with a symmetric, positive definite metric $g \in C^\infty\big(E, (T^*E)^{\otimes 2}\big)$. Newtonian mechanics then postulates that (E, g) is isometric to $(\mathbb{R}^3, \cdot)$, where $\cdot$ denotes the Euclidean inner product. This means that there exists a diffeomorphism $\Phi : E \to \mathbb{R}^3$ so that

$$g_p(v, w) = d\Phi_p(v) \cdot d\Phi_p(w),\ \forall\, v, w \in T_pE.$$

Particles are described by trajectories $[t_0, t_1] \ni t \mapsto \gamma(t) \in E$ and Newton's law states that the force acting on a particle is equal to its acceleration times its mass. In particular, time is absolute in the sense that it is independent of any specific coordinate frame.

Observe that the postulate that space is isometric to $(\mathbb{R}^3, \cdot)$ singles out isometric coordinate frames as preferred coordinates. Such frames correspond to the previously mentioned inertial frames. Indeed, if a trajectory $[t_0, t_1] \ni t \mapsto \gamma(t) \in E$ of a particle of mass $m > 0$ is described in terms of isometric coordinates $y = (y_1, y_2, y_3) : E \to \mathbb{R}^3$ and if no force acts on the particle, then Newton's law states that

$$\begin{aligned} m\frac{d^2\gamma}{dt^2} = 0 \iff m\frac{d^2(y\circ\gamma)}{dt^2} + \sum_{1\le i,j\le 3} \Gamma_{ij}(\gamma)\frac{d(y_i\circ\gamma)}{dt}\frac{d(y_j\circ\gamma)}{dt} = 0, \\ \iff \frac{d^2(y\circ\gamma)}{dt^2} = 0. \end{aligned} \tag{1.43}$$

Here, we used that the metric g is locally constant and diagonal in isometric coordinates so that the Christoffel symbols $(\Gamma_{ij}^k)_{i,j,k=1}^3$ vanish identically (for their definition and other basic notions from (semi-) Riemannian geometry, see e.g. [13, Chapter 3]). (1.43) implies that in isometric coordinates particles travel on straight lines $(y \circ \gamma)(t) = at + b$ for suitable $a, b \in \mathbb{R}^3$. By the last identity in (1.43), this describes equivalently the geodesics of (E, g) in isometric coordinates. As a consequence, the Riemannian distance $d_g(p, q)$ between two points $p = y^{-1}(v) \in E$ and $q = y^{-1}(w) \in E$, defined by

$$d_g(p,q) = \inf\big\{L(\gamma) : [0,1] \ni t \mapsto \gamma(t) \in E \ \text{ piecewise smooth}, \gamma(0) = p, \gamma(1) = q\big\}$$

for $L(\gamma) = \int_0^1 dt\, \sqrt{g_{\gamma(t)}(d\gamma/dt, d\gamma/dt)}$, is equal to the standard Euclidean distance

$$d_g(p,q) = d_g\big(y^{-1}(v), y^{-1}(w)\big) = |v - w|.$$

Now, suppose we consider a change of coordinates from one inertial frame to another. This corresponds to a Riemannian isometry $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ which, by the last observation and the invariance of the Riemannian distance under isometries, satisfies

$$|\Psi(v) - \Psi(w)| = |v - w|, \ \forall\, v, w \in \mathbb{R}^3.$$

In other words, $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ is a Euclidean isometry and thus equal to a Euclidean transformation $\Psi(x) = Rx + a$ for some translation $a \in \mathbb{R}^3$ and some orthogonal matrix $R \in \mathrm{O}(3)$ (*exercise*). If we can reach the new inertial frame continuously from the original frame, we may assume additionally that $R \in \mathrm{SO}(3)$. Put in geometric terms, the isometry group, defined as the group of isometries on (E, g) (with the composition of maps as group multiplication) is isomorphic to the Euclidean group

$$\mathrm{E}(3) = \big\{(a, R) : a \in \mathbb{R}^3, R \in \mathrm{O}(3)\big\} \ \text{ with } \ (a, R)(a', R') = (Ra' + a, RR'). \tag{1.44}$$

Finally, in anticipation of the axiomatic transformation behavior of quantum fields, let us also recall the transformation behavior of classical fields with regards to inertial frames in Newtonian mechanics. By definition, classical fields include all tensor fields on (E, g). Suppose that w.r.t. a fixed inertial frame $u : E \to \mathbb{R}^3$ a field is represented by

$$\Phi = \Phi^{i_1 \ldots i_p; j_1 \ldots j_q}\, du_{i_1} \otimes \ldots \otimes du_{i_p} \otimes \frac{\partial}{\partial u_{j_1}} \otimes \ldots \otimes \frac{\partial}{\partial u_{j_q}} \in C^\infty\big(E, (T^*E)^{\otimes p} \otimes (TE)^{\otimes q}\big), \tag{1.45}$$

where here and in the following we use the Einstein summation convention (indices appearing once as upper and once as lower index are summed over). Recall that the vector fields $\partial/\partial u_j \in C^\infty(E, TE)$, for $j \in \{1, 2, 3\}$, in (1.45) are defined by

$$\frac{\partial}{\partial u_j}_{|p} = \big(du^{-1}\big)_{|u(p)} \frac{\partial}{\partial x_j}_{|u(p)}, \ \forall\, p \in E.$$

This means that $\partial/\partial u_j \in C^\infty(E, TE)$ is equal to the pushforward of the standard Euclidean vector field $\partial/\partial x_j \in C^\infty(\mathbb{R}^3, T\mathbb{R}^3)$ (which as a vector corresponds to the

standard basis vector $e_j \in \mathbb{R}^3$ and as a derivation to the directional derivative in direction $e_j \in \mathbb{R}^3$) by the diffeomorphism $u^{-1} : \mathbb{R}^3 \to E$. Put differently, if $U \in C^\infty(E, TE)$ is a general vector field of the form

$$U = U^j \frac{\partial}{\partial u_j} \in C^\infty(E, TE),$$

its pushforward $(duU) \circ u^{-1} \in C^\infty(\mathbb{R}^3, T\mathbb{R}^3)$ by $u^{-1} : \mathbb{R}^3 \to E$, which is interpreted as the coordinate representation of U with regards to the chart $u : E \to \mathbb{R}^3$, takes values

$$(duU) \circ u^{-1}(v) = du_{|u^{-1}(v)}\Big((U^j \circ u^{-1})(v)\frac{\partial}{\partial u_j}_{|u^{-1}(v)}\Big) = (U^j \circ u^{-1})(v)\frac{\partial}{\partial x_j}_{|v}, \; \forall\, v \in \mathbb{R}^3.$$

The one forms $du_j \in C^\infty(E, (T^*E))$ in (1.45), on the other hand, correspond to the dual elements which are defined so that $du_j(\partial/\partial_{u_k}) = \delta_{jk}$. This means explicitly that

$$(du_j)_{|p} = (dx_j)_{|u(p)}(du)_{|p}, \; \forall\, p \in E,$$

so that the pullback $(u^{-1})^*\omega \in C^\infty(\mathbb{R}^3, (T^*\mathbb{R}^3))$ of a general one form

$$\omega = \omega^j du_j \in C^\infty(E, (T^*E)),$$

interpreted as the coordinate representation of ω w.r.t. the chart $u : E \to \mathbb{R}^3$, satisfies

$$(u^{-1})^*\omega_{|v} = (\omega^j \circ u^{-1})(v)(du_j)_{u^{-1}(v)}du^{-1}_{|v} = (\omega^j \circ u^{-1})(v)(dx_j)_{|v}$$

Problem 1.6. *In the same notation as above, suppose $u : E \to \mathbb{R}^3$ is a global chart and let $U = U^j \frac{\partial}{\partial u_j} \in C^\infty(E, TE)$, $f \in C^\infty(E, (T^*E))$. Verify that*

$$(Uf) \circ u^{-1} = (U^j \circ u^{-1})\, \partial_{x_j}(f \circ u^{-1}).$$

Generalizing the previous remarks to Φ, we see that the coordinate representation Φ_u of the field Φ with regards to the coordinate frame $u : E \to \mathbb{R}^3$ takes the form

$$\Phi_u = \big(\Phi^{i_1\dots i_p; j_1\dots j_q} \circ u^{-1}\big)\, dx_{i_1} \otimes \dots dx_{i_p} \otimes \frac{\partial}{\partial x_{j_1}} \otimes \dots \frac{\partial}{\partial x_{j_q}} \in C^\infty\big(\mathbb{R}^3, (T^*\mathbb{R}^3)^{\otimes p} \otimes (T\mathbb{R}^3)^{\otimes q}\big).$$

Now, consider a different inertial frame $u' : E \to \mathbb{R}^3$ w.r.t. which Φ has the form

$$\Phi = (\Phi')^{i_1\dots i_p; j_1\dots j_q}\, du'_{i_1} \otimes \dots \otimes du'_{i_p} \otimes \frac{\partial}{\partial u'_{j_1}} \otimes \dots \otimes \frac{\partial}{\partial u'_{j_q}} \in C^\infty\big(\mathbb{R}^3, (T^*\mathbb{R}^3)^{\otimes p} \otimes (T\mathbb{R}^3)^{\otimes q}\big).$$

Then, the chain rule, applied to $(u' \circ u^{-1})(\cdot) = R(\cdot) + a$, for suitable $(a, R) \in \mathrm{E}(3)$, implies that the component functions $\big(\Phi^{i_1\dots i_p; j_1\dots j_q} \circ u^{-1}\big)$ and $\big((\Phi')^{i_1\dots i_p; j_1\dots j_q} \circ (u')^{-1}\big)$ of the coordinate representations Φ_u and, respectively, $\Phi_{u'}$ of the field Φ are related by

$$\begin{aligned} &\big(\Phi^{i_1\dots i_p; j_1\dots j_q} \circ u^{-1}\big)(\cdot) \\ &= R_{k_1}{}^{i_1} \dots R_{k_p}{}^{i_p} (R^{-1})^{j_1}{}_{l_1} \dots (R^{-1})^{j_p}{}_{l_p} \big((\Phi')^{k_1\dots k_p; l_1\dots l_q} \circ (u')^{-1}\big)\big(R(\cdot) + a\big). \end{aligned} \tag{1.46}$$

As mentioned earlier, the transformation behavior (1.46) has a quantum field analogue that is collected as part of the Wightman axioms (see Section 1.4 below).

Despite its simplicity and intuitive character, the Newtonian perspective on space and time outlined above led to fundamental, conceptual difficulties (related for instance to the attempts to explain the Michelson-Morley experiment; for a brief historical overview see e.g. [19, Section 1.6]). The resolution of these problems led ultimately to the theory of special and general relativity by Einstein. In special relativity, one still holds up the relativity principle, but one modifies the notion of inertial frame. A central postulate of special relativity is that the speed of propagation of light, or more generally of any interaction between interacting particles (cf. the discussion in [8, Chapter 1]), is a universal constant that is independent of the inertial frame. Note that in the context of Newtonian mechanics, interactions act instantaneously at an arbitrarily large distance which seems counterintuitive. The postulate of the constancy of the speed of light remedies this and it leads to the following, modified geometric formulation of spacetime.

On a heuristic level, suppose a ray of light is described in two different inertial frames with coordinates $x = (x_0, x_1, x_2, x_3)$ and, respectively, $x' = (x'_0, x'_1, x'_2, x'_3)$ where x_0 and x'_0 measure the time in each frame via the identification $t = x_0/c$, $t' = x'_0/c$, $c > 0$ denoting the speed of light. Note in particular that time is now considered relative to the inertial frame, like the spatial coordinates. Special relativity then postulates that

$$c = \Big|\frac{dx}{dt}\Big| = \Big|\frac{dx'}{dt'}\Big| \quad \Longleftrightarrow \quad c^2|dt|^2 - |dx|^2 = c^2|dt'|^2 - |dx'|^2.$$

This is an infinitesimal version of the identity

$$\eta(x,x) = \eta^{ij}x_i x_j = \eta(x',x'), \quad \text{for} \quad \eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in \mathbb{R}^{4\times 4}. \tag{1.47}$$

In special relativity, this observation is generalized to the following point of view on space and time. Spacetime is a four dimensional time-orientied Lorentz manifold (E, g) which is isometric to the (flat) semi-Riemannian manifold $(\mathbb{R}^4, \eta)$. Here, η is identified with the constant metric tensor with components as in (1.47) with regards to the Euclidean standard basis and it is called the Minkowski metric. Accordingly, the space $(\mathbb{R}^4, \eta)$ is called Minkowski space. Identifying the tangent spaces of $(\mathbb{R}^4, \eta)$ canonically with $T_x\mathbb{R}^4 \simeq \mathbb{R}^4$, for all $x \in \mathbb{R}^4$, tangent vectors $v \in \mathbb{R}^4$ are future-oriented if $x_0 > 0$ and past-oriented if $x_0 < 0$. This defines the time-orientation of $(\mathbb{R}^4, \eta)$ (cf. [13, Chapter 5]).

As an aside, let us also mention that the theory of general relativity generalizes the previous postulate and interprets spacetime as a general, time-oriented Lorentz manifold whose curvature determines gravity. Freely falling bodies follow the geodesics of spacetime while its curvature itself is dynamically linked to the distribution of matter within it through Einstein's equation (for more details, see e.g. [13, Chapter 12]).

Following the geometric discussion on Newtonian mechanics above, let us proceed similarly for special relativity and record some of its basic aspects. First of all, notice

that g defines indeed a symmetric, non-degenerate bilinear form. Its non-degeneracy follows e.g. by isometric equivalence to η and by noting that $\eta(x,y)=0$ for all $y\in\mathbb{R}^4$ implies that $|x|^2=\eta(x,Px)=0$, for $P(x_0,x_1,x_2,x_3)=(x_0,-x_1,-x_2,-x_3)$, so that $x=0$. According to its semi-definiteness properties, one calls tangent vectors $v\in T_gE$ timelike if $g_p(v,v)>0$, lightlike if $g_p(v,v)=0$ and spacelike if $g_p(v,v)<0$. Physically, a timelike vector corresponds to a velocity smaller than $c>0$ and a lightlike vector is a velocity of the speed of light. Spacelike vectors correspond to velocities greater than the speed of light. By definition, spacelike tangent vectors are excluded in the description of massive ($m>0$) and massless ($m=0$) particles. Massive particles are described by future directed timelike curves $[\tau_0,\tau_1]\ni\tau\mapsto\gamma(\tau)\in E$, so that $g_{\gamma(\tau)}(d\gamma/d\tau,d\gamma/d\tau)>0$ for all $\tau\in[\tau_0,\tau_1]$. Massless particles are described by future directed lightlike curves $[\tau_0,\tau_1]\ni\tau\mapsto\gamma(\tau)\in E$, so that $g_{\gamma(\tau)}(d\gamma/d\tau,d\gamma/d\tau)=0$ for all $\tau\in[\tau_0,\tau_1]$. Finally, as a generalization of Newton's law of inertia, special relativity postulates that freely falling particles (those on which no external forces act) are described by the geodesics of (E,g).

As in Newtonian mechanics, the isometric coordinates correspond to the preferred inertial frames. Given such a coordinate frame $x=(x_0,x_1,x_2,x_3):E\to\mathbb{R}^4$, geodesics $[\tau_0,\tau_1]\ni\tau\mapsto(x\circ\gamma)(\tau)$ take the simple form

$$(x\circ\gamma)(\tau)=a\tau+b$$

for suitable $a,b\in\mathbb{R}^4$. The defining property $d^2\gamma/d\tau^2=0$ implies furthermore (*exercise*) that, as in the Riemannian setting, geodesics minimize the so called proper time

$$T(\gamma)=\int_{\tau_0}^{\tau_1}d\tau\,\sqrt{g_{\gamma(\tau)}(d\gamma/d\tau,d\gamma/d\tau)}$$

among all piecewise smooth, timelike curves $\varphi:[\tau_0,\tau_1]\to E$ and that the semi-Riemannian distance between two timelike separated points $p=x^{-1}(v),q=x^{-1}(w)\in E$ equals

$$d_g(p,q)=\sqrt{\eta(v-w,v-w)}.$$

Here, timelike, lightlike and spacelike separated points refer to points that can be connected by a timelike, lightlike and, respectively, spacelike curve. In Minkowski space $(\mathbb{R}^4,\cdot)$, this corresponds to points $v,w\in\mathbb{R}^4$ such that $\eta(v-w,v-w)>0$ (timelike), $\eta(v-w,v-w)=0$ (lightlike) and $\eta(v-w,v-w)<0$ (spacelike). Spacelike separated events cannot causally influence each other (if $\eta(x,x)=|ct|^2-|\mathbf{x}|^2<0$ for $x=(ct,\mathbf{x})$, a light ray starting at $0\in\mathbb{R}^3$ travels the distance ct until time t, so it does not reach $\mathbf{x}$).

As in the Riemannian setting, if $\Psi=(x'\circ x^{-1}):\mathbb{R}^4\to\mathbb{R}^4$ describes a change of inertial frames (that is, a semi-Riemannian isometry), the above observations imply

$$\eta(\Psi(v)-\Psi(w),\Psi(v)-\Psi(w))=\eta(v-w,v-w),\ \forall\ v,w\in\mathbb{R}^4.$$

Assuming w.l.o.g. that $\Psi(0)=0$ (by shifting Ψ to $\Psi'=\Psi-\Psi(0)$ if necessary), bilinearity of the metric implies $\eta(\Psi(v),\Psi(w))=\eta(v,w)$, for every $v,w\in\mathbb{R}^4$. From here, it is straightforward to deduce (*exercise*) that $\Psi:\mathbb{R}^4\to\mathbb{R}^4$ is affine-linear and that the isometry group $\mathrm{ISO}(1,3)$ of (E,g) is isomorphic to the Poincaré group $\mathcal{P}$, defined by

$$\mathcal{P}=\big\{(a,L):a\in\mathbb{R}^4,L\in\mathrm{O}(1,3)\big\}\ \text{ with }\ (a,L)(a',L')=(La'+a,LL').\tag{1.48}$$

Here, $\mathrm{O}(1,3)$ denotes the group of linear maps $L : \mathbb{R}^4 \to \mathbb{R}^4$ that satisfy

$$\eta(L(v), L(w)) = \eta(v, w), \ \forall\, v, w \in \mathbb{R}^4. \tag{1.49}$$

The elements of $\mathrm{O}(1,3)$ are called Lorentz transformations. In special relativity, fields transform as in (1.46), replacing $(a, R) \in \mathrm{E}(3)$ by $(a, L) \in \mathcal{P}$.

The analogue of $\mathrm{SO}(3)$ in Newtonian mechanics (rotations that are path-connected to the identity $\mathbf{1}_{\mathbb{R}^3} \in \mathrm{SO}(3)$), consists of the proper Lorentz group $\mathcal{L}_+^\uparrow$, defined by

$$\mathcal{L}_+^\uparrow = \big\{ L = (L_{\mu\nu})_{\mu,\nu=0}^4 \in \mathrm{O}(1,3) : L_{00} > 0, \det L = 1 \big\}. \tag{1.50}$$

The condition $\det L = 1$ is motivated as in Newtonian mechanics, noting the embedding

$$\mathrm{O}(3) \ni R \mapsto \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \in \mathrm{O}(1,3).$$

The condition $L_{00} > 0$ ensures that L maps timelike vectors to timelike vectors. Note here that $L \in \mathrm{O}(1,3)$ implies L^{-1} exists and that $L^T, L^{-1} \in \mathrm{O}(1,3)$ (*exercise*) and

$$\eta^{\mu\nu} L_{\mu\lambda} L_{\nu\kappa} = \eta_{\lambda\kappa} = \eta^{\mu\nu} L_{\lambda\mu} L_{\kappa\nu} \quad \overset{\lambda,\kappa=0}{\Longrightarrow} \quad L_{00}^2 = 1 + \sum_{j=1}^3 L_{j0}^2 = 1 + \sum_{j=1}^3 L_{0j}^2.$$

Hence, if $L, M \in \mathcal{L}_+^\uparrow$, we have in fact $L_{00}, M_{00} \geq 1$ and by Cauchy-Schwarz that

$$\Big| \sum_{j=1}^3 L_{0j} M_{j0} \Big|^2 \leq \Big(\sum_{j=1}^3 L_{0j}^2 \Big) \Big(\sum_{j=1}^3 M_{j0}^2 \Big) = \big(L_{00}^2 - 1 \big) \big(M_{00}^2 - 1 \big) < L_{00}^2 M_{00}^2,$$

which implies

$$(LM)_{00} = L_{00} M_{00} + \sum_{j=1}^3 L_{0j} M_{j0} > L_{00} M_{00} - \Big| \sum_{j=1}^3 L_{0j} M_{j0} \Big| > 0.$$

Since $\det ML = \det M \det L = 1$, this verifies that $\mathcal{L}_+^\uparrow$ is indeed a subgroup of $\mathrm{O}(1,3)$. Further properties of $\mathcal{L}_+^\uparrow$ as well as the associated group $\mathcal{P}_+^\uparrow = \big\{ (a, L) \in \mathcal{P} : L \in \mathcal{L}_+^\uparrow \big\}$ are discussed in detail in Sections 2.2 and 3.2.

Let us conclude this section by connecting special relativity to Newtonian mechanics. As already mentioned, in special relativity freely falling bodies move on geodesics of spacetime. Choosing a fixed inertial frame, suppose that $[\tau_0, \tau_1] \mapsto x(\tau) \in \mathbb{R}^4$ describes such a geodesic. If the particle is massless, we have by definition $\eta(dx/d\tau, dx/d\tau) = 0$. Since linear reparametrizations $\tau \mapsto a\tau + b$ map geodesics to geodesics, we have some freedom to choose a particular parametrization for massive particles. Generalizing the massless case, one parametrizes $[\tau_0, \tau_1] \mapsto x(\tau) \in \mathbb{R}^4$ conventionally in such a way that

$$\eta(dx/d\tau, dx/d\tau) = m^2 c^2, \ \forall\, \tau \in [\tau_0, \tau_1].$$

In this parametrization, one refers to the derivative $p = (p_0, \mathbf{p}) = dx/d\tau$ as the four-momentum. The geodesic equation can then be formulated as the Hamiltonian dynamics

$$\frac{dx}{d\tau} = p, \quad \frac{dp}{d\tau} = 0. \tag{1.51}$$

Recalling that massive particles are described by forward oriented curves, this implies

$$\eta(p(\tau), p(\tau)) = m^2c^2, \;\; p_0(\tau) > 0, \;\; \forall\, \tau \in [\tau_0, \tau_1]. \tag{1.52}$$

The connection to Newtonian mechanics follows from combining the dynamics (1.51) and the relativistic energy relation (1.52) to eliminate p_0 and $x_0 = ct$. This yields

$$p_0 = \omega(\mathbf{p}) = \sqrt{|\mathbf{p}|^2 + m^2c^2}, \quad \frac{dx_0}{d\tau} = \omega(\mathbf{p})\ (> 0).$$

By the inverse function theorem, the second identity implies that we can express the relativistic time τ as a function of $x_0 = ct$, the local time associated to the chosen inertial frame, and the chain rule then implies that the remaining spatial variables $(\mathbf{x}, \mathbf{p})$ satisfy

$$\frac{d\mathbf{x}}{dt} = \frac{d\mathbf{x}}{d\tau}\frac{d\tau}{dt} = \frac{c\,\mathbf{p}}{\omega(\mathbf{p})}, \quad \frac{d\mathbf{p}}{dt} = 0.$$

This corresponds to a Hamiltonian system on $\mathbb{R}^3 \times \mathbb{R}^3$ with Hamiltonian $(\mathbf{x}, \mathbf{p}) \mapsto H(\mathbf{x}, \mathbf{p}) = c\,\omega(\mathbf{p})$. Conservation of the energy $E = H(\mathbf{x}, \mathbf{p})$ under the Hamiltonian flow $t \mapsto (\mathbf{x}(t), \mathbf{p}(t)) \in \mathbb{R}^3 \times \mathbb{R}^3$ implies the identity $p = (p_0, \mathbf{p}) = (E/c, \mathbf{p})$. For this reason, p is also called the energy-momentum vector. Finally, observe that

$$E = c\,\omega(\mathbf{p}) = mc^2\sqrt{1 + \frac{|\mathbf{p}|^2}{m^2c^2}} = mc^2 + \frac{|\mathbf{p}|^2}{2m} + O\big((|\mathbf{v}|/c)^4\big)$$

for $\mathbf{v} = d\mathbf{x}/dt$. In other words, we recover the Newtonian expression for the kinetic energy of a free particle (up to the so called rest mass) to leading order in $|\mathbf{v}|/c$. The classical Newtonian dynamics for massive particles thus follows in the non-relativistic limit $|\mathbf{v}|/c \to 0$. For massless particles, on the other hand, the non-relativistic approximation can not be applied. While the Hamiltonian formulation remains valid, we obtain from $\eta(p, p) = 0$ that $E = |\mathbf{p}|c$ and hence $\mathbf{v} = d\mathbf{x}/dt = c\,\mathbf{p}/|\mathbf{p}|$ so that $|\mathbf{v}|/c = 1$. In other words, massless particles travel with the speed of light.

1.3.1 Examples of Classical Fields

Compared to Newtonian mechanics whose initial focus lies on describing the trajectories of interacting massive particles, modern particle physics focuses rather on fields as fundamental entities while particles are interpreted as quantum excitations of the fields (once the latter are quantized). In Section 3, this viewpoint is illustrated for three important types of fields. In this section, we introduce the corresponding classical fields.

In Example (1.3), we introduced the electromagnetic field so let's discuss it first in the context of special relativity. Throughout this section, we work without loss of

generality directly in Minkowski space $(\mathbb{R}^4, \eta)$ and recall that changes of inertial frames are determined by the Poincaré group $\mathcal{P}$. As mentioned in the previous section, typical fields are tensor fields on $(\mathbb{R}^4, \eta)$. Physically, they are determined by suitable partial differential equations that are invariant under Poincaré transformations. Motivated by the discussion of the previous section, this means that if a field that is represented by components $\left(\Phi^{i_1\dots i_p; j_1\dots j_q}\right)_{i_1,\dots,i_p,j_1,\dots,j_q} \in C^\infty(\mathbb{R}^4)$ on Minkowski space, its transformed components, determined by (1.46) (with $(a, R) \in \mathrm{E}(3)$ replaced by $(a, L) \in \mathcal{P}$), must satisfy the same equations in $\mathbb{R}^4$. This is in particular the case if the field equations admit a geometric (that is, coordinate free) formulation.

In case of the electromagnetic field, the field equations are Maxwell's equations (1.3). Based on a relativistic generalization of the Lorentz force (recall (1.6) and see e.g. [4, Section 7.2.3]), it is natural to associate the field with a rank two tensor (as a relativistic force, both its input and output are smooth vector fields so that, by duality, it can be identified with a two form). It turns out that the identification of the field with a two form $F \in \Omega^2(\mathbb{R}^4)$ leads to a geometric reformulation of Maxwell's equations (1.3).

So, consider smooth (time-dependent) electric $\mathbf{E} = (E_1, E_2, E_3) \in C^\infty(\mathbb{R}^4, \mathbb{R}^3)$ and magnetic $\mathbf{B} = (B_1, B_2, B_3) \in C^\infty(\mathbb{R}^4, \mathbb{R}^3)$ fields as well as smooth charge $\rho \in C^\infty(\mathbb{R}^4)$ and current densities $\mathbf{j} = (j_1, j_2, j_3) \in C^\infty(\mathbb{R}^4, \mathbb{R}^3)$. Then, one defines the electromagnetic field tensor $F \in \Omega^2(\mathbb{R}^4)$ by

$$\begin{aligned} F = {} & E_1 dx_0 \wedge dx_1 + E_2 dx_0 \wedge dx_2 + E_3 dx_0 \wedge dx_3 \\ & + B_1 dx_2 \wedge dx_3 - B_2 dx_1 \wedge dx_3 + B_3 dx_1 \wedge dx_2, \end{aligned} \tag{1.53}$$

its Hodge-$\star$ (see e.g. [12, Section 7.9.2]) dual form $\star F \in \Omega^2(\mathbb{R}^4)$ by

$$\begin{aligned} \star F = {} & -B_1 dx_0 \wedge dx_1 - B_2 dx_0 \wedge dx_2 - B_3 dx_0 \wedge dx_3 \\ & - E_1 dx_2 \wedge dx_3 + E_2 dx_1 \wedge dx_3 - E_3 dx_1 \wedge dx_2 \end{aligned} \tag{1.54}$$

and, finally, the current differential form $j \in \Omega^3(\mathbb{R}^4)$ by

$$\begin{aligned} j = {} & \rho\, dx_1 \wedge dx_2 \wedge dx_3 - j_1 dx_0 \wedge dx_2 \wedge dx_3 \\ & + j_2 dx_0 \wedge dx_1 \wedge dx_3 - j_3 dx_0 \wedge dx_1 \wedge dx_2. \end{aligned} \tag{1.55}$$

Recalling $x_0 = ct$ so that $\partial_{x_0} = \frac{1}{c}\partial_t$, Maxwell's equations take the following elegant form.

Problem 1.7. *Show explicitly that the system* (1.3), (1.4) *is equivalent to*

$$dF = 0, \quad d \star F = j, \quad dj = 0. \tag{1.56}$$

Use this to show that the system (1.3), (1.4) *is invarant under Poincaré transformations (alternatively, verify the invariance explicitly based on* (1.46)*, for* $(a, L) \in \mathcal{P}$*).*

The last problem shows that Maxwell's equations are independent of the inertial frame, which is of fundamental importance in view of the relativity principle.

Generalizing Example 1.3, the equations (1.56) can also be reformulated in terms of the electromagnetic vector potential. Indeed, applying the Poincaré lemma to the closed

form $F \in \Omega^2(\mathbb{R}^4)$, we find that there exists a one form $A \in \Omega^1(\mathbb{R}^4)$ such that $F = dA$. We can identify this with a vector $A = (\Phi, \mathbf{A}) = (\Phi, A_0, A_1, A_3) \in C^\infty(\mathbb{R}^4)$ via

$$A = \Phi dx_0 + A_1 dx_1 + A_2 dx_2 + A_3 dx_3$$

so that

$$F = \sum_{0 \le i < j \le 3} (\partial_{x_i} A_j - \partial_{x_j} A_i)\, dx_i \wedge dx_j. \tag{1.57}$$

This connects Maxwell's equations with the wave equation in a straightforward way. To see this, notice first that if $F = dA$ solves (1.56), so does $F' = dA'$ for $A' = A + d\chi$ for every $\chi \in C^\infty(\mathbb{R}^4)$, because $dA' = dA + d^2\chi = dA$. This is referred to as the gauge invariance of the electromagnetic field. Since electromagnetic forces are encoded by F, a gauge transform does not influence the physics of the system. This observation is further developed in modern gauge theories (for more on this, see Section 6.1).

Abbreviating $\partial_\nu = \partial_{x_\nu}$, $\partial^\nu = \eta^{\nu\lambda}\partial_\lambda$ and choosing $\chi \in C^\infty(\mathbb{R}^4)$ such that

$$\partial^\nu \partial_\nu \chi = -\partial^\nu A_\nu \in C^\infty(\mathbb{R}^4),$$

note that $A' = A + d\chi$ satisfies

$$\partial^\nu A'_\nu = \partial^\nu A_\nu + \partial^\nu \partial_\nu \chi = 0.$$

This choice of χ is called Lorenz gauge and can always be arranged. Note indeed that

$$\partial^\nu \partial_\nu = \eta^{\nu\lambda}\partial_\lambda \partial_\nu = \partial_{x_0}^2 - \sum_{j=1}^3 \partial_{x_j}^2 = \frac{1}{c^2}\partial_t^2 - \sum_{j=1}^3 \partial_{x_j}^2 = \Box$$

corresponds to the d'Alembertian, the generator of solutions of the wave equation. In conclusion, potentials $A \in \Omega^1(\mathbb{R}^4)$ in Lorenz gauge that solve (1.56) satisfy

$$\Box A = j, \quad \partial^\nu A_\nu = 0, \tag{1.58}$$

where $\Box A = \big(\Box A_\mu\big)_{\mu=0}^3$ and $j = (j_\mu)_{\mu=0}^3$. This corresponds componentwise to the wave equation with propagation speed c, the speed of light, and with a vector constraint.

Problem 1.8. *Prove the following statements:*

a) Let $f \in C^\infty(\mathbb{R}^4)$. Define the function $u \in C^\infty(\mathbb{R}^4)$ pointwise by

$$u(x) = u(x_0, \boldsymbol{x}) = \frac{1}{4\pi x_0} \int_0^{x_0} dy_0 \int_{|\boldsymbol{x}-\boldsymbol{y}|=x_0} \sigma(d\boldsymbol{y})\, f(y_0, \boldsymbol{x}),$$

where $\sigma(\cdot)$ denotes the uniform measure on $\{\boldsymbol{y} \in \mathbb{R}^3 : |\boldsymbol{x}-\boldsymbol{y}| = x_0\} \subset \mathbb{R}^3$. Verify that $\Box u = f$. For more on wave equations and a derivation of this formula, see [20].

b) Use Maxwell's equations in Lorenz gauge to deduce the wave equation in (1.58).

c) Let $A \in C^\infty(\mathbb{R}^4, \mathbb{R}^4)$ *satisfy* (1.58) *for* $j = 0$ *and define for* $(a, L) \in \mathcal{P}$

$$A'_\mu(x) = L^\nu{}_\mu A_\nu(Lx + a), \ \forall\, x \in \mathbb{R}^4.$$

Show that $A' \in C^\infty(\mathbb{R}^4, \mathbb{R}^4)$ *also satisfies* (1.58).

Without sources ($j = 0$), the system of equations (1.56) describes the so called free (non-interacting) electromagnetic field. Motivated by the Lorenz gauge formulation, it also turns out to be of independent interest to study scalar valued solutions $\varphi \in C^\infty(\mathbb{R}^4, \mathbb{R})$ (real scalar field) or $\varphi \in C^\infty(\mathbb{R}^4, \mathbb{C})$ (complex scalar field) of the so called Klein-Gordon equation with mass $m > 0$. Such fields solve the generalized wave equation

$$(\Box + m^2)\varphi = (\partial^\nu \partial_\nu + m^2)\varphi = 0. \tag{1.59}$$

In quantum field theory, the quantum excitations of the electromagnetic field describe photons while the quantum excitations of fields solving (1.59) correspond to bosons of mass m and spin zero (the connection to spin as well as between m and the mass of the field quanta is explained in Section 3.1). Notice that (1.59) is invariant under inertial coordinate changes. This follows by noting that $\partial^\nu \partial_\nu = \Delta_\eta$ is equal to the Laplace-Beltrami operator related to the Minkowski metric. Alternatively, we readily compute

$$\begin{aligned}(\partial^\nu \partial_\nu + m^2)\, \varphi\big(L(\cdot) + a\big) &= \eta^{\nu\mu} L^\lambda{}_\nu L^\kappa{}_\mu\, (\partial_\lambda \partial_\kappa)\varphi\big(L(\cdot) + a\big) + m^2 \varphi\big(L(\cdot) + a\big) \\ &= \big((\Box + m^2)\varphi\big)\big(L(\cdot) + a\big) = 0.\end{aligned}$$

The quantization of the scalar field is explained in detail in Section 3.1.

Problem 1.9. *On* $L^2(\mathbb{R}^3, \mathbb{R}) \oplus H^1(\mathbb{R}^3, \mathbb{R})$, *define the field Hamiltonian*

$$H(\phi, \pi) = \frac{1}{2} \int_{\mathbb{R}^3} dx\, \big(|\pi(x)|^2 + |\nabla\phi(x)|^2 + m^2|\phi(x)|^2\big).$$

Show that its partial derivatives $\partial_\pi H \in \big(L^2(\mathbb{R}^3, \mathbb{R})\big)^*, \partial_\phi H \in \big(H^1(\mathbb{R}^3, \mathbb{R})\big)^*$ *satisfy*

$$\big(\partial_\pi H\big)_{|\pi}(\rho) = \langle \pi, \rho\rangle_{L^2(\mathbb{R}^3)}, \quad \big(\partial_\phi H\big)_{|\phi}(\rho) = \langle \nabla\phi, \nabla\rho\rangle_{L^2(\mathbb{R}^3)} + m^2\langle \phi, \rho\rangle_{L^2(\mathbb{R}^3)}.$$

Symbolically, this means that $\partial_{\pi(x)} H = \pi(x)$, $\partial_{\phi(x)} H = (-\Delta + m^2)\phi(x)$ *so that we can view the model as a Hamiltonian system with continuously many canonical momenta and positions. Setting* $c = 1$, *show that the Klein-Gordon* (1.59) *is equivalent to the Hamiltonian dynamics generated by* H.

We close this section with a brief discussion of classical Dirac fields. Dirac fields are spinor valued fields on Minkowski space and as such beyond the classical notion of a tensor field. The detailed mathematical introduction of spinors is beyond the scope of these notes and we refer the interested reader to e.g. [23, Chapter 13] and [11, Section 8.4]. As the name suggests, spinor fields are related to the quantum property of spin. In particular, there is no field in classical physics described as a Dirac field. As a mathematical object, it is nevertheless useful in view of the correct relativistic description

of massive quantum particles of spin-$\frac{1}{2}$ (such as electrons, protons or neutrons). These particles can be described through the quantization of classical Dirac fields. In Section 2.2, we discuss an alternative motivation for the introduction of spinors based on the inclusion of parity as a fundamental symmetry.

Consider the Dirac γ-matrices $(\gamma^\mu)_{\mu=0}^4$, defined by

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1}_{\mathbb{C}^2} \\ \mathbf{1}_{\mathbb{C}^2} & 0 \end{pmatrix} \in \mathbb{C}^4, \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma_\mu \\ -\sigma_\mu & 0 \end{pmatrix} \in \mathbb{C}^4, \ \forall\, \mu \in \{1,2,3\}, \tag{1.60}$$

where we recall the definition of the Pauli spin matrices $\sigma_\mu \in \mathbb{C}^{2\times 2}$ from (1.34). The Dirac matrices form what's called a Clifford algebra.

Problem 1.10. *Denote by $[A,B]_+ = AB + BA$ the anticommutator. Prove that*

$$[\gamma^\mu, \gamma^\nu]_+ = 2\eta_{\mu\nu}, \ \forall\, \mu,\nu \in \{0,1,2,3\}.$$

Conclude that for every $x, y \in \mathbb{R}^4$, we have that

$$[\gamma^\mu x_\mu, \gamma^\nu y_\nu]_+ = 2\eta(x,y).$$

Problem 1.10 implies in particular the isometric property $\big(\gamma^\mu x_\mu\big)^2 = \eta(x,x)$ for every $x \in \mathbb{R}^4$ which leads to the observation that

$$\Box = \big(\gamma^\mu \partial_\mu\big)^2.$$

In other words, the linear operator $\gamma^\mu\partial_\mu$ can be considered a square-root of $\Box$. By definition, classical Dirac fields $\psi : \mathbb{R}^4 \to \mathbb{C}^4$ correspond to solutions of the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi = 0. \tag{1.61}$$

In particular, every solution $\psi \in C^\infty(\mathbb{R}^4, \mathbb{C}^4)$ also solves componentwise

$$0 = (i\gamma^\mu\partial_\mu - m)^2\psi = \big(-\Box + m^2 - 2mi\gamma^\mu\partial_\mu)\psi = -(\Box + m^2)\psi$$

the Klein-Gordon equation.

As mentioned at the beginning of this section, the previous examples describe classical free fields. As one can check explicitly (*exercise*) it turns out that the defining equations for the above examples can be obtained as critical point equations of suitable non-linear functionals (*exercise*). For instance, the functionals $\mathcal{S}_{\mathrm{SC}}$, $\mathcal{S}_{\mathrm{EM}}$ and $\mathcal{S}_{\mathrm{D}}$ for the real scalar, electromagnetic and, respectively, Dirac fields take the form

$$\begin{aligned}
\mathcal{S}_{\mathrm{SC}}(\varphi) &= \int_{[t_0,t_1]\times\mathbb{R}^3} \big(\partial^\nu\varphi\,\partial_\nu\varphi + m^2\varphi^2\big)\,dx, \\
\mathcal{S}_{\mathrm{EM}}(A,j) &= \int_{[t_0,t_1]\times\mathbb{R}^3} \big(dA\wedge\star dA - 2A\wedge j\big), \\
\mathcal{S}_{\mathrm{D}}(\psi) &= \int_{[t_0,t_1]\times\mathbb{R}^3} \overline{\psi}\,\gamma^0\big(i\gamma^\mu\partial_\mu - m)\psi\,dx.
\end{aligned}$$

Classical interacting theories can be obtained by combining fields of different type. The fundamental principle on which possible interactions can be obtained is called the local gauge principle, see e.g. [4, Section 7.4] for a brief introduction. This is discussed in detail in Section 6.1. Of independent mathematical interest are also presumably simpler interacting theories such as the ϕ^4_{d}-theories whose action functional takes the form

$$\mathcal{S}_{\phi^4_{\mathrm{d}}}(\varphi) = \int_{[t_0,t_1]\times\mathbb{R}^d} \left(\partial^\nu\varphi\,\partial_\nu\varphi + m^2\varphi^2 + \lambda\varphi^4\right) dx.$$

Critical points of $\mathcal{S}_{\phi^4_{\mathrm{d}}}$ satisfy a cubic, non-linear wave equation (*exercise*). In these models, the non-linearity is usually referred to as the self-interaction term.

1.4 Quantum Fields and Wightman's Axioms

The previous sections summarized basic concepts from classical and non-relativistic quantum physics. Quantum field theory combines the principles of quantum mechanics with special relativity. Constructive quantum field theory in particular aims at the rigorous construction of non-trivial interacting quantum field theories and the detailed analysis of their properties. To date, it remains a major open challenge in mathematical physics to construct a non-trivial interacting quantum field theory in spacetime dimension $d = 4$. In lack of such examples, we continue our discussion in the following sections within an axiomatic framework for general quantum field theories. The explicit construction of non-interacting theories in Section 3 shows that the axioms are certainly not unreasonable. This is further confirmed by the rigorous construction of non-trivial interacting theories in lower spacetime dimensions $d < 4$, outlined in Section 5.

Let us now list the Wightman axioms that describe the central notion of a quantum field theory discussed in these notes. In the rest of these notes, we largely ignore universal physical constants and set from now on in particular $c = \hbar = 1$.

0. Relativistic Quantum Theory. The possible states of the theory are described by the unit rays in a complex, separable Hilbert space $\mathcal{H}$. The state transformation law with regards to coordinate changes from one inertial frame to another is given by a strongly continuous projective unitary representation

$$\mathcal{P}^\uparrow_+ \ni (a, L) \mapsto U(a, L) \in \{U \in \mathcal{L}(\mathcal{H}) : U^*U = UU^* = \mathbf{1}_{\mathcal{H}}\}$$

of the proper Poincaré group $\mathcal{P}^\uparrow_+$. It turns out that every such representation can be obtained through a strongly continuous unitary representation of the inhomogeneous $\mathrm{SL}(2,\mathbb{C})$ (consisting of pairs $(a, L) \in \mathbb{C}^4 \times \mathrm{SL}(2,\mathbb{C})$, see Section 2.2), where

$$\mathrm{SL}(2,\mathbb{C}) = \left\{L \in \mathbb{C}^{2\times 2} : \det L = 1\right\} \tag{1.62}$$

denotes the special linear group. This is a consequence of the main result of [1] (of which Corollary 1.1 was a special case). In the sequel, we might therefore identify the representation of $\mathcal{P}^\uparrow_+$ directly with a representation of the inhomogeneous $\mathrm{SL}(2,\mathbb{C})$.

The relativistic four momentum operator $(P_\mu)_{\mu=0}^4$ is identified through an application of Stone's Theorem 1.2 with the generators of spacetime translations $a \in \mathbb{R}^4$, that is

$$U(a, \mathbf{1}_{\mathbb{R}^4}) = e^{iP^\mu a_\mu}.$$

To be more precise, one applies the following generalization of Theorem 1.2, whose proof is left as an *exercise* (alternatively, one may consult [14, Chapter VIII]), to the strongly continuous family of unitary operators $\big(U(a, \mathbf{1}_{\mathbb{R}^4})\big)_{a\in\mathbb{R}^4}$.

Theorem 1.5. *Let $\mathbb{R}^n \ni x \mapsto U(x)$ be a strongly continuous map of $\mathbb{R}^n$ into the set of unitary operators on some separable Hilbert space $\mathcal{H}$ and such that*

$$U(x+y) = U(x)U(y), \ \forall\, x, y \in \mathbb{R}^n.$$

Set $D = \mathrm{span}\big(\int_{\mathbb{R}^n} dx\, f(x)U(x)\phi \,:\, f \in C_c^\infty(\mathbb{R}^n), \phi \in \mathcal{H}\big)$. Then D is a domain of self-adjointness for each of the generators A_j corresponding to the strongly continuous unitary groups $\mathbb{R} \ni x_j \mapsto U(0, \ldots, 0, x_j, 0, \ldots, 0) = e^{iA_j x_j}$, $A_j : D \to D$ and $[A_j, A_k] = 0$ in D. Furthermore, there exists a projection-valued measure $(\chi_\Omega)_{\Omega\in\mathcal{B}(\mathbb{R}^n)}$ such that

$$U(x) = \int_{\mathbb{R}^n} e^{ix\cdot y}\, \chi_{dy} = \prod_{j=1}^n \int_{\mathbb{R}} e^{ix_j y_j}\, \chi_{dy}(A_j) = \prod_{j=1}^n e^{ix_j A_j}.$$

Based on this identity, we write symbolically $U(x) = e^{iA^\mu x_\mu}$.

In other words, the four momentum $P = (P_0, P_1, P_2, P_3) : \mathcal{D}_P \to \mathcal{H}^4$ is equal to a densely defined, self-adjoint operator. We typically suppress the P-dependence of its spectral measure $(\chi_\Omega(P))_{\Omega\in\mathcal{B}(\mathbb{R}^4)} = (\chi_\Omega)_{\Omega\in\mathcal{B}(\mathbb{R}^4)}$. The component $P_0 = E$ is interpreted as the field energy (the Hamiltonian) and the operator $P^\mu P_\mu = m^2$ is interpreted as the square of its mass (in the non-interacting examples discussed below, the latter is by construction always a positive constant). On P, one assumes the spectral condition

$$\sigma(P) \subset \big\{p \in \mathbb{R}^4 : p^\mu p_\mu \geq 0, p_0 \geq 0\big\} = \overline{V}_+$$

for the forward light cone V_+, defined by

$$V_+ = \big\{p \in \mathbb{R}^4 : p^\mu p_\mu > 0, p_0 > 0\big\}. \tag{1.63}$$

The spectral constraint on P is motivated by the energy relation (1.52) for a single particle of mass $m > 0$ (see also the related discussions in [21, Section 1.4]).

Finally, one assumes the existence of a unique (up to multiplication by a phase) state $\psi_0 \in \mathcal{H}$, $\|\psi_0\| = 1$, which represents the vacuum and which satisfies

$$U(a, L)\psi_0 = \psi_0, \ \forall\, (a, L) \in \mathcal{P}_+^\uparrow. \tag{1.64}$$

The vacuum state has the lowest possible energy $P_0\psi_0 = 0$ (and, more generally, zero four momentum $P\psi_0 = 0$) and is usually interpreted as describing the state with no

physical particles. As such, it takes the same form in all inertial systems. If $0 \in \sigma(P)$ turns out to be a discrete eigenvalue (that is, isolated and of finite multiplicity) of P, we say that the theory has a mass gap of size $\Delta_{m^2} = \inf \sigma(P^\mu P_\mu) \cap [0, \infty) > 0$.

Following e.g. [15, Section IX.8], it is worth to point out that (1.64) is equivalent to

$$U(a, \mathbf{1}_{\mathbb{R}^4})\psi_0 = \psi_0, \ \forall \ a \in \mathbb{R}^4 \tag{1.65}$$

if $(U(a, L))_{(a,L)\in\mathcal{P}_+^\uparrow}$ forms an ordinary representation. Indeed, based on the commutator

$$U(a, L) = U(a, \mathbf{1}_{\mathbb{R}^4})U(0, L) = U(0, L)U\big(L^{-1}a, \mathbf{1}_{\mathbb{R}^4}\big),$$

for all $(a, L) \in \mathcal{P}_+^\uparrow$, we obtain that

$$\begin{aligned} f(P)U(0, L) = \int_{\mathbb{R}^4} \chi_{dx}\, f(x)U(0, L) &= \int_{\mathbb{R}^4} \chi_{dx} \Big(\int_{\mathbb{R}^4} da\, \widehat{f}(a)e^{2\pi i a\cdot x}\Big)U(0, L) \\ &= \int_{\mathbb{R}^4} da\, \widehat{f}(a)U(2\pi a, \mathbf{1}_{\mathbb{R}^4})U(0, L) \\ &= U(0, L)f(LP) \end{aligned}$$

for every $f \in \mathcal{S}(\mathbb{R}^4)$, $L \in \mathcal{L}_+^\uparrow$. Using a dominated convergence argument to approximate the characteristic function of the point $0 \in \mathbb{R}^4$, this shows in particular (*exercise*) that

$$\big[U(0, L), \chi_{\{0\}}\big] = 0.$$

By the uniqueness assumption on ψ_0, (1.65), we know that the range $\mathrm{ran}\big(\chi_{\{0\}}\big) = \mathrm{span}(\psi_0)$ is one dimensional. Thus, the previous commutator identity implies that

$$U(0, L)_{|\mathrm{ran}(\chi_{\{0\}})} : \mathrm{ran}\big(\chi_{\{0\}}\big) \to \mathrm{ran}\big(\chi_{\{0\}}\big)$$

so that $\mathcal{L}_+^\uparrow \ni L \mapsto U(0, L)_{|\mathrm{ran}(\chi_{\{0\}})}$ is a one-dimensional unitary representation of $\mathcal{L}_+^\uparrow$ (and hence commutative, irreducible). Lemma 1.4 implies that $U(0, L)_{|\mathrm{ran}(\chi_{\{0\}})}$ is a multiple of $\mathbf{1}_{\mathcal{H}}$. Since $\mathcal{L}_+^\uparrow$ (see Section 2.2) is connected and $U(0, 0) = \mathbf{1}_{\mathcal{H}}$, we get (1.64).

I. Regularity of the Field. For $n \in \mathbb{N}$, we consider n-component quantum fields $\Phi = (\Phi_1, \ldots, \Phi_n)$ whose transformation law is related to an n-dimensional representation $S : \mathcal{P}_+^\uparrow \to \mathbb{C}^{n\times n}$ of the proper Poincaré group $\mathcal{P}_+^\uparrow$. This is made more precise and exemplified in Section 3 below. The field components are assumed to form operator-valued distributions which by definition means the following.

For every $f \in \mathcal{S}(\mathbb{R}^4)$, there exist operators $\Phi(f) = (\Phi_1(f), \ldots, \Phi_n(f))$, defined on a linear, dense domain $D \subset \mathcal{H}$ such that $\psi_0 \in D$. The domains of the adjoint components $\Phi(f)^* = (\Phi_1(f)^*, \ldots, \Phi_n(f)^*)$ also contain D and we assume, moreover, that

$$U(a, L)D \subset D, \quad \Phi_k(f)D \subset D, \quad \Phi_k(f)^*D \subset D, \ \forall\ (a, L) \in \mathcal{P}_+^\uparrow, \ k \in \{1, \ldots, n\}.$$

Finally, we assume that for every $\varphi, \psi \in \mathcal{H}$ and $j \in \{1, \ldots, n\}$, the linear functional

$$\mathcal{S}(\mathbb{R}^4) \ni f \mapsto \langle \varphi, \Phi_k(f)\psi\rangle \in \mathbb{C}$$

defines a tempered distribution (see Section 2.1 for the definition and basic properties).

Let us add two comments on the regularity assumptions. First, in view of basic quantum theory, a quantum field $(\Phi_x)_{x\in\mathbb{R}^4}$ should correspond most naturally to a family of self-adjoint operators labeled by the points in Minkowski space (to every spacetime point we attach an observable). We allow for more general fields which are not necessarily self-adjoint. Such fields are useful and occur quite naturally in quantum field theory (for instance, in the form of the so called creation and annihilation operators). Second, as becomes already clear in the simplest case of a massive non-interacting scalar field, it is usually not possible to define a quantum field as an operator-valued function (that is, a self-adjoint operator Φ_x for each spacetime point $x \in \mathbb{R}^4$), but rather as an operator-valued distribution. In this sense, one interprets the operators $(\Phi(f))_{f\in\mathcal{S}(\mathbb{R}^4)}$ via

$$\Phi(f) = \int_{\mathbb{R}^4} dx\, f(x)\Phi_x \tag{1.66}$$

as a smeared version of the field $(\Phi_x)_{x\in\mathbb{R}^4}$, which is only to be understood symbolically. Mathematically, there is of course no necessity to restrict to tempered distributions and one might consider more general fields in this regard as well. The temperedness assumption is nevertheless useful and allows to work with the Fourier transform.

II. Transformation Behavior of the Field. With the same notation as in **I**, the transformation law of the field Φ takes the form

$$U(a,L)\Phi_k(f)U(a,L)^* = S(L)_k{}^l\Phi_l\big((a,L)f\big),\ \forall\ (a,L)\in\mathcal{P}_+^\uparrow,\ f\in\mathcal{S}(\mathbb{R}^4), \tag{1.67}$$

where we set

$$\big((a,L)f\big)(x) = f\big(L^{-1}(x-a)\big),\ \forall\ (a,L)\in\mathcal{P}_+^\uparrow, x\in\mathbb{R}^4. \tag{1.68}$$

The identity (1.67) is assumed to hold in $D \subset \mathcal{H}$. The left hand side in (1.67) is interpreted as representing the field components w.r.t. a transformed inertial frame. With the heuristic identification (1.66), note that (1.67) is equivalent to

$$U(a,L)(\Phi_k)_xU(a,L)^* = S(L)_k{}^l(\Phi_l)_{a+Lx},\ \forall\ (a,L)\in\mathcal{P}_+^\uparrow,\ x\in\mathbb{R}^4. \tag{1.69}$$

III. Local Commutativity (Microscopic Causality). For a fixed choice of $-$ or $+$, the following holds true: let $f, g \in \mathcal{S}(\mathbb{R}^4)$ satisfy $f(x)g(y) = 0$ if $\eta(x-y, x-y) \geq 0$. In this case, we call f, g spacelike separated. Then, for every such pair

$$[\Phi_k(f), \Phi_l(g)]_\pm = [\Phi_k(f), \Phi_l^*(g)]_\pm = 0,\ \forall\ k,l\in\{1,\ldots,n\}, \tag{1.70}$$

where $[A,B]_\pm = AB \pm BA$. The identities (1.70) are motivated by the causality relations of events in $(\mathbb{R}^4,\eta)$. As was noted in Section 1.3, spacelike separated events can not influence one another. It is therefore natural to assume that the corresponding fields can be jointly diagonalized (in case of self-adjoint quantum fields), which is defined mathematically by requiring that their projection valued measures commute. The possibility

for anticommutators in (1.70) is related to the quantum field theoretic discovery that half-inter spin fields can only be appropriately quantized if one imposes anticommutation relations. There is no classical or non-relativistic motivation of this assumption.

IV. Quantum Field Theories. A relativistic quantum theory satisfying **0** with a quantum field satisfying axioms **I**, **II**, **III**, is called a quantum field theory if the vacuum $\psi_0 \in \mathcal{H}$ is cyclic for the smeared fields. By definition, this means that the linear space

$$D_0 = \mathrm{span}\big\{\Phi_{i_1}(f_1)\dots\Phi_{i_k}(f_k)\psi_0 \in D : f_j \in \mathcal{S}(\mathbb{R}^4)\,\forall\, j \in \mathbb{N}, \mathbf{i} \in \{1,\dots,n\}^k, k \in \mathbb{N}\big\}$$

is dense, that is, $\overline{D}_0 = \mathcal{H}$. Loosely speaking, a non-trivial quantum field should have an impact on a sufficiently large class of states in order to be physically relevant.

Axioms **O** to **IV** make precise what we mean by a quantum field theory. In view of particle physics, one should also include axioms that relate the theory to scattering experiments. This topic is not discussed in these notes. References on possible extensions related to scattering can be found in [21, Section 3.1].

2 Some Mathematical Tools

In this chapter, we discuss basic aspects of the theory of distributions and of the Lorentz and Poincaré groups. The relevance of these topics to the description of quantum fields should be clear from the Wightman axioms described in the previous Section 1.4.

2.1 Distributions

This section discusses basic results in the theory of distributions. In addition to [21], we follow mostly [18, Chapters 1, 6 & 7] and [14, Chapter V].

2.1.1 Locally Convex Topological Vector Spaces

From a functional analytic point of view, distributions are most naturally defined in the context of locally convex topological vector spaces. In this section, we first discuss some general definitions and results in this context. In the next Section 2.1.2, we get more concrete by defining distributions and tempered distributions on open subsets $\Omega \subset \mathbb{R}^n$.

A topological vector space is a vector space X equipped with a topology such that the vector space operations $+ : X \times X \to X$ and $\cdot : X \times \mathbb{K} \to X$ are continuous and such that the point sets $\{x\}$ are closed, for every $x \in X$. Throughout this section, we always consider real $\mathbb{K} = \mathbb{R}$ or complex $\mathbb{K} = \mathbb{C}$ vector spaces.

Problem 2.1. *Let X be a topological vector space and define for $x \in X, \alpha \in \mathbb{K}$ the translation $\tau_x : X \to X$ and, respectively, multiplication operators $\mu_\alpha : X \to X$ by*

$$\tau_x(y) = x + y, \quad \mu_\alpha(y) = \alpha \cdot y,\ \forall\, y \in X.$$

Show that both τ_x and μ_α map X homeomorphically onto itself.

We call a family $\mathcal{B}$ of subsets a local base at $0 \in X$ if every $B \in \mathcal{B}$ is open, contains $0 \in B$ and if every other open neighborhood of $0 \in X$ contains an elements $B \in \mathcal{B}$. Given a local base $\mathcal{B}$ at $0 \in X$, Problem 2.1 implies that every open set in X is equal to a union of translates of elements of $\mathcal{B}$. In the following, we therefore refer to a local base at $0 \in X$ simply as a local base. Any such base determines the topology of X.

We say that X is locally convex if there exists a local base $\mathcal{B}$ whose elements are convex. We say that (X, τ) is metrizable if there exists a metric $d : X \times X \to [0, \infty)$ on X whose induced topology is equal to τ. We call X a Fréchet space if it is a locally convex metrizable topological vector space whose metric $d : X \times X \to [0, \infty)$ is translation invariant (that is, $d(x, y) = d(\tau_z x, \tau_z y) = d(x + z, y + z)$ for every $x, y, z \in X$) and such that (X, d) is complete (every Cauchy sequence converges to a limit in X). We say that X has the Heine-Borel property if every closed and bounded subset of X is compact.

Proposition 2.1. *Suppose X is a topological vector space, $K \subset X$ is compact and $C \subset X$ is closed such that $K \cap C = \emptyset$. Then $0 \in X$ has a neighborhood $U \subset X$ such that*

$$(K + U) \cap (C + U) = \emptyset.$$

As a consequence, if $\mathcal{B}$ is a local base for X, then every element in $\mathcal{B}$ contains the closure of some element of $\mathcal{B}$. Furthermore, X is a Hausdorff space.

Proof. By assumption, singletons are closed and they are certainly compact. Applying the claim on K and C to two single point sets in place of K and C thus shows that X is a Hausdorff space. To prove the general claim on K and C, on the other hand, we may assume w.l.o.g. that $K \neq \emptyset$ (so that $K + U = \emptyset$) and proceed as follows.

First of all, suppose $W \subset X$ is an open neighborhood of $0 \in X$. Then W contains another neighborhood U such that $U = -U$ and such that $U + U \subset W$. Indeed, since $0+0 = 0 \in W$ and $+ : X \times X \to X$ is continuous, by assumption, we find open sets V_1, V_2 such that $0 \in V_1, V_2$ and such that $V_1 + V_2 \subset W$. Choosing $U = V_1 \cap V_2 \cap (-V_1) \cap (-V_2)$, we see that $U = -U$ and that $U + U \subset V_1 + V_2 \subset W$, as desired.

Now, let $x \in K$. Then $x \in C^c$ and $C^c - x$ is an open neighborhood of $0 \in X$. By the preceding observation, we find $\widetilde{U}_x \subset X$ open, $0 \in \widetilde{U}_x = -\widetilde{U}_x$ such that $\widetilde{U}_x + \widetilde{U}_x \subset C^c - x$. Applying the preceding observation once more, now for $\widetilde{U}_x$, we find an open neighborhood $U_x \subset X$, $0 \in U_x = -U_x$ such that $U_x + U_x + U_x + U_x \subset \widetilde{U}_x + \widetilde{U}_x \subset C^c - x$. In particular, $x + U_x + U_x + U_x \subset x + (U_x + U_x + U_x + U_x) \subset C^c$ has empty intersection with the closed set C. Moreover, since $U_x = -U_x$, we also have that

$$(x + U_x + U_x) \cap (C + U_x) = \emptyset \quad \big(\iff \quad (x + U_x + U_x - U_x) \cap C = \emptyset \big).$$

Now, K is compact so that we find finitely many points $x_1, \dots, x_n \in K$ such that

$$K \subset \bigcup_{j=1}^{n} (x_j + U_{x_j}).$$

Thus, setting $U = \bigcap_{j=1}^{n} U_{x_j}$, we conclude that

$$K + U \subset \bigcup_{j=1}^{n} (x_j + U_{x_j} + U) \subset \bigcup_{j=1}^{n} (x_j + U_{x_j} + U_{x_j})$$

has empty intersection with $C+U$. Finally, for the statement about $\mathcal{B}$, notice that

$$C+U=\bigcup_{x\in C}(x+U)$$

is open, because the topology of X is translation invariant. The previous observation thus shows that $K+U\subset(C+U)^c$ so that also its closure $\overline{K+U}\cap(C+U)=\emptyset$. Thus, if $\mathcal{B}$ is a local base for X and $\widetilde{B}\in\mathcal{B}$ is an open neighborhood of $0\in X$, choose $K=\{0\}$ and $C=\widetilde{B}^c$ to find an open neighborhood $U\subset X$ of $0\in X$ and thus, by definition of a local base, an element $B\subset U$ with $0\in B$ such that

$$\overline{B}\subset(\widetilde{B}^c+U)^c=\Big(\bigcup_{x\in U}(x+\widetilde{B}^c)\Big)^c=\bigcap_{x\in U}(x+\widetilde{B})\subset\widetilde{B}.$$

□

We call a subset $S\subset X$ balanced if $\alpha S\subset S$ for every $\alpha\in\mathbb{K}$ with $|\alpha|\leq 1$.

Proposition 2.2. *Let X be a topological vector space. Then, every neighborhood of $0\in X$ contains a balanced neighborhood of $0\in X$ and every convex neighborhood $0\in X$ contains a balanced convex neighborhood of $0\in X$. In particular, every locally convex space X admits a balanced convex local base.*

Proof. Let $B\subset X$ be an open neighborhood of $0\in X$. First we show that B contains a balanced neighborhood W. To see this, recall that multiplication by $\alpha\in\mathbb{K}$ is a homeomorphism and in particular continuous. This means that $\cdot(V\times U_\delta)\in B$ for some $U_\delta\subset\mathbb{K}$ or, in other words, that $\alpha V\subset B$ for all $|\alpha|\leq\delta$ for some $\delta>0$. Thus

$$W=\bigcup_{0\leq|\alpha|<\delta:\alpha V\subset B}\alpha V\subset B$$

is equal to a balanced open neighborhood contained in B.

In the next step, assume additionally that B is convex and let us construct a balanced convex neighborhood $S\subset B$. To this end, consider $A=\bigcap_{|\alpha|=1}\alpha B$ and choose $W\subset B$ as above. Then, for $|\alpha|=1$, we have that $\alpha^{-1}W\subset W$ so that $W\subset\bigcap_{|\alpha|=1}\alpha B=A\subset B$. Now, W is open, so $W\subset A^\circ$ is contained in the interior of A which, as an intersection of convex sets, is convex. This means that also A° is convex (*exercise*). A° (containing $0\in X$) is also balanced if A is (*exercise*). To show that A is balanced, on the other hand, let $|\beta|\leq 1$ and notice that $\beta\alpha B=|\beta|\alpha B\subset\alpha B$ for every $|\alpha|=1$, because αB is by assumption convex and contains $0\in X$. Thus, we have for all $|\beta|\leq 1$

$$\beta A=\bigcap_{|\alpha|=1}|\beta|\alpha B\subset A.$$

This implies that $A^\circ\subset B$ is an open convex balanced neighborhood of $0\in X$. □

In a topological vector space X, we call a set $S\subset X$ bounded if for every neighborhood $U\subset X$ of $0\in X$ we have that $S\subset tU$ for every $t>s$, for some $s>0$.

Proposition 2.3. *Let B be an open neighborhood of $0 \in X$ in the topological vector space X. Assume that $(a_n)_{n\in\mathbb{N}}$ is increasing with $\lim_{n\to\infty} a_n = \infty$. Then*

$$X = \bigcup_{n\in\mathbb{N}} a_n B.$$

If $(b_n)_{n\in\mathbb{N}}$ is decreasing with $\lim_{n\to\infty} b_n = 0$ and B is in addition bounded, then

$$\mathcal{B} = \{b_n B : n \in \mathbb{N}\}$$

forms a local base for X.

Proof. For the first part, fix $x \in X$ and notice that $\{\alpha \in \mathbb{K} : \alpha x \in B\}$ is open, by continuity of scalar multiplication, and contains $0 \in X$, by assumption on B. Since $\lim_{n\to\infty} a_n^{-1} = 0$, we must have $a_n^{-1} x \in B$, that is $x \in a_n B$, for $n \in \mathbb{N}$ large enough.

For the second part, let U be an open neighborhood of $0 \in X$. Since B is assumed to be bounded, we have that $B \subset tU$ for all $t > 0$ large enough. This shows that for all $n \in \mathbb{N}$ large enough, we have $B \subset b_n^{-1} U$, that is $b_n B \subset U$. □

Problem 2.2. *Let X be a topological vector space. Show that the closure of a bounded set is bounded and that every compact set is bounded.*

Next, let us recall some results on metrizability in the context of topological vector spaces.

2.1.2 Distributions and Tempered Distributions

2.1.3 Regularity and Nuclear Theorems for Tempered Distributions

2.1.4 The Laplace Transform of Tempered Distributions

2.2 The Lorentz and Poincaré Groups

3 Construction of Free Quantum Fields

3.1 The Free Massive Scalar Field

3.2 Representation Theory of the Lorentz and Poincaré Groups

3.3 The Free Massive Vector and Dirac Fields

4 General Properties of Quantum Field Theories

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