Summary of the course

Analysis on metric spaces

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This is only a summary of the main results and arguments discussed in class and *not* a complete set of lecture notes. These notes can thus not replace the careful study of the literature. In particular, these notes only contain selected proofs. As discussed in class, among others the following book is recommended:

- Juha Heinonen, Lectures on Analysis on Metric Spaces, Springer, 2001.

These notes are based on the book mentioned above and further sources which are not always mentioned specifically.

These notes are only for the use of the students in the class 'Analysis on metric spaces' at Bonn University, Winter term 2019/20

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Contents

[8.10. 2019, Lecture 1]

1. Introduction

1.1. **Motivation.** Overarching theme in analysis and geometry: pass from smooth to nonsmooth objects.

Theory of PDE: weak solutions and Sobolev spaces have revolutionized the theory. Existence 'easy'. Then develop regularity theory to show that weak solutions are everywhere or at least one large sets better, even smooth.

Geometry: just one example. Gromov's (pre)compactness theorem states that the set of compact Riemannian manifolds of a given dimension, with Ricci curvature $\geq c$ and diameter $\leq D$ is relatively compact in the Gromov–Hausdorff metric [3, 4]. Limit spaces are in general no longer smooth manifolds.

Goals of this lecture:

(1) Extend concept of Sobolev functions to functions defined on non-smooth spaces
(2) Rigidity results and differentiability
(3) Interesting maps between non-smooth spaces, in particular, quasiconformal and quasisymmetric maps

Main reference: [8]

Ad 1. Usual definition relies on weak derivatives. This requires a differential structure. What can one do if there is no differential structure.

First hint: For nice sets in $\mathbb{R}^n$ we ahve $W^{1,\infty}$ = Lipschitz and the number $|\nabla u(x)|$ = is the 'optimal local Lipschitz constant'.

Second hint: The Sobolev embedding theorem for $u \in W^{1,p}$ and $p > q > n$ gives the following estimate. If $r = 2|x-y|$ then

$$\begin{aligned} |u(x) - u(y)| &\leq C|x-y| \left(|B_r(x)|^{-1} \int_{B_r(x)} |\nabla u|^q \, dz \right)^{1/q} \\ &\leq C|x-y| \underbrace{(M(|\nabla u|^q))^{1/q}}_{L^p \text{ function}}(x). \end{aligned}$$

Third hint: A function belongs to the Sobolev space $W^{1,p}((0,1)^n)$ if and only if it absolutely continuous on a.e. line and the directional derivatives agree a..e with a function in $L^p((0,1)^n)$. The latter space is sometimes denoted as ACL_p.

Ad 2. Lip maps are differentiable a.e.
My favourite example of rigidity ('Liouville's theorem'): if $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ and $Df(x) \in SO(n)$ a.e. then Df locally constant and f locally affine.
In words: infinitesimal isometries are isometries; isometries are affine (the latter implication strongly uses the Hilbert structure)

Conformal maps: $f \in W^{1,n}$, $p \geq n$, $Df(x) = \lambda(x)R(x)$ a.e., $\lambda(x) \geq 0$, $R(x) \in SO(n)$. For $n = 2$ such maps are holomorphic and hence analytic. For $n \neq 3$ they are even more rigid. They must be Möbius maps, i.e., compositions of rigid motions, dilations and reflections on the unit sphere. This conclusion fails for $p < n/2$. In even dimensions the positive result still holds for $p = n/2$. In odd dimensions the optimal p for which the conclusion holds is not known (we only know that it is strictly less than n and $\geq n/2$).
Reformulation of the assumption without derivatives (if $\lambda > 0$ a.e) : for a.e. x

$$\lim_{r \to 0} \frac{\max_{y \in B(x,r)} d(f(y), f(x))}{\min_{y \in B(x,r)} d(f(y), f(x)} = 1$$

and f 'orientation preserving' (the can be expressed, for example, by using the degree which is defined for continuous functions).

Quasiconformal maps: f homeomorphism, $f \in W^{1,n}$, $\det Df \geq 0$ a.e. $|Df|^n \leq K \det Df$. In $\mathbb{R}^n$ qc maps have higher regularity properties: $f \in W^{1,p}$ for some $p > n$. Alternative: either $f =$ const or $\det Df > 0$ a.e.
There are many quasiconformal maps on $\mathbb{R}^n$ (for $n = 2$ one can construct them by solving a Beltrami equation of the form $\partial_{\bar{z}} f = \mu(z) \partial_z f$ for some measurable μ with $\|\mu\|_{L^\infty} < 1$. We shall see later that in spaces different from $\mathbb{R}^n$ qc maps can be much more rigid.
Reformulation without derivatives (for $\lambda > 0$ a.e.)

$$\limsup_{r \to 0} \frac{\max_{y \in B(x,r)} d(f(y), f(x))}{\min_{y \in B(x,r)} d(f(y), f(x)} \leq k$$

for a.e. x.

1.2. **Outline of the course.** Today: some general remarks and examples

Then

- First theme: Covering theorems, maximal functions, (usual) Sobolev spaces, Poincaré inequality, Sobolev spaces on metric spaces (Hajlasz version) (Chapters 1–5 in [8])
- Lipschitz functions, upper gradients, modulus of a curve family, Loewner spaces on Poincaré inequality

- Quasiconformal and quasisymmetric maps (Chapters and 6-12 in [8])
- Recent results for the Heisenberg groups and more general Carnot groups

1.3. **Example 1: fractal spaces.** Look at $[0,1]$ with metric $d_\alpha(x,y) = |x-y|^\alpha$, $\alpha \in (0,1)$.
$f : ([0,1], d_\alpha) \to ([0,1], d_1)$ Lipschitz same as f Hölder
$f : ([0,1], d_1) \to ([0,1], d_\alpha)$ Lipschitz implies f constant
Are there non-trival Sobolev maps?

Fractal spaces give a a good first hint what non-Euclidean may look like, but it is nicer to look at examples with a bit more structure.

1.4. **Example 2: the Heisenberg group $\mathbb{H}$.** We will take a rather hands-on approach. Some calculations which may look a bit miraculous have their roots in more general facts about Lie groups and the subclass of Carnot groups. We will study these more systematically later in the course.

Definition 1.1. *The Heisenberg group $\mathbb{H}$ consists of upper triangular 3×3 matrices for which all diagonal entries are 1:*

$$\mathbb{H} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

The group operation is given by the usual matrix multiplication.

The group $\mathbb{H}$ is trivially a 3-dimensional smooth manifold and group operation is smooth. Thus $\mathbb{H}$ is a Lie group.

Alternative view: $\mathbb{H} = \mathbb{R}^3$ with the group operation

$$(x,y,z) * (x',y',z') = (x+x', y+y', z+z'+xy').$$

Often we drop the $*$

It is convenient to denote the left group action by ℓ_g and the right group action by r_g. Thus

$$\ell_g(h) = g * h, \quad r_g(h) = h * g.$$

We view ℓ_g as a map from $\mathbb{R}^3$ to $\mathbb{R}^3$. This map is polynomial and in particular smooth.

So for $\mathbb{H}$ is still just $\mathbb{R}^3$ with a group operation. We look for naturally interact with the group action. This will eventually lead as to a new metric on $\mathbb{H}$ which is invariant under the group action.

Left-invariant vectorfields. On a general manifold a tangent vector can be defined as the derivative of a curve $X = \frac{d}{dt}\gamma(t)$ or as 1st order differential operator L_X. The two notions are connected by the identity Connection: $L_X f = \frac{d}{dt} f(\gamma(t))$. In $\mathbb{R}^3$ we can thus describe tangent vectors as

$$X = (a_1, a_2, a_3), \quad \text{or} \quad L_X = a_1 \frac{\partial}{\partial_{x_1}} + a_2 \frac{\partial}{\partial_{x_2}} + a_3 \frac{\partial}{\partial_{x_3}} = a_1 \frac{\partial}{\partial_x} + a_2 \frac{\partial}{\partial_y} + a_3 \frac{\partial}{\partial_z}.$$

One usually writes X instead of L_X.

Special vector fields on a Lie group: left-invariant fields. Move a tangent vector by the group action to produce a vector field. Example: $\gamma : (-1, 1) \to \mathbb{H}$ curve with $\gamma(0) = 0$. Let $X(g) = \frac{d}{dt} g * \gamma(t)$. The standard left-invariant vectorfields X_1, X_2, X_3 on $\mathbb{H}$ are obtained by taking $X_1(0) = (1, 0, 0)$ $X_2(0) = (0, 1, 0)$ and $X_3 = (0, 0, 1)$. A short calculation gives

$$\begin{aligned} X_1(x, y, z) &= \frac{d}{dt}_{t=0}(x + t, y, z + x) = (1, 0, 0), \\ X_2(x, y, z) &= \frac{d}{dt}_{t=0}(x, y + t, z + xt) = (1, 0, x), \\ X_3(x, y, z) &= \frac{d}{dt}_{t=0}(x, y, z + t) = (0, 0, 1). \end{aligned}$$

These vectorfields satisfy

$$X_i(g) = D\ell_g(0) X_i(0).$$

Since $\ell_{gh} = \ell_g \ell_h$ it follows from the chain rule that we also have

$$X_i(gh) = D_\ell g(h) X_i(h) \tag{1.1}$$

(this is usually taken as the formal definition of left-invariance).

At each point $g = (x, y, z)$ the vectorfields $X_1(g)$, $X_2(g)$ and $X_3(g)$ form a basis of (the tangent space) $\mathbb{R}^3$. The corresponding differential operators are

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \quad X_3 = \frac{\partial}{\partial z}, \tag{1.2}$$

Commutators. Given two (smooth) vectorfields X and Y we define the commutator by $[X, Y]f = X(Yf) - Y(Xf)$. Note that the commutator is again given by a first order differential operator, i.e., a vectorfield. Geometrically $[X, Y]$ measure the amount of non-commutativity of flows Φ_t and Ψ_t, defined by the vectorfields X and Y, respectively. More precisely $\Phi_t(g)$ is defined as $\gamma(t)$ where γ is the solution of the ODE $\gamma'(s) = X(\gamma(s))$ with initial value $\gamma(0) = g$.

For the Heisenberg group we have

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = 0, \quad [X_2, X_3] = 0. \tag{1.3}$$

Note that $X_1, X_2, [X_1, X_2]$ span tangent space at each point.

Idea: derivatives wrt X_1 and X_2 are more important.

Example: if f is smooth and $X_1 f = X_2 f = 0$. Then f is constant.

Does a similar argument work for Lipschitz functions? Is it true that if $|X_1 f| \le C$ and $|X_2 f| \le C$ then f is locally Lipschitz. Counterexample $f(x,y,z) = \sqrt{x^2 + y^2 + |z|}$ (details: exercise).

[8.10. 2019, Lecture 1]

[9.10. 2019, Lecture 2]

The Carnot-Caratheodory metric on $\mathbb{H}$. We look for a new metric on $\mathbb{H}$ which emphasizes the special role of X_1 and X_2.

Definition 1.2. *We say that $\gamma : [a,b] \to \mathbb{H}$ is a horizontal curve if*

$$\gamma'(t) \in \operatorname{span}(X_1(\gamma(t)), X_2(\gamma(t))) \quad \forall t \in (a,b).$$

Let $g = (x,y,z)$ and $g' = (x',y',z')$. If there exists a horizontal curve with $\gamma(a) = g$ and $\gamma(b) = g'$ we define the Carnot-Caratheodory distance as

$$\begin{aligned} &d_{\mathbb{H}}(g,g') := d_{CC}(g,g') \\ &:= \inf \left\{ \int_a^b |\gamma'(t)|_{\mathbb{H}}\, dt : \gamma \text{ horizontal curve}, \gamma(a) = g,\ \gamma(b) = g' \right\}. \end{aligned}$$

Here define length of a horizontal vectorfield $X = \alpha_1 X_1 + \alpha_2 X_2$ at a point $g \in \mathbb{R}^3$ by

$$|X(g)|_{\mathbb{H}} := \Big(\sum_{i=1}^{2} \alpha_i(g)^2 \Big)^{1/2}. \tag{1.4}$$

Remarks. Equivalently one can consider piecewise C^1 curves. The new distance is left-invariant, i.e. $d_{\mathbb{H}}(g*h, h*h') = d_{\mathbb{H}}(h,h')$ for all $g,h,h' \in \mathbb{H}$. This follows easily from the property (1.1) which implies that ℓ_g maps horizontal curves to horizontal curves of equal length.

It follows easily from the following result that two points in $\mathbb{H}$ can always be connected by a horizontal curve.

Proposition 1.3 (Horizontal lifts)**.** *Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the map given by $\pi(a_1,a_2,a_3) = (a_1,a_2)$. Let $\eta : [a,b] \to \mathbb{R}^2$ be a C^1 curve and let $c_3 \in \mathbb{R}$. Then there exists a unique horizontal curve C^1 curve $\gamma : [a,b] \to \mathbb{H}$ such that*

(1) $\pi \circ \gamma = \eta$,
(2) $\gamma_3(a) = c_3$.

If κ is a closed curve then

(1.5) $\gamma_3(b) - \gamma_3(a) =$ *oriented area (with multiplicity) enclosed by* κ.

Moreover

$$\int_a^b |\gamma'(s)|_{\mathbb{H}}\, ds = \int_a^b |\eta'(s)|_{\mathbb{R}^2}\, ds \tag{1.6}$$

where $|b|_{\mathbb{R}^2} = \sqrt{b_1^2 + b_2^2}$ *is the usual Euclidean length.*

Proof. If $g = (x, y, z)$

$$(a_1, a_2, a_3) = \alpha_1 X_1(g) + \alpha_2 X_2(g)$$

if and only if

$$\alpha_1 = a_1, \quad \alpha_2 = a_2, \quad a_3 = x a_2. \tag{1.7}$$

Thus a curve γ is horizontal if and only if

$$\gamma_3'(t) = \gamma_1(t)\gamma_2'(t).$$

Thus the unique horizontal lift γ is given by

$$\gamma_1(t) = \eta_1(t), \quad \gamma_2(t) = \eta_2(t)$$

and

$$\gamma_3(t) = c_3 + \int_a^t \eta_1(s)\eta_2'(s)\, ds$$

Consider the two-dimension vectorfield $v(x, y) = (0, x)$ The integral on the right hand side can be rewritten as

$$\int_a^t \eta_1(s)\eta_2'(s)\, ds = \int_a^t v(\eta(s)) \cdot \eta'(s)\, ds.$$

Thus if η is closed, encloses the set U and goes around A once in the anticlockwise sense then Stokes theorem implies that

$$\int_a^b v(\eta(s)) \cdot \eta'(s)\, ds = \int_\eta v \cdot \tau = \int_U \mathrm{curl} v\, dx\, dy = \mathrm{area}(U)$$

since $\mathrm{curl} v = \partial_x v_2 - \partial_y v_1 = 1$.

Finally (1.6) follows from the expression of α_1 and α_2 in (1.7) and the definition of $|a|_{\mathbb{H}}$. $\square$

Application:

$$d_{CC}(0, (0, 0, z)) = \sqrt{4\pi |z|}.$$

Sketch of proof: Let γ be a curve connecting zero to $g = (0, 0, z_0)$ and let $\eta = \pi \circ \gamma$. The η is a closed curve. Assume first that η is simple closed and encloses the set U. Then it follows from (1.5), the isoperimetric inequality and (1.6)

$$|z| = |\mathrm{area}(U)| \leq \frac{1}{4\pi}(\mathrm{length}(\eta))^2 = \frac{1}{4\pi}\left(\int_a^b |\gamma'(s)|_{\mathbb{H}}\, ds\right)^2.$$

Optimising over all curves γ which connect 0 and g we see that

$$|z| \leq \frac{1}{4\pi} d_{CC}^2(0, g).$$

One can show that the estimate only improves if η is not simply closed an one take the weighted area. On the other hand the inequality becomes sharp if we choose η as circle of radius $R =$ which passes through $(0,0)$ and choose γ as the horizontal lift with $\gamma_3(a) = 0$.

Moreover we have

$$\sqrt{x^2+y^2} \leq d_{CC}(0,(x,y,0)) \leq |x| + |y| \leq \sqrt{2}\sqrt{x^2+y^2}$$

Indeed the lower bound follows from we first deduce from (1.6). For the upper bound we consider a piecewise C^1. The first piece is the straight line from 0 to $(0,y,0)$ along this curve $\gamma_1 = 0$ and this curve is horizontal and has length $|y|$. The second piece is the straight line from $(0,y,0)$ to $(x,y,0)$. Along this curve $\gamma_2' = 0$ and hence this curve is horizontal and has length $|x|$.

Thus the metric is comparable to the Euclidean metric in the (x,y) plane and comparable to the fractal metric in the z-direction. We can make this more precise by introducing an anisotropic dilation.
Scaling of the metric. For $r > 0$ consider the dilation map defined by

$$\delta_r(x_1, x_2, x_3) := (rx_1, rx_2, r^2 x_3).$$

This maps is a group homomorphism, i.e.

$$\delta_r(g * g') = \delta_r g * \delta_r g'.$$

Check:

$$\begin{aligned} \delta_r g * \delta_r g' &= (rg_1, rg_2, r^2 g_3) * (rg_1', rg_2', r^2 g_3') \\ &= (r(g_1+g_1'), r(g_2+g_2'), r^2(g_3+g_3'+g_1 g_2')) = \delta_r(g*g'). \end{aligned}$$

Consequence: the differential of δ_r scales the the standard left-invariant vectorfields X_1 and X_2 by r (and X_3 by r^2). Indeed, $(D\delta_r)(g)X_2 = \frac{d}{dt}_{|t=0}(\delta_r(g*(0,t,0)) = \frac{d}{dt}_{|t=0}(\delta_r g) * \delta_r((0,t,0)) = X_2(\delta_r g)$ Thus it follows directly from the definition of the CC-metric that

$$d(\delta_r g, \delta_r g') = r d(g, g'). \tag{1.8}$$

Scaling of natural measure on $\mathbb{H}$. The Lebesgue measure on $\mathbb{R}^3$ is left-invariant and right-invariant, i.e. the left and right translations are volume-preserving. Thus the Lebesgue measure is the Haar measure on the group.

Proof:

$$\begin{gathered}\ell_g g' = g * g' = (x + x', y + y', z + z' + xy') \\ (d\ell_g)(0)(x', y', z') = (x', y', z' + xy') \\ d\ell_g(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{pmatrix}\end{gathered}$$

This is a lower triangular matrix with 1's on the diagonal. Hence $\det d\ell_g = 1$.

For right translation one can use that $r_{g'} g = \ell_g g'$ and one gets

$$\begin{gathered}dr_{g'}(0)(x, y, z) = (x, y, z + xy') \\ dr_{g'}(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y' & 0 & 1 \end{pmatrix}.\end{gathered}$$

Thus $\det dr_{g'}(0) = 1$.

Together with (1.8) this implies that

$$\mu(B_{CC}(r)) = r^4 \mu(B_{CC}(1)).$$

Note the the (topological, smooth) dimension of $\mathbb{H}$ is 3 not 4.

Outlook: rigidity results. Bilip maps on $\mathbb{H} \times \mathbb{H}$ factor; bilip maps on $\mathbb{H}^{\mathbb{C}}$ are holomorphic

Strategy of proof: 'algebraic step' plus 'analytic step'.

Define notion of derivative which is adapted to the group structure and the dilation ('Pansu derivative'). Differentiability at x_0 for maps $\mathbb{R}^n \to \mathbb{R}^m$: there exists a linear map $L : \mathbb{R}^m \to \mathbb{R}^n$

$$\frac{1}{r}(f(x_0 + ry) - f(x_0)) \to Ly \quad \text{uniformly for } y \text{ in a compact set.}$$

Pansu derivative for a map $f : \mathbb{H} \to \mathbb{H}$. Define left-translation $\ell_g(g') = g * g'$. The map f is Pansu differentiable at x_0 if there exists a group homomorphism $\Phi : \mathbb{H} \to \mathbb{H}$ such that

$$\delta_{r^{-1}} \circ \ell_{f(x_0)^{-1}} f(\ell_{x_0} \delta_r y) \to \Phi(y) \text{uniformly for } y \text{ in a compact set.}$$

Strategy of proof

- Key results of Pansu: Lipschitz maps are Pansu differentiable a.e.

- Study first group homomorphisms (the counterpart of linear maps on $\mathbb{R}^n$)

- Then show local 'no-switching' of Pansu derivative

Model problem for 'no-switching'.
$f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$ bilipschitz
For a.e. x: $Df(x)$ block-diagonal or anti-diagonal
Then $Df(x)$ always block-diagonal or always block anti-diagonal and $f(x) = (f_1(x_1, x_2), f_2(x_3, x_4))$ or $f(x) = (f_1(x_3, x_4), f_2(x_1, x_2))$.
Idea of proof:
$df^*(dy_1 \wedge dy_2) = 0$ (in distributions).

$$f^*(dy_1 \wedge dy_2) = a dy_1 \wedge dy_2 + b dy_3 \wedge dy_4.$$

There exists a (measurable) set E such that

$$\text{on } E: \quad a \neq 0, \quad b = 0,$$

$$\text{on } \mathbb{R}^4 \setminus E: \quad a = 0, \quad b \neq 0.$$

Now

$$\begin{aligned} 0 =& df^*(dy_1 \wedge dy_2) \\ =& (\sum_{i=1}^{4} \partial_i a dy_i) \wedge dy_1 \wedge dy_2 + (\sum_{i=1}^{4} \partial_i b dy_i) \wedge dy_3 \wedge dy_4 \\ =& \partial_3 a dy_3 \wedge dy_1 \wedge dy_2 + \dots \end{aligned}$$

Hence $\partial_3 a = \partial_4 a = 0$ and $a = a(x_1, x_2)$. Thus $\chi_E = \chi_E(x_1, x_2)$. Similarly $\chi_{U \setminus E} = \chi E(x_3, x_4)$. Hence $\chi_E = \text{const}$. This proves the result.

Take home message: it is important to develop a rigorous framework to carry out such calculations beyond the Euclidean setting.

Slightly different coordinates on $\mathbb{H}$. The coordinates chosen above are the simplest ones as they directly reflect the interpretation of $\mathbb{H}$ as a group of upper triangular matrices. On can also use the matrix exponential map to define coordinates on $\mathbb{H}$. This leads a definition which is slightly more symmetric in x and y. Recall that for a matrix A one defines

$$\exp A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

with the convention $A^0 = \mathrm{Id}$. For
$$A = \begin{pmatrix} 0 & \tilde{x} & \tilde{z} \\ 0 & 0 & \tilde{y} \\ 0 & 0 & 0 \end{pmatrix}.$$
we get
$$\exp A = \begin{pmatrix} 1 & \tilde{x} & \tilde{z} + \frac{1}{2}\tilde{x}\tilde{y} \\ 0 & 1 & \tilde{y} \\ 0 & 0 & 1 \end{pmatrix}.$$
Thus the relation between the old and new coordinates is
$$x = \tilde{x}, \quad y = \tilde{y}, \quad z = \tilde{z} + \frac{1}{2}\tilde{x}\tilde{y}$$
or
$$\tilde{x} = x, \quad \tilde{y} = y, \quad \tilde{z} = z - \frac{1}{2}xy.$$
The group action is then defined by $\exp(A * A') = \exp A \exp A'$ which gives
$$(\tilde{x}, \tilde{y}, \tilde{z}) * (\tilde{x}', \tilde{y}', \tilde{z}') = (\tilde{x} + \tilde{x}', \tilde{y} + \tilde{y}', \tilde{z} + \tilde{z}' + \frac{1}{2}(\tilde{x}\tilde{y}' - \tilde{x}'\tilde{y})).$$
Dropping the tilde again this leads to the left-invariant vectorfields
$$X_1 = \partial_x - \frac{1}{2}y\partial_z, \quad X_2 = \partial_y + \frac{1}{2}x\partial_z, \quad X_3 = \partial_z.$$
We have again
$$[X_1, X_2] = X_3$$
and the remaining calculations proceed as before.

1.5. **Example 3: sub-Riemannian manifolds.** Let M be a smooth n-dimension manifold. Let $X_1, \dots X_k$ be vector fields. Assume that there exists an integer $s \geq 2$, such the vectorfields obtained by taking commutators up to order s span the tangent space at each point.

Example: $s = 2$. The vectorfields $X_1, \dots, X_k, [X_i, X_j]$ span the tangent space at each point.
$s = 3$ $X_1, X_k, [X_i, X_j], [X_i, [X_j, X_k]$ span the tangent space at every point.

Consequence if $f : M \to \mathbb{R}$ is smooth and $X_1 f = \ldots = X_k f = 0$ then f is locally constant.

Horizontal curves and the Carnot-Caratheodory metric can be define as before.

When is a map $u : U \subset \mathbb{R}^m \to M$ Lipschitz, Sobolev, ...?

[9.10. 2019, Lecture 2]

[15.10. 2019, Lecture 3]

2. Covering theorem

This section follows very closely Chapter 1 of [8].

In the rest of these notes X will always denote a metric space.

By a ball in a metric space X we mean a pair of a centre x and a radius $r > 0$, i.e. we distinguish between two balls $B(x, r)$ and $B(y, s)$ even if they agree as sets. We refer to the set $\{z : d(x, z) < r\}$ as the open ball and $\{z : d(x, z) \leq r\}$ as the closed ball. If $B = B(x, r)$ then λB denotes the ball $B(x, \lambda r)$.

Theorem 2.1 (basic covering theorem)**.** *Let $\mathcal{F}$ be a family of balls of uniformly bounded radius. Then there exists a disjointed subfamily $\mathcal{G}$ with the following property: for every ball $B \in \mathcal{F}$ there exists a ball $B' \in \mathcal{G}$ such that*

$$B' \cap B \neq \emptyset \quad \text{radius}(B') \geq \frac{1}{2}\text{radius}B. \tag{2.1}$$

In particular

$$\bigcup_{B \in \mathcal{F}} B \subset \bigcup_{B' \in \mathcal{G}} 5B' \tag{2.2}$$

Proof. The second assertion follows from the first by the triangle inequality. The first assertion is proved by Zorn's lemma. Let Ω be the set of disjointed subcollections ω with the following property. If a ball $B \in \mathcal{F}$ meets any ball in ω then there exists a ball $B' \in \omega$ which satisfies (2.1). The collection Ω is not empty. Indeed let

$$R := \sup\{\text{radius}(B) : B \in \mathcal{F}\}.$$

Then there exists $B' \in \mathcal{F}$ such that $\text{radius}(B') \geq \frac{1}{2}R$. The collection $\omega = \{B'\}$ belongs to Ω.

By Zorn's lemma one easily sees that Ω contains a maximal element $\mathcal{G}$ (see [8]). We claim that $\mathcal{G}$ has the desired properties. Define

$$\mathcal{H} = \{B \in \mathcal{F} : B \text{ does not meet any ball in } \mathcal{G}\}.$$

If $\mathcal{H}$ is empty we are done. If not set

$$R_0 = \sup\{\text{radius}(B) : B \in \mathcal{H}\}.$$

Then there exist $B_0 \in \mathcal{H}$ such that $\text{radius}(B_0) \geq \frac{1}{2}R_0$. By definition of $\mathcal{H}$ the collection $\mathcal{G} \cup \{B_0\}$ is disjointed. It follows from the definition of R_0 that $\mathcal{G} \cup \{B_0\} \in \Omega$. This contradicts the maximality of $\mathcal{G}$. $\square$

Definition 2.2. *A measure μ is a subadditive map $\mu : 2^X \to [0, \infty]$. A set A is μ-measurable if*

$$\mu(E) = \mu(E \cap A) + \mu(E \setminus A) \quad \textit{for all sets } E.$$

A measure is Borel regular if all open sets are measurable and every set is contained in a Borel set of the same measure.

In the following we will always consider Borel regular measures. The have the following additional regularity properties, see [2, 2.2.2, 2.2.3].

Proposition 2.3. *Let μ be a Borel regular measure. And let A be a μ measurable set with $\mu(A) < \infty$. Then:*

- $\mu(A) = \sup\{\mu(C) : \ C \subset A, \ C \ \textit{closed}\}$;
- *if all metric balls have finite measure then* $\mu(A) = \inf\{\mu(U) : U \supset A, \ U \ \textit{open}\}$;

If f is a non-negative function on X (not necessarily measurable) we denote by

$$\int_X f \, d\mu$$

the upper integral.

We say that a measure is doubling if every ball has finite measure and there exists a constant C such that

$$\mu(2B) \leq C\mu(B). \tag{2.3}$$

Examples: Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue measurable with $0 < c \leq f \leq C$ a.e. and define $\mu(A) = \int_A f \, d\mathcal{L}^n$. Then μ is doubling; the volume measure on a compact Riemannian manifold is doubling; the Lebesgue measure on the Heisenberg group is doubling, in fact $\mathcal{L}^3(B(a, r)) = cr^4$.

Non-examples: The Dirac measure δ_0 is not doubling, the Hausdorff measure on lower dimensional manifold is not doubling; volume in a hyperbolic space is not globally doubling $\mu(B_r) \sim e^{Cr}$ for $r \gg 1$.

Theorem 2.4 (Vitali)**.** *Let $A \subset X$ and let $V \supset A$ be open. Let μ be a doubling measure. Let $\mathcal{F}$ be a fine cover of A, i.e., for every $a \in A$ there exists a ball $B(a, r) \in \mathcal{F}$ and*

$$\inf\{r : B(a, r) \in \mathcal{F}\} = 0.$$

Then there exists a disjointed countable subcollection $\mathcal{G}$ which covers A μ a.e., i.e.,

$$\mu(A \setminus \bigcup_{B \in \mathcal{G}} B) = 0.$$

Proof. Consider the subcollection

$$\mathcal{F}' = \{B \in \mathcal{F} : B \subset V, \text{radius}(B) \leq 1\}.$$

Then $\mathcal{F}'$ is still a fine cover of A. For bounded sets (with $\mathcal{F}$ replaced by $\mathcal{F}'$) see [8].

Modification for unbounded sets A. If V is countable union of disjoint open sets V_j then there exist subfamilies $\mathcal{F}_j$ such that $\mathcal{F}_j$ is a fine cover of $A \cap V_j$ and the balls in $\mathcal{F}_j$ are contained in V_j. Thus there exists a countable disjointed collection $\mathcal{G} = \bigcup \mathcal{G}_j$ with $\mu((V \cap A) \setminus \bigcup_{B \in \mathcal{G}} B) = 0$.

Finally note that the spheres $S_r = \{x : d(x_0, x) = r\}$ are mutually disjoint and thus $\mu(S_r) = 0$ for all but at most countably many $r \in (0, \infty)$. In particular there exists a strictly increasing sequence r_j such that $r_j \to \infty$ and $\mu(S_{r_j}) = 0$. Now apply the previous reasoning with $V_j = B(0, r_{j+1}) \setminus (B(0, r_j) \cup S_{r_j})$. Since $\mu(X \setminus \bigcup_j V_j) = 0$ we get $\mu(A \setminus \bigcup_{B \in \mathcal{G}} B) = 0$. $\square$

The Besicovitch covering theorem avoids the doubling condition but is not very useful in our context, because it does not even hold for the Heisenberg group, see [14], Section 1.4.

[15.10. 2019, Lecture 3]

[22.10. 2019, Lecture 4]

A typical application of the Vitali covering theorem is the following result.

Theorem 2.5 (Lebegue's differentiation theorem)**.** *If f is a non-negative locally integrable function on a doubling metric measure space (X, μ) then*

$$\lim_{r \to 0} \fint_{B(x,r)} f \, d\mu = f(x) \tag{2.4}$$

for a.e. $x \in X$.

Here a function is called locally integrable in X if for each $x \in X$ there exists a ball $B(x, r)$ such that f is integrable in $B(x, r)$.

We use the abbreviation

$$\fint_E f \, d\mu = \frac{1}{\mu(E)} \int_E f \, d\mu.$$

Some ideas in the proof. Let E denote the set of points where (2.4) fails. Cover E by closed balls with centers at E and radiii so small that f is integrable in each ball. By the Vitali covering theorem there is a countable union of balls of this kind containing a.e. point of E. Thus it suffices to show that E has measure zero in every ball B where f is integrable.

Main claim: if $A \subset B$ is measurable and

$$\liminf_{r \to 0} \fint_{B(x,r)} f \, d\mu \le t \quad \forall x \in A$$

then

$$\int_A f \, d\mu \le t\mu(A).$$

By applying this result to every measurable subset of A we set that in fact $f \le t$ μ a.e. in A.

Proof: Let $\varepsilon > 0$. By Proposition 2.3 there exists an open set $U \supset A$ such that $\mu(U) < \mu(A) + \varepsilon$. Let $\mathcal{F}$ we the collection of closed balls $B(a,r)$ such that $a \in A$, $B(a,r) \subset U$ and

$$\fint_{B(x,r)} f \, d\mu \le t + \varepsilon.$$

Then $\mathcal{F}$ is a fine cover of A. Let $\mathcal{G}$ be the set in the Vitali covering theorem. Since $\mathcal{G} \subset \mathcal{F}$ we have for all $B \in \mathcal{G}$

$$\int_B f \, d\mu \le (t+\varepsilon)\mu(B) \qquad \text{for all } B \in \mathcal{G}.$$

Since $\mathcal{G}$ is disjointed, $B \subset U$ and $\mu(A \setminus \bigcup_{B \in \mathcal{G}} B) = 0$ we get

$$\int_A f \, d\mu \le (t+\varepsilon)\mu(U) \le (t+\varepsilon)(\mu(A)+\varepsilon)$$

and the claim follows by taking $\varepsilon \downarrow 0$.

Similarly if

$$\limsup_{r \to 0} \fint_{B(x,r)} f \, d\mu \ge t \quad \forall x \in A$$

then

$$\int_A f \, d\mu \ge t\mu(A).$$

In particular we get $\limsup_{r \to 0} \fint_{B(x,r)} f \, d\mu < \infty$ a.e. in B.

Now let $s < t$ and consider the set $A_{s,t}$ be the set of points in B such that

$$\liminf_{r \to 0} \fint_{B(x,r)} f \, d\mu \le s < t \le \limsup_{r \to 0} \fint_{B(x,r)} f \, d\mu$$

Then $A_{s,t}$ is measurable and

$$t\mu(A_{s,t}) \leq \int_{A_{s,t}} f\,d\mu \leq s\mu(A_{s,t}).$$

This implies that $\mu(A_{s,t}) = 0$. Letting s and t for the rational numbers we see that the limit $g(x) = \lim_{r\to 0} \fint_{B(x,r)} f\,d\mu$ exists a.e. in B. A similar argument shows that the limit agrees with $f = g$ a.e. $\square$

3. Maximal functions

This section follows very closely Chapter 2 of [8].

For a locally integrable function f define the maximal function by

$$Mf(x) := \sup_{r>0} \fint_{B(x,r)} |f| \, d\mu.$$

Theorem 3.1 (Maximal function theorem [8], Thm. 2.2)**.** *Let μ be a doubling measure. Then the maximal function operator satisfies a weak $(1,1)$ estimate and a (p,p) estimate. More precisely the following estimates hold. If $f \in L^1(X)$ then*

$$\mu(\{Mf > t\}) \leq \frac{C(\mu)}{t} \int |f| \, d\mu. \tag{3.1}$$

If $p \in (1, \infty]$ then

$$\int_X |Mf|^p \, d\mu \leq C_p(\mu) \int_X |f|^p \, d\mu. \tag{3.2}$$

The constants $C(\mu)$ and $C_p(\mu)$ can be bounded in terms of the doubling constant of μ.

Proposition 3.2. *Let $\varphi : [0, \infty) \to \mathbb{R}$ be locally integrable and let*

$$\Phi(t) = \int_0^t \varphi(s) \, ds.$$

Let $f : X \to [0, \infty)$ be measurable. Then

$$\int_X \Phi(f) \, d\mu = \int_0^\infty \phi(s) \, \mu(\{f > s\} \, ds. \tag{3.3}$$

Proof. This follows directly from Fubini's theorem. Consider the set

$$E : \{(x, s) : f(x) > s\} \subset X \times \mathbb{R}.$$

Then

$$\mu(\{f > s\} = \int_X 1_E \, d\mu.$$

and

$$\int_0^\infty \phi(s) 1_E \, ds = \int_0^{f(x)} \phi(s) = \Phi(f(x)).$$

Now apply Fubini. □

Proof of Thm. 3.1. The first estimate follows from the basic covering theorem and the fact that μ is doubling. First consider the local maximal function $M_R f(x) = \sup_{r<R} \ldots$ and then pass to the limit $R \to \infty$.

For the second estimate write $f = g + b$ with $g = f1_{\{f\leq t/2\}}$. Then $Mg \leq t/2$ and hence $\{Mf > t\} \subset \{Mb > t/2\}$. Apply (3.3) to Mf with $\Phi(t) = t^p$ and estimate

$$\begin{aligned}
\mu(\{Mb > t/2\}) &\leq \int_X b\,d\mu \\
&\leq \frac{C}{t}\int_{|f|>t/2} |f|\,d\mu \\
&= \frac{C}{t}\left(\int_{|f|>t/2}\left(|f| - \frac{t}{2}\right)d\mu + \frac{t}{2}\mu(\{|f| > t/2\})\right) \\
&= \frac{C}{t}\int_{t/2}^{\infty} \mu(\{|f| > s\})\,ds + C\mu(\{|f| > t/2\}.
\end{aligned}$$

In the last step we applied (3.3) with $\Phi(s) = \max(s - \frac{t}{2}, 0)$ and $\varphi = 1_{(t/2,\infty)}$. Now for the integral

$$\int_0^\infty t^{p-2}\int_{t/2}^\infty \mu(\{|f| > s\})\,ds\,dt$$

integrate by parts in t (Homework: check that there are no boundary terms at $t = 0$ and $t = \infty$) and apply (3.3) with $\Phi(s) = s^p$. For the second contribution we can directly apply (3.3) with $\Phi(s) = s^p$. $\square$

4. Sobolev spaces on subsets of $\mathbb{R}^n$

This section follows very closely Chapter 3 of [8].

4.1. **Definition and basic properties.** We often write dx for the Lebesgue measure in $\mathbb{R}^n$.

Definition 4.1. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We say that f is weakly differentiable if for $i = 1, \dots, n$ there exist $g_i \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that*

$$\int_{\mathbb{R}^n} f \partial_i \varphi \, dx = - \int_{\mathbb{R}^n} g_i \varphi \, dx \quad \forall \varphi \in C^\infty_c(\mathbb{R}^n). \tag{4.1}$$

The functions g_i are called the weak derivatives of f and we write $\partial_i f := g_i$ and $\nabla f = g = (g_1, \dots, g_n)$.

We say that a weakly differentiable function belongs to $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ or to $W^{1,p}(\mathbb{R}^n)$ if $f, g_1, \dots, g_n$ in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ or in $L^p(\mathbb{R}^n)$, respectively. We set $\|f\|_p = \|f\|_{L^p}$ and

$$\|f\|_{1,p} :== \|f\|_p + \|\nabla f\|_p$$

Proposition 4.2. *For $1 \le p < \infty$ the space $C^\infty_c(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$. For every $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ there exist $f_k \in C^\infty(\mathbb{R}^n)$ such that*

$$f_k \to f, \quad \partial_i f_k \to \partial_i f \quad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^n).$$

Proof. This follows easily by convolution with scaled smooth functions with compact support. Use that $\partial_i(\varphi * f) = (\partial_i \varphi) * f$ and the definition of the weak derivative. □

In one dimension: $W^{1,1}_{\mathrm{loc}}$ = absolutely continuous
General: $W^{1,p} = \mathrm{ACL}_p$

Chain rules, left composition with Lipschitz functions, for $u^+ = \max(u, 0)$ we have $\nabla u^+ = \nabla u \, 1_{u>0}$ for the weak derivative.

[22.10. 2019, Lecture 4]

[23.10. 2019, Lecture 5]

4.2. **Sobolev inequalities.** Let $u \in W^{1,p}(\mathbb{R}^n)$. We have

$$\|u\|_{p^*} \le C(n,p) \|\nabla u\|_p \quad \text{if } 1 \le p < n \text{ and } p^* = \frac{np}{n-p} \tag{4.2}$$

and if $p > n$ then u has a continuous representative, which satisfies

$$|u(x) - u(y)| \le C(n,p) |x-y|^{1-n/p} \|\nabla u\|_p \quad \forall x, y \in \mathbb{R}^n. \tag{4.3}$$

Alternative characterisation of p^*:

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.$$

To prove the Sobolev inequality there are at least three strategies:

- Start from $|u(x_1,\dots,x_n)| \le \int_{-\infty}^{\infty} |\partial_1 u_i(x_1,x_2,\dots,x_n)\,dx_1$ permute coordinates, obtain and estimate for $|u(x)|^{n/(n-1)}$ and use the generalized Hölder inequality;
- use the isoperimetric inequality;
- estimate u by a convolution of $|\nabla u|$ with a suitable kernel.

In all case it suffices to show the result for functions in $C_c^\infty(\mathbb{R}^n)$.

We will follow the third approach. For $u \in C_c^\infty(\mathbb{R}^n)$ we have

$$u(x) = -\int_0^\infty D_r u(x+r\omega)\,dr.$$

$$D_r u(x+r\omega) = \nabla u(x+r\omega)\cdot\omega$$

Change of variables

$$y = x + r\omega, \quad r = |y-x|, \quad \omega = \frac{y-x}{r}.$$

$$u(x) = -\frac{1}{H^{n-1}(S^{n-1}}\int_0^\infty \int_{S^{n-1}} \nabla u(x+r\omega)\cdot\omega \frac{1}{r^{n-1}} H^{n-1}(d\omega) r^{n-1}\,dr$$

$$u(x) = C(n)\int_{\mathbb{R}^n} \frac{\nabla u(y)\cdot(y-x)}{|y-x|^n}\,dy.$$

The Riesz potential of order 1 of a locally integrable function f is defined by

$$I_1 f = |\cdot|^{1-n} * f,$$

or more explicitly

$$(I_1 f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-1}}\,dy.$$

Thus

$$|u| \le C I_1(|\nabla u|). \tag{4.4}$$

Proposition 4.3 ([8], Prop. 3.19)**.** *Let $n \ge 2$. Then the sublinear operator $f \mapsto I_1(|f|)$ maps L^1 to weak $L^{n/(n-1)}$ and L^p to $L^{np/(n-p)}$ for $1 < p < n$.*

Idea of proof. General strategy: estimate a nonlocal quantity pointwise by using the maximal function.

First consider the case $p = 1$. We want to show that

$$\mathcal{L}^n(\{I_1 f > t\}) \le \frac{C}{t^{n/(n-1)}}\|f\|_1^{n/(n-1)}.$$

Since $f \mapsto I_1 f$ is homogeneous of degree 1 it suffices to show this result for $\|f\|_1 = 1$.

Local contribution: integration over annuli gives

$$\int_{B(x,\rho)\setminus B(x,\rho/2)} \frac{f(y)}{|x-y|^{n-1}}\,dy \le C\rho\, Mf(x)$$

and summing the geometric series $2^{-k}\rho$ we obtain

$$\int_{B(x,\rho)} \frac{f(y)}{|x-y|^{n-1}}\,dy \le C\rho\, Mf(x).$$

Far field contribution:

$$\int_{\mathbb{R}^n\setminus B(x,\rho)} \frac{f(y)}{|x-y|^{n-1}}\,dy \le \rho^{1-n}\|f\|_1 \le C\rho^{1-n}.$$

Choose $\rho = \Big(Mf(x)\Big)^{-1/n}$ to balance the two contributions. This gives

$$I_1 f(x) \le C(Mf)^{1-1/n}(x)$$

Thus

$$\begin{aligned}\mathcal{L}^n(\{I_1 f > t\} &\le \mathcal{L}^n(\{Mf > t^{n/(n-1)}\} \\ &\le Ct^{-n/(n-1)}\|f\|_1 = Ct^{-n/(n-1)}\end{aligned}$$

Now assume $p \in (1,n)$. Again by homogeneity we may assume $\|f\|_p = 1$. The local estimate is the same. For the far field estimate we use that the dual exponent satisfies $p' \in (\frac{n}{n-1},\infty)$. Hence $y \mapsto |x-y|^{1-n}$ is in $L^{p'}(\mathbb{R}^n \setminus B(x,\rho))$ and

$$\int_{\mathbb{R}^n\setminus B(x,\rho)} \frac{f(y)}{|x-y|^{n-1}}\,dy \le \left(\int_{\mathbb{R}^n\setminus B(0,\rho)} |z|^{(1-n)p'}\,dz\right)^{1/p'} \le C\rho^{1-n}\rho^{n/p'}$$
$$=C\rho^{1-n/p}.$$

Choose $\rho^{-n/p} = Mf(x)$. Then

$$I_1 f(x) \le CMf(x)^{1-p/n} = CMf(x)^{(n-p)/n}$$

and thus

$$\int_{\mathbb{R}^n} |I_1 f|^{p*}\,dx \le C\int_{\mathbb{R}^n} |Mf|^p\,dx \le C\int_{\mathbb{R}^n} |f|^p\,dx = C.$$

□

Remark 4.4. *The weak estimate for $p=1$ is optimal, we can in general not control the $L^{n/(n-1)}$ norm. To see this let $f_k(x) = k^n g(kx)$, $g \ge 0$, $\int g = 1$. Then $(I_1 f_k)(x) \to x^{1-n}$.*

Nonetheless Sobolev inequality holds for $p = 1$. This follows from the following estimate

$$2^{kn/(n-1)}\mathcal{L}^n(\{2^k < |u| \le 2^{k+1}\} \le C \int_{2^{k-1} \le |u| < 2^k} |\nabla u| \, dx$$

by summing over k. To prove the estimate we apply the weak $L^{n/(n-1)}$ estimate to the function

$$v = (\min(|u|, 2^k) - 2^{k-1})^+$$

with $t = 2^{k-1}$. Note that $|u| > 2^k$ if and only if $v > 2^{k-1}$ and

$$|\nabla v| = |\nabla u| \; 1_{2^{k-1} < |u| < 2^k} \quad a.e.$$

Exercise 4.5. *Show that for $p \in (n, \infty)$*

$$|(I_1 f)(x) - (I_1 f)(z)| \le C(p, n)|x - z|^{1-n/p}\|f\|_p.$$

Hint 1: Let $r = |x - y|$. Distinguish the cases $y \in B(x, 2r)$ and $y \notin B(x, r)$ and the the latter case use that $\left| |x - y|^{1-n} - |z - y|^{1-n} \right| \le C|x - z||x - y|^{-n}$.
Hint 2: argue first that by translation and scaling we may assume $x = 0$ and $|z| = 1$. Then distinguish $y \in B(0, 2)$ and $y \notin B(0, 2)$.

The arguments used to prove Proposition 4.3 are very flexible. In particular the same reasoning yields the following result.

Theorem 4.6 ([8], Thm. 3.22)**.** *In a doubling metric measure space (X, μ) define*

$$I_1 f(x) := \int_X \frac{f(y)d(x, y)}{\mu(B(x, d(x, y)))} \, d\mu(y)$$

for a nonnegative measurable f. If there are constants $s > 1$, and $C \ge 1$ such that

$$\mu(B_r) \ge C^{-1} r^s$$

for every ball of radius $r < \operatorname{diam}(X)$ then

$$\|I_1(f)\|_{sp/(s-p),\mu} \le C(s, p, \mu)\|f\|_{p,\mu} \tag{4.5}$$

for $1 < p < s$ and

$$\mu(\{I_1(f) > t\}) \le C(s, \mu) t^{-s/(s-1)} \|f\|_1^{s/(s-1)}. \tag{4.6}$$

The proof and the rest of the subsection was not discussed in class.

Proof. Again we may assume $\|f\|_p = 1$. The main point is to show

$$I_1 f(x) \le C M f(x)^{1-p/s}.$$

Then we can proceed as before. The proof of the pointwise inequality for $I_1(f)$ is very similar to the Euclidean case. The main difference is that for $p \neq 1$ we use annuli also to estimate the far field contribution.

Local estimate: since μ is doubling we have $\mu(B(x,r/2) \geq c\mu(B(x,r)$ and thus

$$\int_{B(x,r)\setminus B(x,r/2)} \frac{f(y)d(x,y)}{\mu(B(x,d(x,y)))}\,d\mu(y) \leq CrMf(x).$$

Far field estimate: First assume that $p \in (1,s)$.

$$\begin{aligned}
&\int_{B(x,R)\setminus B(x,R/2)} \left(\frac{d(x,y)}{\mu(B(x,d(x,y)))}\right)^{p'} d\mu(y)\\
\leq &R^{p'}\mu(B(x,R/2))^{-p'}\mu(B(x,R))\\
\leq &CR^{p'}\mu(B(x,R/2))^{1-p'}\\
\leq &CR^{p'(1-s/p)}
\end{aligned}$$

Thus summing the estimate for $2^k r$ from $k=1$ to ∞ we get

$$\int_{X\setminus B(x,r)} \left(\frac{d(x,y)}{\mu(B(x,d(x,y)))}\right)^{p'} d\mu(y) \leq Cr^{p'(1-s/p)}.$$

Hence

$$\int_{X\setminus B(x,r} \frac{f(y)d(x,y)}{\mu(B(x,d(x,y)))}\,d\mu(y) \leq Cr^{1-s/p}.$$

Since $\frac{f(y)d(x,y)}{\mu(B(x,d(x,y)))} \leq Cd(x,y)^{1-s}$ this estimate also holds for $p=1$. Finally taking $r^{-s/p} = Mf(x)$ we get the desired pointwise estimate for $I_1(f)$. $\square$

The proof via the isoperimetric inequality uses the following result, see [8], equation (3.34).

Proposition 4.7. *Let $F:[0,\infty)\to[0,\infty)$ be measurable and decreasing. Then for all $\alpha\in(0,1)$*

$$\int_0^\infty F(t)^\alpha\,dt \geq \left(\frac{1}{\alpha}\int_0^\infty t^{1/\alpha-1}F(t)\,dt\right)^\alpha. \tag{4.7}$$

Proof. Assume first the F is continuous, decreasing and has support in a compact set $[0,T]$. Set

$$L(t) = \int_0^t F^\alpha(s)\,ds, \quad M(t) = \frac{1}{\alpha}\int_0^\infty t^{1/\alpha-1}F(s)\,ds, \quad R(t) = M^\alpha(t).$$

Clearly $L(0) = R(0) = 0$. It suffices to show that $L' \le R'$ on $(0, \infty)$. We have

$$L'(t) = F^\alpha(t).$$

Moreover

$$R'(t) = M^{\alpha-1}(t)\ t^{1/\alpha-1} F(t).$$

Since F is decreasing we have

$$\begin{aligned} M(t) &\ge \int_0^t \frac{1}{\alpha} s^{1/\alpha-1} F(s)\, ds \\ &\ge \int_0^t \frac{1}{\alpha} s^{1/\alpha-1}\, ds\ F(t) \\ &\ge t^{1/\alpha} F(t). \end{aligned}$$

Taking into account that $\alpha - 1 < 0$ this implies that

$$M^{\alpha-1}(t) \le t^{1-1/\alpha} F^{1-\alpha}(t).$$

Thus

$$R'(t) \le F^\alpha(t) = L'(t).$$

This proves the result for continuous functions, which are decreasing and have support in a compact set $[0, T]$. The general case follows by approximation. $\square$

[23.10. 2019, Lecture 5]

[29.10. 2019, Lecture 6]

5. The Poincaré inequality

This section follows very closely Chapter 4 of [8].

Let $B \subset \mathbb{R}^n$ be a ball. Assume that $u \in W^{1,p}(B)$ and set

$$u_B = \fint_B u \, dx.$$

If $1 \le p < n$ then

$$\|u - u_B\|_{\frac{np}{n-p},B} \le C(n,p)\|\nabla u\|_{p,B}. \tag{5.1}$$

This inequality if often refereed to us the Poincaré-Sobolev inequality. Using Hölder's inequality we deduce the Poincaré inequality

$$\|u - u_B\|_{p,B} \le C(n,p)r\|\nabla u\|_{p,B} \tag{5.2}$$

where r is the radius of the ball.

To prove the Poincaré inequality it suffices to consider smooth functions since they are dense in $W^{1,p}(B)$. Let $x, y \in B$. One starts from the identity

$$u(x) - u(y) = \int_0^{|x-y|} D_r u(x + r\omega) \, dr \quad \text{where } \omega = \frac{x-y}{|x-y|}.$$

Integrating over $y \in B$ and using polar coordinates we deduce that

$$|u(x) - u_B| \le C(n) \int_B \frac{\nabla u|(y)}{|x-y|^{n-1}} \, dy$$

or , in the language of Riesz potentials

$$|u(x) - u_B| \le C(n) I_1(|\nabla u|)(x)$$

where $|\nabla u|$ is extended by zero outside $B(x,r)$.

From the triangle inequality we get the symmetric version

$$|u(x) - u(y)| \le C(I_1(|\nabla u|)(x) + I_1(|\nabla u|)(y)) \tag{5.3}$$

valid for a smooth function u in B and all $x, y \in B$. Again we extend $|\nabla u$ by zero outside B.

By approximation we see that (5.3) holds for a.e. $x, y \in B$ if $u \in W^{1,p}(B)$. If $p > n$ and u is the continuous representative then the estimates holds for every $x, y \in B$.

Definition 5.1 (Chain condition)**.** *Given numbers $\lambda \ge 1$, $M \ge 1$ and $a > 1$ and a ball B_0 in a metric space X a bounded set $A \subset X$ in a metric space X is said to satisfy a (λ, M, a)-chain condition with respect to B_0 if for each point $x \in A$ there is a sequence of balls B_i : $i = 1, 2, \ldots\}$ such that*

(1) $\lambda B_i \subset A$ for all $i \ge 0$;

(2) B_i *is centered at* x *for sufficiently large* i;
(3) *the radius* r_i *of* B_i *satisfies*

$$M^{-1}a^{-i}\operatorname{diam} A \le r_i \le M a^{-i} \operatorname{diam} A$$

for all $i \ge 0$ *and*
(4) *the intersection* $B_i \cap B_{i+1}$ *contains a ball* B_i' *such that* $B_i \cup B_{i+1} \subset MB_i'$.

Note that the conditions are unchanged if we multiply the metric by a positive constant.

If μ is a doubling measure on X then there exists an $s > 1$ such that

$$\frac{\mu(B_r)}{\mu(A)} \ge 2^{-s}\left(\frac{r}{\operatorname{diam} A}\right)^s. \tag{5.4}$$

Theorem 5.2 (Hajlasz-Koskela, [8], Thm. 4.18)**.** *Let* (X,μ) *be a doubling space and let* $A \subset X$ *be a bounded set satisfying a* (λ, M, a)*-chain condition. Suppose that* (5.4) *holds for some* $s > 1$. *Let* u *and* g *are locally integrable function on* A *with* $g \ge 0$. *If*

$$\fint_B |u - u_B|\, d\mu \le C \operatorname{diam} B \left(\fint_{\lambda B} g^p\, d\mu\right)^{1/p} \tag{5.5}$$

for some $p \in [1, s)$, *some* $C \ge 1$ *and for all balls* B *in* X *for which* $\lambda B \subset A$, *then for each* $q < ps/(s-p)$ *there exists a constant* $C' \ge 1$ *depending only on* $q, p, s, \lambda, M, a, C$ *and the doubling constant of* μ, *such that*

$$\left(\fint_A |u - u_A|^q\, d\mu\right)^{1/q} \le C \operatorname{diam} A \left(\fint_A g^p\, d\mu\right)^{1/p}. \tag{5.6}$$

If the pair (u, g) satisfies a truncation property (satisfied, e.g., by $u \in W^{1,1}_{\mathrm{loc}}$ and $g = |\nabla u|$ then one can choose $q = ps/(s-p)$, see [7], Thm 5.1 and 9.7.

Lemma 5.3. *Let* (X,μ) *be a measure space with* $\mu(X) = 1$ *and let* u *be a measurable function on* X. *If* $s > 1$ *and if*

$$\mu(\{|u| < t\} \le C_0 t^{-s}$$

then for each $q < s$ *we have*

$$\|u\|_q \le \left(\frac{s}{s-q}\right)^{1/q} C_0^{1/s}.$$

Proof. Apply Proposition 3.2 with $\Phi(t) = t^q$ and use $\mu(\{|u| < t\} \le C_0 t^{-s}$ for $t > t_0 = C_0^{1/s}$ and $\mu(\{|u| < t\}) \le 1$ for $t \le t_0$. $\square$

Proof of Theorem 5.2. The assumption and the conclusion do not change if we multiply the measure by a positive scalar (note that condition (5.4) as well as the doubling condition are invariant under that change). Thus we may assume $\mu(A) = 1$.

We claim that we can also assume without loss of generality that $\operatorname{diam}(A) = 1$. Assume that we have shown the theorem under the additional assumption $\operatorname{diam} A = 1$. For a general metric d consider a new metric $\tilde{d} = (\operatorname{diam}_d A)^{-1} d$ and set $\tilde{g} = \operatorname{diam}_d A g$. Then $\operatorname{diam}_{\tilde{d}} A = 1$ and the assumption of the theorem holds if we replace $\operatorname{diam}_d B$ by $\operatorname{diam}_{\tilde{d}} B$ and g by $\tilde{d}$. Hence the conclusion holds with the same replacements and we easily conclude the desired result.

Note that the chain-condition and (5.4) are invariant under the passage from d to $\tilde{d}$.

Let B_0 be the fixed ball in the (λ, M, a)-chain condition. Since the assumption and conclusion are invariant under adding a constant to u we may assume that $u_{B_0} = 0$. Moreover it suffices to bound $\displaystyle\fint_A |u|^q \, d\mu$. Then the bound for u_A follows from Jensen's inequality.

In view of the previous lemma the assertion thus is easily deduced from the following

Claim. For each $\varepsilon \in (0,1)$ and $\rho = p(1-\varepsilon)/s$ we have the estimate

$$\mu(\{|u| > t\}) \le C(\varepsilon) \left(t^{-p} \int_A g^p \, d\mu \right)^{1/(1-\rho)} \tag{5.7}$$

To see that the claim implies the assertion not that for $q < p/(1-\rho) = sp/(s-(1-\varepsilon)p)$ we get from the lemma with decay exponent $\sigma = p/(1-\rho)$ and

$$C_0 = \left(C'(\varepsilon) \left(\int_A g^p \, d\mu \right)^{1/p} \right)^{\sigma}$$

the estimate

$$\|u\|_{q,A} \le C''(\varepsilon, q) \left(\int_A g^p \, d\mu \right)^{1/p}.$$

To prove the claim let A_t denote the set of Lebesgue points of $|u|$ in $\{|u| > t\}$. It suffice to show the estimate for $\mu(A_t)$. For $x \in A_t$ let B_i the chain of balls in the definition of the (λ, M, a)-chain condition. Then for each $x \in A_t$ we have

$$t \le u(x) = \lim_{i \to \infty} u_{B_i}.$$

From the fact that $B_i' \subset B_i \cap B_{i+1}$ and $B_i \cup B_{i+1} \subset MB_i$, the doubling condition and the estimate $|u_{B_{i+1}} - u_{B_i}| \le |u_{B_{i+1}} - u(y)| + |u(y) - u_{B_i}|$ we easily deduce that

$$|u_{B_{i+1}} - u_{B_i}| \le C \left(\fint_{B_{i+1}} |u - u_{B_{i+1}}| \, dy + \fint_{B_i} |u - u_{B_i}| \, dy \right).$$

Since $u_{B_0} = 0$ it follows that

$$t \le C \sum_{i=0}^{\infty} r_i \left(\fint_{\lambda B_i} g^p \, d\mu \right)^{1/p}$$

where r_i is the radius of B_i. Let we have $\sum a^{-i\varepsilon} = C'(\varepsilon)$ and thus we get

$$\sum r_i^{\varepsilon} t \le C(\varepsilon) t.$$

Thus for every $x \in A_t$ there exists an index i_x such that

$$\boxed{r_{i_x}^{\varepsilon} \, t \le C(\varepsilon) r_{i_x} \left(\fint_{\lambda B_{i_x}} g^p \, d\mu \right)^{1/p}}$$

Since $\rho = p(1-\varepsilon)/s$ this is equivalent to

$$r_{i_x}^{-s\rho} \le C(\varepsilon) t^{-p} \fint_{\lambda B_{i_x}} g^p \, d\mu.$$

By (5.4) we have

$$\frac{C}{\mu(B_{i_x})} \le r_{i_x}^{-s}$$

Thus

$$\mu(B_{i_x})^{1-\rho} \le C(\varepsilon) t^{-p} \int_{\lambda B_{i_x}} g^p \, d\mu.$$

Let $\mathcal{F}$ be the family of balls λB_{i_x}. Since $x \in B_{i_x}$ the basic covering implies that there exists a disjointed subcover $\mathcal{G}$ such that $A_t \subset \bigcup_{B \in \mathcal{G}} 5B$. Since $\mu(\lambda B_{i_x}) > 0$ and $\mu(A) < \infty$ the family $\mathcal{G}$ consists of at most countable many balls λB_j where $B_j = B_{i_{x_j}}$. Moreover Since μ is doubling we have

$$\mu(A_t) \le \sum_j \mu(5\lambda B_j) \le C \sum_j \mu(B_j).$$

Now we use that for $\beta \in (0,1)$

$$(\sum_j a_j)^\beta \leq \sum_j a_j^\beta$$

Thus

$$\mu(A_t)^{1-\rho} \leq C(\varepsilon) t^{-p} \int_A g^p \, d\mu$$

as claimed. $\square$

[29.10. 2019, Lecture 6]

[30.10. 2019, Lecture 7]

6. Sobolev spaces on metric spaces

The main references for this section are [5, 6]. This presentation follows very closely Chapter 5 of [8]. The first definition of a Sobolev space on a metric space is based on the following inequality

$$|u(x) - u(y)| \le d(x,y)(g(x) + g(y)) \quad \text{for a.e. } x, y \in X. \tag{6.1}$$

Definition 6.1 (Hajlasz Sobolev space)**.** *Let (X, μ) be a metric measure space. For $1 \le p < \infty$ define*

$$M^{1,p}(X) = \{u \in L^p(X) : \textit{there exists } g \in L^p(X) \textit{ such that } (6.1) \textit{ holds}\}$$

We define

$$\|u\|_{M^{1,p}} := \|u\|_p + \inf\{\|g\|_p : (6.1) \textit{ holds}\}.$$

If there is no danger of confusion of write $\|u\|_{1,p}$ instead of $\|u\|_{M^{1,p}}$. If $p > 1$ then one can easily show that there exists a unique $g_u \in L^p(x)$ for which the infimum in the definition of the norm of u is achieved.

Theorem 6.2. *For all $p \in [1, \infty)$ the map $u \mapsto \|u\|_{M^{1,p}}$ is a norm and with this norm $M^{1,p}$ is a Banach space.*

Theorem 6.3. *Let $p \ge 1$ and let $U \subset \mathbb{R}^n$ be open. Then the space $M^{1,p}(U)$ is continuously embedded into $W^{1,p}(U)$.*

Proof. Let $u \in M^{1,p}(U)$. Let $V \subset U$ be open and bounded and suppose that $\bar{V} \subset U$. Then there exists $h_0 > 0$ such that $B_{h_0}(V) \subset U$. For $0 < h < h_0$ and $x \in V$ consider the difference quotients

$$\Delta_i^h u(x) = \frac{u(x + he_i) - u(x)}{h}.$$

Then

$$|\Delta_i^h u(x)| \le g(x) + g(x + he_i).$$

Thus $\Delta_i^h u$ is bounded in L^p. If $p \in (1, \infty)$ then there exists a weakly convergenc subsequence in L^p with limit h_i. Moreover $|h_i| \le 2g$. Let $\varphi \in C_c^\infty(U)$. Then there exist V as above such that $\varphi \in C_c^\infty(V)$. Multiplying φ and passing the difference quotient to the test function we see that u is weakly differentiable in U with derivative h_i.

If $p = 1$ we use the fact that the difference quotients are in addition equiintegrable. Hence a subsequence converges weakly in L^1. □

Corollary 6.4. *If $X = \mathbb{R}^n$ or X is a ball in $\mathbb{R}^n$ and $p \in (1, \infty)$ then $M^{1,p}(X) = W^{1,p}(X)$ with equivalent norms.*

Proof. This follows from the estimate

$$|u(x) - u(y)| \le C(n) d(x,y)(M(|\nabla u|)(x) + M(|\nabla u|)(y))$$

and the L^p estimate for the maximal function. □

The same conclusion holds if $X \subset \mathbb{R}^n$ is an extension domain, i.e., if there exist a bounded linear extension operator $E : W^{1,p}(X) \to W^{1,p}(\mathbb{R}^n)$.

Theorem 6.5 (Poincaré inequality in $M^{1,p}$)**.** . *Let (X, μ) be a metric measure space with* $\operatorname{diam} X < \infty$ *and $\mu(X) < \infty$. Then, for all $p \geq 1$ and for all functions $u \in M^{1,p}$ we have*

$$\int_X |u - u_X|^p \, d\mu \leq 2^p (\operatorname{diam} X)^p \int_X g^p \, d\mu$$

whenever $g \geq 0$ is a function such that (6.1) *holds.*

Proof. Integrate (6.1) first in y to get a pointwise estimate for $u - u_X$. Then take the L^p norm. □

One possible approach to Sobolev spaces in an arbitrary metric measure space is to consider functions u for which an L^p function g can be found so that a Poincare inequality such as inequality (5.16) holds (not just globally but uniformly on all balls in the space). This approach has been pursued in [7].

Note that there are open bounded sets $U \subset \mathbb{R}^n$ and functions in $W^{1,p}(U)$ which do not satisfy a Poincaré inequality. Thus for those sets $W^{1,p}(U) \neq M^{1,p}(U)$.

Theorem 6.6 (Approximation by Lipschitz functions)**.** . *Let $u \in M^{1,p}(X)$ and $\varepsilon > 0$. Then there exists a Lipschitz function $v : X \to \mathbb{R}$ such that*

$$\mu(\{u \neq v\}) < \varepsilon \quad \textit{and} \quad \|u - v\|_{M^{1,p}} < \varepsilon.$$

Proof. Let

$$E_\lambda = \{x : |u(x)| \leq \lambda, g(x) \leq \lambda\}.$$

Then $\lambda^p \mu(X \setminus E_\lambda) \to 0$ as $\lambda \to \infty$. Moreover the restriction of u to E_λ is 2λ-Lipschitz. Thus there exists a 2λ-Lipschitz function u_λ which agrees with u on E_λ. Let $T_\lambda(s) = \min(\max(s, -\lambda), \lambda)$. Then T_λ is 1-Lipschitz and hence $v_\lambda := T_\lambda u_\lambda$ is 2λ-Lipschitz. Moreover $v_\lambda = u$ on E_λ. We easily deduce that $v_\lambda \to 0$ in $L^p(X)$ as $\lambda \to \infty$. We claim that

$$|(u - v_\lambda)(x) - (u - v_\lambda)(y)| \leq d(x, y)(h_\lambda(x) + h_\lambda(y))$$

where

$$h_\lambda = g 1_{X \setminus E_\lambda} + 3\lambda 1_{X \setminus E_\lambda}.$$

Indeed if both x and y are in E_λ the left hand side is zero. Assume that $x \in X \setminus E_\lambda$ then

$$|(u - v_\lambda)(x) - (u - v_\lambda)(y)| \leq d(x, y)(g(x) + g(y) + 2\lambda).$$

If $y \in X \setminus E_\lambda$ the right hand side is bounded by $h_\lambda(x) + h_\lambda(y)$. If $y \in E_\lambda$ then $g(y) \leq \lambda$ and hence the right hand side is bounded by $h_\lambda(x)$. Similar reasoning applies if $y \in X \setminus E_\lambda$. Now $h_\lambda \to 0$ in $L^p(X)$ as $\lambda \to \infty$. □

Remark 6.7. *The definition of $M^{1,p}$ is global in the following sense. If u vanishes on an open V we can in general not find a function g which vanishes on V and still satisfies* (6.1). *For an example take $X = (0,3)$ and assume that u is Lipschitz on $(0,3)$ with $u = 0$ on $(0,1)$ and $u = 1$ on $(2,3)$. Then g cannot vanish on $(0,1) \cup (2,3)$, see* [8] *for further comments. We will later introduce the concept of an 'upper gradient' which in contrast to g is local. Upper gradients, however, are only useful if the space admits sufficiently many rectifiable curves.*

7. Lipschitz functions

This section follows very closely Chapter 6 of [8]. Let X and Y be metric spaces. A map $f : X \supset A \to Y$ is Lipschitz if there exists and L such that

$$d(f(x), f(y)) \leq Ld(x, y) \quad \text{for all } x, y \in A.$$

If this inequality holds we say that f is L-Lipschitz.

Theorem 7.1. *Let X be a metric space and $A \subset X$. If $f : A \to \mathbb{R}$ is L-Lipschitz then it can be extended to an L-Lipschitz map from X to $\mathbb{R}$.*

Lemma 7.2. *Let X be metric space and let $\mathcal{F}$ be a family of L-Lipschitz maps from X to $\mathbb{R}$. If the function F defined by*

$$F(x) := \inf\{f(x) : f \in \mathcal{F}\}$$

is finite at one point then it is L-Lipschitz.

[30.10. 2019, Lecture 7]

[6.11. 2019, Lecture 8]

Theorem 7.3. *Every uniformly continuous bounded function in a metric space is a uniform limit of Lipschitz functions.*

Proof. Consider the inf-convolutions

$$f_j(x) = \inf\{f(y) + j|y - x| : y \in X\}.$$

□

Theorem 7.4. *Let $U \subset \mathbb{R}^n$ be open. Then the space $W^{1,\infty}(U)$ consists precisely of bounded functions that are locally uniformly Lipschitz on U.*

The functions need not be globally Lipschitz: consider a set which is not connected or a slit domain.

Proof. 'Locally uniformly Lipschitz' means: there exists a representative u and a number $L \geq 0$ such that for every $x \in U$ there exists a ball $B(x, r)$ such that the restriction of u to $B(x, r)$ is Lipschitz.

Thus it suffices to show that for a ball B the space $W^{1,\infty}(B)$ consists of (equivalence classes of) bounded L-Lipschitz functions and

$$L(u) := \sup_{x,y \in B, x \neq y} \frac{|u(x) - u(y)|}{|x - y|} = \|\nabla u\|_{L^\infty}. \tag{7.1}$$

Here on the right hand side we use the Euclidean norm for $|\nabla u(x)|$.

Indeed if $u \in W^{1,\infty}(B)$ then u has a continuous representative and approximating u by convolution and integrating along the line from x to y we get $L(u) \leq \|\nabla u\|_{L^\infty}$. Instead of approximation we can use a narrow pencil of lines which connect x and y and use that for $\mathcal{L}^{n-1}$ a.e. line in this pencil the restriction of u to that line is in $W^{1,\infty}$ with the same gradient bounds. In particular for each $\varepsilon > 0$ there exists such a line of length $< |x-y| + \varepsilon$.

Conversely if u is L-Lipschitz then looking at difference quotients as in the proof of Theorem 6.3 we see that u is weakly differentiable and the weak partial derivatives are bounded by L. This shows that $\|\nabla u\|_{L^\infty} \leq \sqrt{n}L(u)$. To get the optimal estimate we can consider difference quotient of the form $(u(x+ta) - u(x))/t$ for all $a \in \mathbb{Q}^n$. □

Theorem 7.5 (Rademacher's theorem)**.** *Every (locally) Lipschitz function on an open set in $\mathbb{R}^n$ is differentiable almost everywhere.*

Two lines of proof:

- Start from one dimensional case
- Use blow-up and Lebesgue points of the weak derivative

Two refinements:
Stepanoff showed that it suffices to assume

$$\limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|} < \infty$$

for almost every x in the domain of definition.

Theorem 7.6. *Let $p > n$ and let $U \subset \mathbb{R}^n$ be open. Every function in the Sobolev space $W^{1,p}(U)$ is almost everywhere differentiable. For $n = 1$ the result also holds for $p = 1$.*

Idea of proof. Key idea: look at Lebesgue points of ∇u. □

8. Modulus of a curve family, capacity and upper gradients

This section follows very closely Chapter 7 of [8].

In a general metric space we have no smooth structure and no counterpart of differentiability. We can, however, still do one dimensional calculus along rectifiable curves, i.e., curves of finite length.

8.1. **Line integrals.** A curve in a metric space X is a *continuous* map γ from an interval I to X. We often abuse terminology and call γ both the map and the image $\gamma(I)$. Curves are special for two reasons: one can define a length without any differentiable structure and a curve can always be reparametrized so that it becomes a Lipschitz map.

If $I = [a, b]$ is a closed interval then the length of a curve $\gamma : I \to X$ is

$$\ell(\gamma) = \text{length}(\gamma) = \sup \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i+1})|,$$

where the supremum is taken over all increasing sequences t_i with $a = t_1$ and $b = t_{n+1}$. If I is not closed we defined the length to be the supremum of the lengths of all closed subcurves of γ. A curve γ is *rectifiable* if its length is finite, and it is *locally rectifiable* if all its closed subcurves are rectifiable. If I is not closed then a rectifiable curve has a unique extension to a rectifiable curve on the closure of I.

The length function of a a curve $\gamma : [a, b] \to X$ is defined by

$$s_\gamma(t) := \ell(\gamma_{|[a,t]}).$$

Any rectifiable curve can be parametrized by arc length, i.e., it factors as

$$\gamma = \gamma_s \circ s_\gamma,$$

where $\gamma_s : [0, \ell(\gamma)] \to X$ is the unique 1-Lipschitz map such that the factorization holds. The curve γ_s is called the *arc length parametrization* of γ.

If γ is a rectifiable curve in X, then the line integral over γ of a Borel function $\rho : X \to [0, \infty]$ is defined as

$$\int_\gamma \rho := \int_0^{\ell(\gamma)} \rho \circ \gamma_s(t)\, dt.$$

Note that if $\varphi : I' \to I$ is an increasing (not necessarily strictly increasing) map the $(\gamma \circ \varphi)_s = \gamma_s$ and thus

$$\int_{\gamma\circ\varphi} \rho = \int_\gamma \rho.$$

If γ is only locally rectifiable we define the line integral by taking the supremum over all rectifiable subcurves.

If X is an open subset of $\mathbb{R}^n$ or of a Riemannian manifold and $\gamma \in W^{1,1}(I; X)$ then

$$\ell(\gamma) = \int_I |\gamma'(t)|\, dt$$

and

$$\int_\gamma \rho = \int_I (\rho \circ \gamma)(t)\, |\gamma'(t)|\, dt.$$

To prove the first identify one uses that the length function s_γ is additive and satisfies $s_\gamma(t') - s_\gamma(t) \leq \int_t^{t'} |\gamma'(\tau)|\, d\tau$ for $t' > t$. Hence s_γ is absolutely continuous. Now γ is differentiable a.e. and a a point t_0 of differentiability the function s_γ is also differentiable with $s'_\gamma(t_0) = |\gamma'(t_0)|$. This yields the assertion. In particular we get $|\gamma'_s| = 1$ a.e. This proves the second identity of $\gamma = \gamma_s$. The general case follows by the chain rule.

There is one subtle point: a rectifiable curve $\gamma : I \to \mathbb{R}^n$ need not be a $W^{1,1}$ map. Indeed consider the Cantor function $\gamma : [0,1] \to [0,1] \subset \mathbb{R}$. Then $\ell(\gamma) = 1$, but γ is not in $W^{1,1}(I)$. Indeed $\gamma' = 0$ almost everywhere.

8.2. **Modulus of a curve family.** Let (X, μ) be a metric measure space.

Definition 8.1. *Let Γ be a family of curves in X and let $p \in [1, \infty)$. We define the p-modulus of Γ by*

$$\operatorname{mod}_p \Gamma = \inf \int_X \rho^p \, d\mu, \tag{8.1}$$

where the infimum is taken over all Borel functions $\rho : X \to [0, \infty]$ which satisfy

$$\int_\gamma \rho\, ds \geq 1 \quad \textit{for all rectifiable } \gamma \in \Gamma.$$

Function ρ which satisfy are called admissible functions, or metrics, for the the family Γ. If Γ contains no rectifiable curve then $\rho = 0$ is admissible and $\operatorname{mod}_p \Gamma = 0$. If Γ contains a constant curve then no function ρ is admissible and hence $\operatorname{mod}_p \Gamma = \infty$ (since the infimum over an empty set is ∞).

The set function mod_p is an outer measure, i.e.,

$$\begin{aligned} \mathrm{mod}_p\emptyset &= 0, \\ \Gamma_1 \subset \Gamma_2 &\implies \mathrm{mod}_p\Gamma_1 \le \mathrm{mod}_p\Gamma_2, \\ \mathrm{mod}_p\Big(\bigcup_{i=1}^{\infty}\Gamma_i\Big) &\le \sum_{i=1}^{\infty}\mathrm{mod}_p\Gamma_i. \end{aligned}$$

Moreover we have the following property: if Γ and Γ_0 are families of curves such that each curve $\gamma \in \Gamma$ has a subcurve $\gamma_0 \in \Gamma$ then

$$\mathrm{mod}_p\Gamma \le \mathrm{mod}_p\Gamma_0 \tag{8.2}$$

since every Γ_0 admissible ρ is also Γ admissible.

Comparison to capacity: consider a compact set $K \subset \mathbb{R}^n$ and a large open ball $B \supset K$. Let Γ be the set of all rectifiable curves from K to ∂B. Let $u : \overline{B} \to \mathbb{R}$ be a C^1 function such that

$$u = 1 \text{ on } K \text{ and } u = 0 \text{ on } \partial B. \tag{8.3}$$

Then $\rho = |\nabla u|$ is admissible since

$$1 = \int_0^1 \frac{d}{dt}(u \circ \gamma)(t)\,dt \le \int_0^1 |\nabla u|(\gamma(t))\,|\gamma'(t)|\,dt = \int_\gamma |\nabla u|.$$

Here we used that a rectifiable curve in arclength parametrisation is a $W^{1,1}$ map. Thus

$$\mathrm{mod}_p\Gamma \le \inf \int_X |\nabla u|^p\,dx$$

where the infimum is taken over all C^1 function u which satisfy (8.3). The right hand side is the p-capacity of the set X (with respect to B). We will come back to the relation between modulus and capacity later.

[6.11. 2019, Lecture 8]

[12.11. 2019, Lecture 9]

In $\mathbb{R}^n$ the most important modulus from the point of view of (quasi)conformal geometry is the n-modulus $\mathrm{mod}_n\Gamma$ which is conformally invariant. This is more generally true on n-dimensional Riemannian manifolds.

A diffeomorphism $f : M^n \to N^n$ between two n-dimensional Riemannian manifolds is conformally invariant if at each points its tangent map is homothety, i.e.,

$$\langle Df(x)X, Df(x)Y\rangle_{f(x)} = \lambda(x)\langle X, Y\rangle_x$$

for all $X, Y \in T_xM^n$.

Theorem 8.2. *If $f : M^n \to N^n$ is a conformal diffeomorphism then*

$$\mathrm{mod}_n\Gamma = \mathrm{mod}_n f(\Gamma)$$

for all curve families Γ in M.

Proof. First note that f is locally Lipschitz. Thus, if $\gamma : I \to M^n$ is rectifiable, then $f \circ \gamma$ is rectifiable. We first claim that

$$\begin{aligned} \rho : N^n \to [0,\infty] \quad & f(\Gamma) \text{ admissible} \\ \Longrightarrow \rho \circ f\,|Df| \quad & \Gamma\text{-admissible.} \end{aligned}$$

Here $|Df(x)|$ denotes the operator norm of the differential as a map from T_xM^n to $T_{f(x)}N^n$. To see this let I be a compact interval and let $\gamma : I \to M^n$ be rectifiable. Then $f \circ \gamma : I \to N^n$ is rectifiable since f is Lipschitz on compact sets. Since the line integral is invariant under monotone reparametrization and since γ_s is Lipschitz we have for every $f(\Gamma)$ admissible ρ

$$\begin{aligned} 1 &\le \int_{f\circ\rho} \rho = \int_{f\circ\gamma_s} \rho \\ &= \int_I \rho \circ (f \circ \gamma_s)(t)\,|(f\circ\gamma_s)'(t)|\,dt \\ &\le \int_I (\rho\circ f)\circ\gamma_s(t)\,|Df|\circ\gamma_s(t)\,|\gamma_s'(t)|\,dt \\ &= \int_{\gamma_s} \rho\circ f\,|Df| \end{aligned}$$

This prove the claim.

Conformality of Df implies that

$$|\det Df| = |Df|^n.$$

Thus for every $f(\Gamma)$ admissible ρ we have

$$\begin{aligned} \int_{N^n} \rho^n \,\mathrm{dvol}_N^n &= \int_{M^n} \rho\circ f\,|\det Df|\,\mathrm{dvol}_{M^n} \\ = \int_{M^n} (\rho\circ f\,|Df|)^n\,\mathrm{dvol}_{M^n} &\ge \mathrm{mod}_p\Gamma. \end{aligned}$$

Hence $\mathrm{mod}_p f(\Gamma) \ge \mathrm{mod}_p\Gamma$. Applying the same reasoning to f^{-1} we get equality. $\square$

Definition 8.3. *Let $f : M^n \to N^n$ be a homeomorphism between n-dimensional Riemannian manifolds. We say that f is K-quasiconformal if and only if*

$$\frac{1}{K}\mathrm{mod}_n\Gamma \leq \mathrm{mod}_n f(\Gamma) \leq K\mathrm{mod}_n\Gamma$$

for all families of curve Γ in M^n.

Proposition 8.4. *If $f : M^n \to N^n$ is a diffeomorphism of n-dimensional Riemannian manifolds then f is K-quasiconformal if and only if*

$$\frac{1}{K}|Df|^n \leq |\det Df| \leq K|Df|^n. \tag{8.4}$$

Proof. To see that (8.4) implies that f is K-quasiconformal one proceeds exactly as in the proof of Theorem 8.2. For the converse implication one considers small cylinders C and the family Γ of straight line segments which connect the bottom and top surface of the cylinder. □

Basic example: let $0 < r < R$. Let $\Gamma_{r,R}$ be the set of all curves which connect $B(0,r) \subset \mathbb{R}^n$ and $\mathbb{R}^n \setminus B(0,R)$. Then

$$\mathrm{mod}_p\Gamma_{r,R} = \begin{cases} C(p,n)|R^{\frac{n-p}{p-1}} - r^{\frac{n-p}{p-1}}|^{1-p} & \text{if } 1 < p \neq n, \\ n\mathcal{H}^{n-1}(S^{n-1})\frac{1}{\log^{n-1}\frac{R}{r}} & \text{if } p = n. \end{cases} \tag{8.5}$$

In particular

$$\mathrm{mod}_p\Gamma_{r,R} \sim r^{n-p} \quad \text{if } p < n \text{ and } r \ll R, \tag{8.6}$$

$$\mathrm{mod}_p\Gamma_{r,R} \sim R^{p-n} \quad \text{if } p > n \text{ and } r \ll R, \tag{8.7}$$

$$\lim_{r\to 0}\mathrm{mod}_n\Gamma_{r,R} = 0. \tag{8.8}$$

Indeed, to obtain an upper bound for the p-modulus one may use radialsymmetric functions $\rho(x) = c|x|^{-\gamma}$. To get a lower bound one use the Hölder inequality to get for each $\omega \in S^{n-1}$ and each admissible ρ

$$1 \leq \int_r^R \rho(t\omega)\,dt = \int_r^R \rho(t\omega)t^{\frac{n-1}{p}}t^{-\frac{n-1}{p}}\,dt \leq \left(\int_r^R \rho^p(t\omega)t^{n-1}\,dt\right)^{\frac{1}{p}}\left(\int_r^R t^{-\frac{n-1}{p-1}}\,dt\right)^{\frac{p-1}{p}}.$$

Rewriting this as a lower bound for $\int_r^R \rho(t\omega)t^{n-1}\,dt$ and integration over ω yields the desired lower bound.

Lemma 8.5. *Suppose that (X,μ) is a metric measure space such that*

$$\mu(B_R) \leq CR^n \tag{8.9}$$

for some constant $C > 0$, exponent $n > 1$ and for all balls of radius $R > 0$. Then for all $x_0 \in X$ and all $r \in (0, \frac{R}{2})$ we have that

$$\mathrm{mod}_n \Gamma \leq C' \left(\log \frac{R}{r} \right)^{1-n}$$

where Γ denotes the family of curves joining $\overline{B}(x_0, r)$ to $X \setminus B(x_0, R)$ and C' is a constant depending only on the values of C and n in (8.9)

Proof. Exercise. Hint: consider the function

$$\rho(x) = \log^{-1} \frac{R}{r} \frac{1}{d(x, x_0)}.$$

and consider the integral of ρ^n over dyadic balls. $\square$

Corollary 8.6. *In the setting of Lemma 8.5 the n-modulus of the family of non-constant curves passing through a point is zero. This holds in particular in $\mathbb{R}^n$.*

8.3. **Upper gradients.**

Definition 8.7. *Let X be a metric space and let $u : X \to \mathbb{R}$. We say that a Borel function $\rho : X \to [0, \infty]$ is an upper gradient of u if for all $x, y \in X$ with $x \neq y$ we have*

(8.10)

$$|u(x) - u(y)| \leq \int_\gamma \rho \quad \textit{for all rectifiable curves } \gamma \textit{ connecting } x \textit{ and } y.$$

The function $\rho \equiv \infty$ is always an upper gradient. If X contains no rectifiable curves then $\rho \equiv 0$ is an upper gradient. For an L-Lipschitz function $\rho \equiv L$ is an upper gradient, but this is rarely optimal (see the following exercise).

Exercise. Assume that $u : X \to \mathbb{R}$ is Lipschitz. Show that the function ρ defined by

$$\rho(x) := \liminf_{r \to 0} \sup_{y \in B(x,r)} \frac{|u(y) - u(x)|}{r}$$

is an upper gradient of u.
Hint: consider first the case that X is an interval and construct a suitable fine cover. Then use the arc length parametrization γ_s.

One can define a Sobolev space $N^{1,p}(X)$ ('Newtonian space') as the space those L^p functions u for which are exists a $\rho \in L^p(X)$ such that (8.10) except for a family of curves of p-modulus zero, see [16, 10]. Closely related are Cheeger's $H^{1,p}(X)$ spaces, see [1], Section 2.

8.4. **Capacity.**

Definition 8.8. *Let (X,μ) be a metric measure space and let $E, F \subset X$. Then*

$$\operatorname{cap}_p(E,F) := \inf\{\int_X \rho^p \, d\mu : \rho \textit{ is an upper gradient of a function } u : X \to \mathbb{R}$$
$$\textit{with } u \leq 0 \textit{ on } E \textit{ and } u \geq 1 \textit{ on } F\} \tag{8.11}$$

Theorem 8.9. *We have that*

$$\operatorname{cap}_p(E,F) = \operatorname{mod}_p(E,F)$$

where the modulus on the right hand side is the modulus of all curves joining the sets E and F in X.

Proof. This follows essentially directly from the definitions. Note that we may assume that $E \cap F = \emptyset$ since otherwise both sides are ∞. For the estimate '$\leq$' one first considers the case that every point $x \in X$ can be joined to E by a rectifiable curve. For a ρ which is admissible for the family of curves joining E and F one then defines

$$u(x) := \inf \int_{\gamma_x} \rho \tag{8.12}$$

where the infimum is taken over all curves γ_x joining E to the point x. It is easy to see that ρ is an upper gradient of u. Indeed let γ by a rectifiable curve joining x and y, γ_x be a curve joining E to x and let γ_y be the curve γ followed by γ_x. Then

$$u(y) \leq \int_\gamma \rho + \int_{\gamma_x} \rho.$$

Taking the infimum over all γ_x we get

$$u(y) - u(x) \leq \int_\gamma \rho$$

and reversing the roles of x and y we see that ρ is an upper gradient for u.

Now assume that the set

$$A := \{x : \text{there is no rectifiable curve joining } E \text{ to } x\}$$

is non-empty. Note that $A \cap E = \emptyset$. Define $\tilde{u} : X \setminus A$ and set $u = \min(\tilde{u}, 1)$ on $X \setminus A$ and $u = 1$ on A. Now assume that there exists a rectifiable curve from x to y. Then either both x and y belong to $X \setminus A$ or they both belong to A. In the first case the previous calculation gives the desired bound for $|u(x) - u(y)|$ (since the map $t \mapsto \min(t, 1)$ is 1-Lipschitz). In the second case we have $u(x) = u(y) = 1$. □

8.5. **Sobolev function in $\mathbb{R}^n$ and upper gradients.** *This subsection was discussed on Nov 19, 2019*

Proposition 8.10. *Let $I = (a, b)$ be a bounded open interval in $\mathbb{R}$. Let $u : I \to \mathbb{R}$ be measurable.*

(1) If $u \in W^{1,1}(I)$ then the continuous representative $\bar{u}$ has the upper gradient $\rho = |u'|$ where $|u'|$ denotes the weak derivative.
(2) If $u : I \to \mathbb{R}$ has an upper gradient $\rho \in L^1(I)$ then $u \in W^{1,1}(I)$. Moreover $\rho \geq |u'|$ a.e. where u' denotes the weak derivative.

Comment: we require that ρ is Borel measurable. Every Lebesgue measurable function on I has a Borel measurable representative. Thus in the first assertion we take ρ as the Borel measurable representative of $|u'|$.

Proof. The absolutely continuous representative $\bar{u}$ satisfies

$$\bar{u}(x) = c + \int_{x_0}^{x} u'(z)\, dz.$$

It easily follows that $\rho = |u'|$ is an upper gradient.

If $\rho \in L^1(I)$ is an upper gradient one easily sees that u is absolutely continuous in I. Hence $u \in W^{1,1}$ and

$$u(x) = c + \int_{x_0}^{x} u'(z)\, dz.$$

On the other hand we have for $x - x_0 > 0$

$$c - \int_{x_0}^{x} \rho(z)\, dz \leq u(x) \leq c + \int_{x_0}^{x} \rho\, dz.$$

Thus $\rho \geq u'$ and $\rho \geq -u'$ on $(x_0, \infty) \cap I$ and we get the same assertion on $(-\infty, x_0) \cap I$. $\square$

For a locally integrable function $f : \mathbb{R}^n \to \mathbb{R}$ we define the restricted maximal function by

$$(M_R f)(x) = \sup_{0 < r \leq R} \fint_{B(x,r)} |f|\, dy \tag{8.13}$$

Note that the supremum does not change if we only consider only rational r in $(0, R]$. Since $x \mapsto \fint_{B(x,r)} |f|\, dy$ is continuous it follows that $M_R f$ is Borel measurable.

Proposition 8.11. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be measurable.*

(1) *There exists a constant C with the following property. If $u \in W^{1,p}(\mathbb{R}^n)$ then $\rho = CM_R|\nabla u|$ is an upper gradient where ∇u is the weak derivative.*
(2) *If $\rho \in L^p(\mathbb{R}^n)$ is an upper gradient of u then $u \in W^{1,p}(\Omega)$ and $\rho \geq |\nabla u|$ a.e.*

Proof. For the first assertion consider the standard mollifier $\varphi_\varepsilon = \varepsilon^{-n}\varphi(\frac{\cdot}{\varepsilon})$ with $\varphi \in C_c^\infty(B(0,1))$, $\varphi \geq 0$, $\int \varphi\, dx = 1$. For for $0 < \varepsilon < R$ the function ρ is an upper gradient of u_ε if $C = \sup \varphi$. By the Lebesgue point theorem $u_\varepsilon \to u$ a.e. Thus there exists a null set N such that

$$|u(x) - u(y)| \leq \int_\gamma \rho$$

for all $x, y \in \mathbb{R}^n \setminus N$ and all rectifiable curves joining x and y. Then a standard argument (see below) shows that u has an extension $\bar{u}$ which satisfies the same estimate for all $x, y \in \mathbb{R}^n$.

The second assertion follows from the second assertion in Proposition 8.10, the fact that L^p functions which are absolutely continuous on a.e. coordinate line with derivative in L^p belong to $W^{1,p}(I)$ and Fubini's theorem applied to ρ. □

Extension argument:
Suppose that there exists a Borel measurable function $\rho : \mathbb{R}^n \to [0, \infty]$ such that

$$|u(x) - u(y)| \leq \int_\gamma \rho$$

for all $x, y \in \mathbb{R}^n \setminus N$ and all rectifiable curves connecting x and y.

We would like to define and extension $\bar{u}$ of u which has the same property for all $x, y \in \mathbb{R}^n$. Set

$N_1 := \{x \in N :$ there exists a rectifiable curve joining x to $\mathbb{R}^n \setminus N\}$,
$N_2 := N \setminus N_1$.

and define

$$\bar{u}(x) = \inf\{u(y) + \int_\gamma \rho : y \in \Omega \setminus N,\ \gamma \text{ rectifiable curve joining } x \text{ to } y\} \quad \forall x \in N_1,$$
$$\bar{u}(x) = 0 \quad \forall x \in N_2.$$

Note that there is no rectifiable curve which joins N_2 and N_1. Indeed, otherwise there would be rectifiable curve joining N_2 and $\mathbb{R}^n \setminus N$. Thus every rectifiable curve starting in N_2 has to stay in N_2. Hence if $x \in N_2$ or $y \in N_2$ we only need to check the condition on $|u(x) - u(y)|$ for $x \in N_2$ and $y \in N_2$. In that case we have $|u(x) - u(y)| = 0$ so the desired condition holds trivially. If $y \in \mathbb{R}^n \setminus N$ and $x \in N_1$ then $u(x) \geq u(y)$ and the bound on $u(x) - u(y)$ follows from the definition of $u(x)$. Finally assume that $x, x' \in N_1$ and wlog $u(x) \geq u(x')$. Let γ be a rectifiable curve connecting x' to x. Let $\varepsilon > 0$ Then there exists a $y \in \mathbb{R}^n \setminus N$ and a curve σ connecting y to x' such that

$$u(x') \geq u(y) + \int_\sigma \rho - \varepsilon.$$

Consider the composition of γ and σ. By definition of $u(x)$ we have

$$u(x) \leq u(y) \int_{\gamma \cup \sigma} \rho \leq u(x') + \varepsilon + \int_\sigma \rho.$$

Since ε was arbitrary this concludes the proof.

[12.11. 2019, Lecture 9]

[13.11. 2019, Lecture 10]

9. Löwner spaces

This section follows very closely Chapter 8 of [8].

9.1. The Löwner p-function.

Definition 9.1. *Let X be a topological space. A subset E is a continuum if it is connected and compact. We say that a continuum is nondegenerate if it is not a point.*

We will often tacitly assume that all our continua are nondegenerate. For two nondegenerate continua E and F in a metric space X we define

$$\operatorname{dist}(E, F) = \inf\{d(x, y) : x \in E, y \in F\} = \min\{d(x, y) : x \in E, y \in F\}$$

and

$$\Delta(E, F) = \frac{\operatorname{dist}(E, F)}{\min(\operatorname{diam} E, \operatorname{diam} F)}.$$

The quantity $\Delta(E, F)$ is a measure how close the sets E and F are which is invariant under rescaling of the metric.

Let (X, μ) be a metric measure space. For $p > 1$ we define the Löwner p-function $\phi_p : (0, \infty) \to [0, \infty]$ by

$$\phi_p(t) = \inf\{\operatorname{mod}_p(E, F) : \Delta(E, F) \leq t\}. \tag{9.1}$$

Here $\operatorname{mod}_p(E, F)$ denotes the p-modulus of the family of curves which join E and F.

Definition 9.2. *Let $n > 1$. We say that (X, μ) is an n-Löwner space if $\phi_n(t) > 0$ for all $t > 0$.*

Here n need not be an integer.

Proposition 9.3. *Suppose that $\mu(X) < \infty$, X is a Löwner n-space and $p \geq n$. Then X is a Löwner p-space.*

Proof. This follows from the definition of the modulus and the Hölder inequality

$$\int_X \rho^n \, d\mu \leq \left(\int_X \rho^p \, d\mu\right)^{\frac{n}{p}} \mu(X)^{1-\frac{n}{p}}.$$

□

Proposition 9.4. *If $p \neq n$ the space $\mathbb{R}^n$ is not a Löwner p-space.*

Proof. This follows by scaling. We have

$$\Delta(\lambda E, \lambda F) = \Delta(E, F)$$

and

$$\operatorname{mod}_p(\lambda E, \lambda F) = \lambda^{n-p} \operatorname{mod}_p(E, F).$$

Taking $\lambda \to 0$ for $p < n$ and $\lambda \to \infty$ for $p > n$ we get the assertion. □

Theorem 9.5. *The space $\mathbb{R}^n$ is a Löwner n-space.*

Proof. This follows from a general result for spaces with a lower volume bound and a Poincaré inequality, see Theorem 10.11 below. Löwner's original four page proof [13] is also worth looking at. □

9.2. **Hausdorff measure, Hausdorff dimension, topological dimension and Löwner spaces.** Heinonen [8] defines the Hausdorff measure as follows.

Definition 9.6. *Let X be a metric space. For $\alpha > 0$ and $\delta > 0$ the Hausdorff premeasure $\mathcal{H}^\alpha_\delta$ of a set $E \subset X$ is defined as*

$$H^\alpha_\delta(E) = \inf\{\sum_i (\operatorname{diam} B_i)^\alpha : E \subset \bigcup_i B_i, \ B_i \textit{ closed ball}, \operatorname{diam} B_i \leq \delta\}.$$

For $\alpha > 0$ the Hausdorff measure is defined

$$\mathcal{H}^\alpha(E) = \sup_{\delta > 0} H^\alpha_\delta(E) = \lim_{\delta \to 0} H^\alpha_\delta(E).$$

For $\alpha = 0$ the measure $H^0(E)$ is the counting measure. The Hausdorff dimension is defined as

$$\dim_{\mathcal{H}} E = \inf\{\alpha > 0 : \mathcal{H}^\alpha(E) = 0\}.$$

Finally the Hausdorff content is defined as

$$H^\alpha_\infty(E) = \inf\{\sum_i (\operatorname{diam} B_i)^\alpha : E \subset \bigcup_i B_i, \ B_i \textit{ closed ball}\}.$$

Usually one allows arbitrary closed sets in the definition of the Hausdorff (pre)measure rather than closed balls. This changes the value of the Hausdorff measure at most by a factor since every closed set A is contained in a closed ball with $\operatorname{diam} B \leq 2 \operatorname{diam} A$.

In general the Hausdorff content can be much smaller than the Hausdorff measure but one easily verifies the following useful fact

$$H^{\alpha}_{\infty}(E) = 0 \quad \Longleftrightarrow \quad H^{\alpha}(E) = 0 \tag{9.2}$$

Proposition 9.7. *Assume that bounded sets in X have finite measure. If there exists a constant $C > 0$ such that*

$$\mu(B_R) \geq C^{-1} R^n$$

for all balls for radius $R \leq \operatorname{diam} X$ then $\dim_{mathcalH} X \leq n$.

Proof. First assume that X is bounded. Use the basic covering theorem to find a family of closed balls B_i of radius R_i with $\operatorname{diam} B_i \leq \delta$ which covers X such that the ball $\frac{1}{5} B_i$ are disjointed. Then for $\alpha > n$

$$\begin{aligned} H^{\alpha}_{\delta}(X) &\leq \delta^{\alpha - n} \sum_i (\operatorname{diam} B_i)^n \\ &\leq \delta^{\alpha-n} 10^n \sum (R_i/5)^n \\ &\leq C\delta^{\alpha-n} 10^n \sum : i\mu(\frac{1}{5} B_i) \\ &\leq C\delta^{\alpha-n} 10^n \mu(X). \end{aligned}$$

The assertion follows by taking $\delta \to 0$. For general X write X as a countable union of bounded sets $X = \bigcup_{k\in\mathbb{N}} (X \cap B(x_0, k))$. $\square$

Definition 9.8. *We say that a metric measure space (X, μ) is (Ahlfors) n-regular if there exists a $C \geq 1$ and $n > 0$ such that*

$$C^{-1} R^n \leq \mu(B_R) \leq C R^n \tag{9.3}$$

for all closed balls B_R of radius $R \in (0, \operatorname{diam} X)$.

Exercise. Prove that if μ is a Borel regular measure on a metric space X satisfying then there is a constant C' such that

$$(C')^{-1} \mathcal{H}_n(E) \leq \mu(E) \leq C' \mathcal{H}_n(E).$$

for all Borel sets E in X.
Hint: use approximation by open and closed sets, see [2], 2.2.2.

Proposition 9.9. *If*

$$\mu(B_R) \geq C^{-1} R^n$$

then $\dim_{\mathcal{H}} X \geq n$, in fact $\mathcal{H}^n(X) > 0$. In particular, if (X, μ) is Ahlfors n-regular then it has Hausdorff dimension n.

Proof. Suppose that $\mathcal{H}^n(X) = 0$ and let $\varepsilon > 0$. Then there exist closed balls B_i of radius R_i Then

$$\varepsilon \geq \sum_i (\operatorname{diam} B_i)^n = 2^n \sum R_i^n \geq 2^n C \sum \mu(B_i) \geq 2^n C\mu(X).$$

Thus $\mu(X) = 0$, a contradiction. $\square$

The topological dimension of a space if defined recursively. One set

$$\dim \emptyset = -1.$$

Definition 9.10. *The topological dimension n of a separable metric space X is the smallest integer n such that for each point $x \in X$ there exist arbitrarily small neighbourhoods of x whose boundary has topological dimension at most $n - 1$.*

Proposition 9.11. *The Hausdorff dimension of a metric space is at least its topological dimension.*

Proof. This follows from Theorem 9.12 below. $\square$

Theorem 9.12. *Let X be a metric space. Let n be an integer such that $\mathcal{H}^{n+1}(X) = 0$. Then the topological dimension of X is at most n.*

Proof. Let $x_0 \in X$. The main point is to show that

$$\mathcal{H}^{n+1}(X) = 0 \quad \Longrightarrow \quad \mathcal{H}^n(\partial B(x_0, r)) = 0 \quad \text{for } \mathcal{L}^1 \text{ a.e. } r > 0.$$

Then the result follows easily by induction. For the proof of this implication see [8]. Two comments on the proof: for $n = 0$ one uses the convention that $a^0 = 1$ if $a > 0$ and $0^0 = 0$. It suffices to show that $\mathcal{H}^n_\infty(\partial B(x_0, r) = 0$ (since this implies $\mathcal{H}^n(\partial B(x_0, r) = 0$. $\square$

Proposition 9.13. *If (X, μ) is an n-Löewner space, then there exists $C > 0$ such that*

$$\mu(B_R) \geq CR^n \quad \textit{for all balls of radius } R \in (0, \operatorname{diam} X).$$

Proof. It suffices to show the results for $R < \frac{1}{2} \operatorname{diam} X$. Let $x \in X$, then there exists a point $y \notin B(x, R)$. Consider a path σ which connects x and y. Then σ contains a subpath σ_1 which connects x to $\partial B(x, R/4)$ in $\overline{B}(x, R/4)$ and a subpath σ_2 which connects $\partial B(x, R/2)$ to $\partial B(x, R)$ in $\overline{B}(x, R) \setminus B(x, R/2)$. One easily sees that $\Delta(\sigma_1, \sigma_2) \leq 2$ and that $\rho = \frac{4}{R} 1_{B(x,R)}$ is $\sigma_1 - \sigma_2$ admissible. Then the assertion follows from the definitions of the modulus and the Löwner function. $\square$

Corollary 9.14. *No space can be Löwner for an exponent less than its topological dimension.*

Corollary 9.15. *An n-Löwner space of topological dimension n has Hausdorff dimension n.*

[13.11. 2019, Lecture 10]
[19.11. 2019, Lecture 11]

10. Poincaré inequalities and Löwner spaces

In this section we discuss various conditions which are equivalent to the validity of a Poincaré inequality for all balls (under suitable assumptions on (X, μ)) and how they are related to the Löwner condition.

This section is based on Chapter 9 of [8]. More details can be found in [9]. Subsection10.1 is partially based on lecture notes by Urs Lang, see https://people.math.ethz.ch/˜lang/LengthSpaces.pdf. Subsection 10.4 follows [11].

10.1. **Length spaces, geodesic spaces and the Hopf-Rinow theorem.** We begin with some purely metric properties of the space X.

Definition 10.1. *Let (X, d) be a metric space.*

(1) The space (X, d) is called locally compact *if for every $x \in X$ there exists an $r > 0$ such that the closed ball $\overline{B}(x, r)$ is compact.*
(2) The space (X, d) is called proper *if every closed and bounded set is compact;*
(3) the space (X, d) is called quasiconvex *if there exists a constant $C \geq 1$ such that for any two points $x, y \in X$ there exists a curve of length $\leq Cd(x, y)$ which joins x and y;*
(4) the space (X, d) is called a length space *if for any two points $x, y \in X$ the distance $d(x, y)$ is the infimum of the length of all curves connecting x and y.*
(5) The space (X, d) is called geodesic *is for any points x and y there exists a curve γ which joins x and y and has length $d(x, y)$. Such a curve is called a (lenth-minimizing) geodesic.*

The following lemma is taken from lecture notes by Urs Lang.

Lemma 10.2 (Mid-points)**.** *Let X be a complete metric space.*

(1) X is length space if and only if for all $x, y \in X$ and every $\varepsilon > 0$ there exists a $z \in X$ such that

$$d(x, z), d(z, y) \leq \frac{1}{2} d(x, y) + \varepsilon;$$

(2) *X is a geodesic space if and only if for all $x, y \in X$ there exists a $z \in X$ such that*

$$d(x,z), d(z,y) \leq \frac{1}{2} d(x,y).$$

Proof. Necessity is easy in both cases (and does not require completeness): take a suitable rectifiable curve which connects x and y, parametrize by arc-length and define z to be the mid-point of the curve. For sufficiency define a Lipschitz curve $\gamma : [0,1] \to X$ connecting x and y as follows.First definine γ on the dense set of dyadic points $\{j2^{-k} : k \in \{1,2,\ldots\}, j = \{1,\ldots,2^{k-1}\}\}$ by iteratively applying the condition (for the assertion about length space use the condition with $\varepsilon_k = 2^{-k}\varepsilon$). The function thus obtained is Lipschitz and hence has a Lipschitz extension which is easily seen to have the desired properties. $\square$

Theorem 10.3 (Hopf-Rinow theorem for length spaces)**.** *Assume that X is a length space which is locally compact and complete. Then*

(1) *X is proper;*
(2) *X is geodesic*

A locally compact length space need not be proper or geodesic. Consider the set $B(0,1) \setminus \{0\}$ in $\mathbb{R}^n$. Then there is no geodesic from x to $-x$.

Proof. This proof is taken from Lecture notes by Urs Lang.
To show that X is proper it suffices to show that for every $z \in X$ the closed balls $\overline{B}(z,r)$ are compact for all r. Fix z and let $I = \{r \geq 0 : \overline{B}(z,r) \text{ is compact}\}$. It suffices to show that I is open and closed in $[0,\infty)$. It is easy to see that local compactness implies that I is open in $[0,\infty)$. Thus it remains to show that $[0,R) \subset I$ implies $[0,R] \subset I$ for all $R > 0$. We show sequential compactness of $\overline{B}(z,R)$.

Let y_j be a sequence in $B(z,R)$. Choose a decreasing sequence ε_i converging to 0, with $\varepsilon_i < R$. Since X is a length space, for all $i, , j$ there exists an $x_i \in B(z, R - \frac{\varepsilon_i}{2})$ with $d(x^i_j, y_j) \leq \varepsilon_i$. The sequence x^1_j has a convergent subsequence $x^1_{j(1,k)}$. Consider the corresponding sequence $x^2_j(1,k)$ and pick a convergent subsequence $x^2_{j(2,k)}$.

Continue in this manner. Finally, put $j(k) := j(k,k)$ for $k \in N$; the sequence $x^i_j(k)$, $k \in N$ converges for all $i \in N$. We claim that the associated sequence $y_{j(k)}$ is Cauchy. Let $\varepsilon > 0$ and choose i with $\varepsilon_i \leq \varepsilon$. Then $d(x^i_{j(k)}, x^i_{j(l)}) \leq \varepsilon$ for k,l sufficiently large. It follows that $d(y_{j(k)}, y_{j(l)}) \leq d(y_{j(k)}, x^i_{j(k)}) + d(x^i_{j(k)}, x^i_{j(l)}) + d(x^i_{j(l)}, y_{j(l)}) \leq \varepsilon_i + \varepsilon +$

$\varepsilon_i \leq 3\varepsilon$. Since X is complete, $y_{j(k)}$ converges. This shows that every sequence y_j in $\overline{B}(z, R)$ has a convergent subsequence.

The fact that X is geodesic now follows from Lemma 10.2 and compactness. □

Proposition 10.4. *If (X, d) is quasiconvex then there exist a new metric d_{in} such that (X, d_{in}) is a length space and $d \leq d_{\text{in}} \leq Cd$. In particular (X, d) is bi-Lipschitz equivalent to a length space. If, in addition, (X, d) is complete and locally compact then $X, d_{\text{in}})$ is geodesic space.*

Proof. Define the inner metric by

$$d_{\text{in}}(x, y) = \inf\{\text{length}_d(\gamma) : \gamma \text{ is a curve which joins } x \text{ and } y.\}$$

By quasiconvexity we have $d \leq d_{\text{in}} \leq Cd$. It is easy to see that d_{in} satisfies the triangle inequality. Hence d_{in} is a metric.

If X is complete in the fact that X is a length space follows easily from Lemma 10.2. In general we can argue as follows. We need to show that for any $\varepsilon > 0$ and any $x, y \in X$ there exists a curve $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$ and

$$\text{length}_{d_{\text{in}}}(\gamma) \leq d_{\text{in}}(x, y) + \varepsilon.$$

By definition there exists a curve $\gamma : [a, b] \to X$ such that

$$\text{length}_d(\gamma) \leq d_{\text{in}}(x, y) + \varepsilon.$$

Let $a = t_1 < t_1 < \ldots < t_{N+1} = b$ and $I_i = [t_i, t_{i+1}]$. It follows from the definition of the length that

$$\text{length}_{d_{\text{in}}}(\gamma) = \sum_{i=1}^{n} \text{length}_{d_{\text{in}}}(\gamma_{I_i}).$$

The definition of d_{in} implies that

$$d_{\text{i}}(\gamma(t_i), \gamma(t_{i+1}) \leq \text{length}_{d_{\text{in}}}(\gamma_{I_i}).$$

Thus

$$\sum_{i=1}^{n} d_{\text{i}}(\gamma(t_i), \gamma(t_{i+1}) \leq d_{\text{in}}(x, y) + \varepsilon.$$

Taking the supremum over all N and all choices $t_1, \ldots t_{N+1}$ we get the desired assertion.

To see that (X, d_{in}) is geodesic use the Hopf-Rinow theorem. □

Proposition 10.5. *For each $\lambda \geq 1$ there exist $a > 1$ and $M \geq 1$ with the following property. Let A be an open ball in a geodesic space (X, d). Then A satisfies an (λ, M, a)-chain condition (with respect to the concentric ball $B_0 = \frac{1}{2\lambda}A$) in the sense of Definition 5.1.*

For a possible choice of a and M see (10.1) and (10.2) below.

Proof. By scaling the metric if needed, we may assume that $A = B = B(x_0, 1)$. Let $x \in B$ and set $s = d(x, x_0)$. Let $\gamma : [0, s] \to X$ be a geodesic from x_0 to x, parametrized by arc-length. Then we have $d(x_0, \gamma(t)) = t$. We consider balls

$$B_i = B(\gamma(t_i), r_i)$$

With

$$r_0 = \frac{1}{2\lambda}, \quad t_0 = 0,$$

$$r_i = a^{-i} r_0, \quad t_{i+1} - t_i = c_i r_i \quad \text{with } c_i \in (0, 1].$$

We want

$$t_i = s \quad \text{for } i \geq i_0$$

. Thus we take

$$c_i = c \text{ for } i \leq i_0 - 1 \quad \text{ and } \quad c_i = 0 \text{ for } i \geq i_0.$$

Note that $B_i \cap B_{i+1}$ contains a ball for radius $r_{i+1}/2 = (2a)^{-1} r_i$ and is contained in a ball for of radius $r_i + r_{i+1} = (1 + a^{-1}) r_i$.

Then the sequence of balls satisfies the (λ, M, a) chain condition if

(1) $t_i + \lambda r_i \leq 1$;
(2) $2M^{-1} a^{-i} \leq r_i \leq 2M a^{-i}$;
(3) $M \geq 2a1 + a^{-1} = 2a + 2$.

The second condition is satisfied if

$$M \geq 4\lambda.$$

We have, for $i \leq i_0$

$$t_i = c \frac{1}{2\lambda} \frac{1 - a^{-i}}{1 - a^{-1}}.$$

Thus the first condition is equivalent to

$$c \frac{1}{2\lambda} \frac{1 - a^{-i}}{1 - a^{-1}} + \frac{1}{2} a^{-i} \leq 1.$$

A sufficient condition is

$$c \frac{1}{2\lambda} \frac{1}{1 - a^{-1}} \leq 1.$$

Since $0 < c \leq 1$ this condition in turn is satisfied if we choose a such that

$$\frac{1}{2\lambda} \frac{1}{1 - a^{-1}} = 1 \tag{10.1}$$

In addition we want that
$$s = t_{i_0} = c\frac{1}{2\lambda}\frac{1-a^{-i_0}}{1-a^{-1}} = c(1-a^{-i_0}).$$
Choose i_0 such that
$$1-a^{-i_0+1} < s \le 1-a^{-i_0}$$
and set
$$c = \frac{s}{1-a^{-i_0}}.$$
Then all conditions are satisfied if we choose a as in (10.1) and

$$M = \max(2a+2, 4\lambda). \tag{10.2}$$

□

[19.11. 2019, Lecture 11]

[20.11. 2019, Lecture 12]

10.2. **The Poincaré inequality and equivalent conditions.** Let (X,μ) be a metric measure space. Throughout the rest of this section we assume that
$$0 < \mu(B) < \infty$$
for all balls.

Definition 10.6. *Let (X,μ) be a metric measure space and let $p,q \in (1,\infty)$. We say that X admits weak (p,q) Poincaré inequality if there are constants $0<\lambda\le 1$ and $C>0$ such that*

$$\left(\fint_{\lambda B} |u-u_B|^p \, d\mu\right)^{1/p} \le C \operatorname{diam} B \left(\fint_B \rho^q \, d\mu\right)^{1/q} \tag{10.3}$$

for all balls B in X, all bounded continuous functions u and all upper gradient ρ of u. We say that X satisfies a Poincaré inequality if the above statement holds with $\lambda = 1$.

In the following we focus on sufficient and necessary conditions for the validity of a (weak) $(1,p)$ Poincaré inequality. For a locally integrable function $f : X \to \mathbb{R}$ we define the the Riesz potential

$$(I_1 f)(x) := \int_X \frac{d(x,z)f(z)}{\mu(B(x,d(x,z)))} \, d\mu(z), \tag{10.4}$$

the localized Riesz potential for a subset A

$$(I_{1,A} f)(x) := \int_A \frac{d(x,z)f(z)}{\mu(B(x,d(x,z)))} \, d\mu(z) \tag{10.5}$$

and the restricted maximal function

$$(M_R f)(x) := \sup_{0<r\leq R} \fint_{B(x,r)} |f(z)|\, d\mu(z). \tag{10.6}$$

Definition 10.7. *Let $p \in [1,\infty)$. We consider the following five conditions which are supposed to hold for all balls all continuous functions $u : X \to \mathbb{R}$ and all upper gradient ρ of u. The constants $C_1, \dots C_6$ below are assumed to be independent of B, u, ρ and the points x and y.*

(1) There exists a $C_1 > 0$ such that for all $x \in \frac{1}{2}B$

$$|u(x) - u_B|^p \leq C_1 (\operatorname{diam} B)^{p-1} (I_{1,B}\rho^p)(x)$$

;

(2) There exist $C_2 > 0$ and $C_3 \geq 1$ such that for all $x, y \in (4C_3)^{-1}B$:

$$|u(x) - u(y)|^p \leq C_2 |x-y|^{p-1}((I_{1,B}\rho^p)(x) + (I_{1,B}\rho^p)(y));$$

(3) There exist constants $C_4 > 0$, $C_5 \geq 1$ and $C_6 \geq 1$ such that for all $x, y \in (4C_6)^{-1}B$

$$|u(x) - u(y)|^p \leq C_4 |x-y|^p((M_{C_5|x-y|}\rho^p)(x) + (M_{C_5|x-y|}\rho^p)(y);$$

(4) X admits a a weak $(1,p)$-Poincaré inequality;
(5) X admits a $(1,p)$-Poincaré inequality.

Theorem 10.8. *If (X,μ) is doubling then we have*

$$(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (4).$$

If, in addition, X is geodesic then

$$(4) \Longrightarrow (5) \Longrightarrow (1).$$

All implications are quantitative in the usual sense.

Proof. We only sketch the argument and refer to [8] for the details.

(1)$\Longrightarrow$ (2): This follows with $C_2 = 3$ from the triangle inequality

$$|u(x) - u(y)|^p \leq 2^{p-1}(|u(x) - u_{B_x}|^p + |u(y) - u_{B_x}|^p)$$

and application of (1) on $B_x = B(x, 2d(y,x))$.

(2)$\Longrightarrow$ (3): We estimate the Riesz potential in terms of the maximal function by summing over $A_j := \{z : 2^{-j-1}R \leq d(z,x) \leq 2^{-j}\}$ from $j = -1$ to ∞. Here the doubling property is used. This gives and estimate for the form $d^{p-1}(x,y)R\dots$ Now if $d(x,y) \geq (100C_3)^{-1}R$ we are done. If $d(x,y) < (100C_3)^{-1}R$ we apply (2) to the smaller ball $\tilde{B} = B(x, 4C_3 d(y,x)) \subset B$.

(3)$\Longrightarrow$ (4): This is the most delicate estimate. One can actually show a weak (p,p)-Poincaré inequality. One first shows the corresponding weak-L^p estimate and then uses a truncation argument similar to the one in Remark 4.4 adjusted for upper gradients, see the proof of Lemma 5.15 in [9] for the details. For truncation in the context of upper gradients see Proposition 10.9 below.

(4)$\Longrightarrow$ (5): This follows from Theorem 5.2 and Proposition 10.5 (note that the λ in Theorem 5.2 corresponds to λ^{-1} in the definition of the weak Poincaré inequality).

(5)$\Longrightarrow$ (1): Let $x \in \frac{1}{2}B$. Pick $y \in B$ with $d(x,y) = \frac{1}{4}R$ where R is the radius of B. Set $r_1 = \frac{1}{2}d(x,y)$. Along the geodesic from x to y pick points x_i with $d(x,x_i) = r_i := 2^{-i+1}r_1$ and consider the balls $B_i = B(x_i, \frac{r_i}{2})$. Since u is continuous we have $u(x) = \lim_{i\to\infty} u_{B_i}$ and to estimate the telescoping sum for $u(x) - u_B$ we use that $B_{i+1} \subset \frac{3}{2}B_i$, Jensen's inequality and the Poincaré inequality. Finally we use that $d(x,x_i) \geq \frac{1}{4}r_i$ for $x \in \frac{3}{2}B_i$ and the only a fixed number of the ball $\frac{3}{2}B_i$ intersect non-trivially. $\square$

Proposition 10.9. *Let (X,d) be a metric space.*

1. *Let U_1 and U_2 be open subsets. Suppose that $v : U_1 \cup U_2 \to \mathbb{R}$ is continuous and that ρ_i is and upper gradient of $v_{|U_i}$ for $i = 1,2$. Extend ρ_1 by zero to $U_2 \setminus U_1$ and ρ_2 by zero to $U_1 \setminus U_2$. Then $\rho := \max(\rho_1, \rho_2)$ is an upper gradient for v.*
2. *Let $W \subset X$ be open, let $A \subset W$ be closed and let $U \supset A$ be open. Assume that $v : W \to \mathbb{R}$ is continuous and locally constant on $W \setminus A$. If ρ is an upper gradient for v the $\rho 1_U$ is also an upper gradient for v.*

Proof. Let $\gamma : [a,b] \to X$ be a rectifiable curve such that $\int_\gamma \rho < \infty$. We want to show that

$$|v(\gamma(b)) - v(\gamma(a))| \leq \int_\gamma \rho.$$

To see this define $\gamma_\tau = \gamma_{[a,\tau]}$ and

$$T := \sup\{t \in [a,b] : |v(\gamma(\tau)) - v(\gamma(a))| \leq \int_{\gamma_\tau} \rho \text{ for all } \tau \in [a,t]\}.$$

It follows from the definition of T that

$$|v(\gamma(\tau)) - v(\gamma(a))| \leq \int_{\gamma_\tau} \rho \text{ for all } \tau \in [a,T).$$

Since both sides of the inequality are continuous in τ the inequality also for all $\tau \in [a,T]$. If $T = b$ we are done. Thus assume $T < b$ and, without loss of generality, $\gamma(T) \in U_1$. Then there exists and $\varepsilon > 0$ such that $\gamma([T, T+\varepsilon]) \subset U_1$. Thus

$$|v(\gamma(\tau) - v(\gamma(T))| \leq \int_{\gamma_{[T,\tau]}} \rho_1 \leq \int_{\gamma_{[T,\tau]}} \rho$$

for all $\tau \in [T, T+\varepsilon]$. Hence by the triangle inequality

$$|v(\gamma(\tau) - v(\gamma(a))| \leq \int_{\gamma_\tau} \rho$$

for all $\tau \in [T, T+\varepsilon]$. This contradicts the definition of T.

To prove the second assertion note that a locally constant function has 0 as an upper gradient and apply the first assertion with $U_1 = U$, $U_2 = W \setminus A$, $\rho_1 = \rho_U$ and $\rho_2 = 0$. □

[20.11. 2019, Lecture 12]

[26.11. 2019, Lecture 13]

10.3. **The Poincaré inequality and Löwner spaces.** This subsection is based on [9]. Recall from Definition 9.8 that the space (X, μ) is n-regular if there exists $C \geq 1$ and $n \geq 1$ such that

$$C^{-1}R^n \leq \mu(B_R) \leq CR^n$$

for all ball B_R with radius $R \in (0, \operatorname{diam} X)$.

The purpose of this subsection is to prove the following result.

Theorem 10.10. *Suppose that (X, μ) is proper, n-regular and quasiconvex. Then X is an n-Löwner space if and only if X admits a weak $(1, n)$-Poincaré inequality.*

We first show that a weak $(1, n)$-Poincaré inequality is sufficient.

Theorem 10.11 ([9], Thm. 5.7)**.** *Let (X, μ) be a proper, doubling and quasiconvex metric measure space. Assume that for some $n > 1$ and some $C > 0$ we have the lower mass bound*

$$\mu(B_R) \geq \frac{1}{C} R^n \quad \text{for all balls of radius } R < \operatorname{diam} X. \tag{10.7}$$

If X admits a weak $(1, n)$-Poincaré inequality then X is an n-Löwner space.

This implies in particular that $\mathbb{R}^n$ is an n-Löwner space, see Theorem 9.5 above.

The proof of Theorem 10.11 is based on two ideas:

(1) A lower bound for the Hausdorff content of E and F and a Poincaré inequality imply a lower bound on the capacity.

(2) The one-dimensional Hausdorff content of continuum is at least its diameter

The following result gives a precise formulation of the first idea.

Theorem 10.12 ('Large Hausdorff content implies large capacity', see [9], Thm. 5.9)**.** *Suppose that (X, μ) is a doubling space where (10.7) holds for some $n \geq 1$. Suppose further that X admits a weak $(1, p)$-Poincaré for some $p \in [1, n]$. Let E, F be two compact subsets of a ball B_R in X and assume that for some $s \in (n-p, n]$ and some $\lambda \in (0, 1]$ we have*

$$\min(\mathcal{H}^s_\infty(E), \mathcal{H}^s_\infty(F)) \geq \lambda R^{s-n}\mu(B_R). \tag{10.8}$$

Then there is a constant $C' \geq 1$, depending only on s and on the data associated with X such that

$$\int_{C'B_R} \rho^p \, d\mu \geq \frac{1}{C'}\lambda\mu(B_R)R^{-p} \tag{10.9}$$

whenever u is a continuous function on the ball $C'B_R$ with $u_{|E} \leq 0$ and $u_{|F} \geq 1$, and ρ is an upper gradient of u in $C'B_R$.

The theorem can be seen as a quantitative statement of the following fact in $\mathbb{R}^n$. If $p \in (1, n]$ then the set of points where a function in $W^{1,p}$ is not well-behaved has p-capacity zero and hence Hausdorff dimension $n - p$ (see, for example, [19], Theorem 2.6.16 and Section 3).

The reason that Theorem 10.12 is formulated in terms of an individual function u and not in terms of capacity is that we have defined capacity by using arbitrary test functions, while the Poincaré inequality if required for continuous functions only.

We will also use the following refinement of Theorem 8.9. We define $\mathrm{cap}^{\mathrm{L}}_p(E, F; X)$ and $\mathrm{cap}^{\mathrm{c}}_p(E, F; X)$ in the same way as the capacity, but restricting to Lipschitz functions or continous functions u, respectively.

Theorem 10.13. *Suppose that X is quasiconvex and proper.Let E and F be disjoint closed sets in X with compact boundaries. Then*

$$\mathrm{cap}^{\mathrm{c}}_p(E \cap B, F \cap B; B) \leq \mathrm{cap}^{\mathrm{L}}_p(E \cap B, F \cap B; B) \leq \mathrm{mod}_p(E, F)$$

for each ball B in X.

We only need the bound $\mathrm{cap}^{\mathrm{c}}_p(E \cap B, F \cap B; B) \leq \mathrm{mod}_p(E, F)$ and for this bound quasiconvexity can be replaced by the weaker condition of φ-convexity, see [9], Section 2.15 and Proposition 2.17.

Proof. See the proof of Proposition 2.17 in [9]. The main idea is to define u as in (8.12), but with ρ replaced by an approximation by bounded functions. $\square$

Finally we make use of a standard comparison of diameter and one-dimensional Hausdorff content for continua.

Proposition 10.14. *Let E be a continuum in a metric space X. Then*

$$\mathcal{H}^1_\infty(E) \geq \operatorname{diam} E. \tag{10.10}$$

Proof. We may assume $\operatorname{diam} E > 0$ since otherwise there is nothing to show. Since E is compact there exist $x, y \in E$ such that $d(x,y) = \operatorname{diam} E$. Assume that $\mathcal{H}^1_\infty(E) < \operatorname{diam} E$. Consider the map $z \mapsto f(z) = d(z,x)$. Then f is a 1-Lipschitz function and in particular f maps a closed ball in X to an interval of length no exceeding $\operatorname{diam} B$. Thus

$$\mathcal{L}^1(f(E)) \leq \mathcal{H}^1_\infty(E) < \operatorname{diam} E.$$

Hence there exists an $r \in (0, \operatorname{diam} E)$ such that $r \notin f(E)$. Therefore E does not intersect the sphere $\{z \in X : d(z,x) = r\}$ and hence E is the disjoint union of the compact sets $E \cap \overline{B}(x,r)$ and $E \setminus B(x,r)$. Both sets are nonempty since the former set contains x and the latter contains y. This contradicts the connectedness of E. $\square$

Proof of Theorem 10.11. Let E and F be two disjoint continua in X. We write $d = \operatorname{dist}(E,F)$ and assume without loss of generality that

$$\delta = \operatorname{diam} E = \min(\operatorname{diam} E, \operatorname{diam} F).$$

Fix t such that

$$t \geq \Delta(E,F) = \frac{\operatorname{dist}(E,F)}{\min(\operatorname{diam} E, \operatorname{diam} F)} = \frac{d}{\delta}.$$

There exist a point $x \in E$ such that the closed ball $\bar{B}(x,d)$ meets F. Then consider the ball

$$B = B(x, d + 2\delta).$$

The compact sets E and $F' := F \cap \bar{B}(x, d+\delta)$ both lie in B. We claim that they both have Hausdorff 1-content at least δ. For E this follows from Proposition 10.14. If $F' = F$ then the proposition also applies to F'. So assume $F \setminus F' \neq \emptyset$. To see that $\mathcal{H}^1_\infty(F') \geq \delta$ we can argue as in the proof or Proposition 10.14. Define $f(z) = d(z,x)$. If $\mathcal{H}^1_\infty(F') < \delta$ then $\mathcal{L}^1(f(F')) < \delta$ and hence there exists $r \in (d, d+\delta)$ such the sphere $\{z : d(z,x) = r\}$ does not intersect F' and hence does not intersect F. Thus F is the disjoint union of the compact nonempty sets $F \cap \overline{B}(x,r) \supset F \cap B(x,d)$ and $F \setminus B(x,r) \supset F \setminus F'$. This contradicts the connectness of F.

Now write

$$\delta = \frac{\delta (d+2\delta)^{n-1}}{\mu(B)} (d+2\delta)^{1-n} \mu(B).$$

and apply Theorem 10.13 and Theorem 10.12 with $p = n$, $s = 1$ and $\lambda = \delta(d + 2\delta)^{n-1}/\mu(B)$ to get

$$\operatorname{mod}_n(E, F) \geq \operatorname{cap}^{\mathrm{c}}_n(E, F; C'B) \geq \frac{1}{C'} \frac{\delta(d + 2\delta)^{n-1}}{\mu(B)} \mu(B)(d+2\delta)^{-n} \geq \frac{1}{C'} \frac{1}{t + 2}.$$

Thus Theorem 10.11 follows. □

Proof of Theorem 10.12. The main idea is to use the Poincaré inequality on dyadic balls and the basic covering argument. For a first reading one may simplify the algebra by taking $R = 1$. This is actually no loss of generality since the Löwner function and the quantity $\inf\{r^{-n}\mu(x, r) : x \in X, r > 0\}$ are invariant if we replace d by $R^{-1}d$ and μ by $R^{-n}\mu$.

Let u be a continuous function on the ball $B_{C'R}$ where $C' = 10\lambda^{-1}$ and λ is the constant in the weak Poincaré inequality (10.3). Assume that $u_{|E} \leq 0$ and $u_{|F} \geq 1$ and the ρ be an upper gradient of u in $B_{C'R}$.

Case 1: There exist points $x \in E$ and $y \in F$ such that

$$|u(x) - u_{B(x,R)}| \leq \frac{1}{5} \quad \text{and} \quad |u(y) - u_{B(y,5R)}| \leq \frac{1}{5}.$$

Then

$$1 \leq |u(x) - u(y)| \leq \frac{1}{5} + |u_{B(x,R)} - u_{B(y,5R)}| + \frac{1}{5}.$$

Note that $B(x, R) \subset B(y, 5R) \subset C'B_R$ by our choices. Thus

$$1 \leq C \fint_{B(x,R)} |u - u_{B(y,5R)}| \, d\mu.$$

Using that $B(y, 5R) \subset B(x, 7R)$ and the fact that X is doubling we deduce that

$$1 \leq C \fint_{B(y,5R)} |u - u_{B(y,5R)}| \, d\mu \leq CR \left(\fint_{C'B_R} \rho^p \, d\mu \right)^{1/p}.$$

Thus (10.9) holds.

By symmetry of the problem in E and F it thus suffices to consider

Case 2: For all points in $x \in E$ we have

$$\frac{1}{5} \leq |u(x) - u_{B(x,R)}|.$$

Since u is continuous this implies that with $B_j(x) = 2^{-j}B(x,R)$

$$1 \le \sum_{j=0}^{\infty} |u_{B_j(x)} - u_{B_{j+1}}(x)| \le C \sum_{j=0}^{\infty} \fint_{B_j(x)} |u - B_j(x)|\, d\mu$$

$$\le \sum_{j=0}^{\infty} (2^{-j}R) \left(\fint_{C_0 B_j(x)} \rho^p \, d\mu \right)^{1/p}.$$

We claim that there exists an $\varepsilon_0 = \varepsilon_0(s,n,p) > 0$ such that for each x there exists a j_x with

$$\int_{C_0 B_{j_x}(x)} \rho^p \, d\mu \ge \varepsilon_0 R^{n-s-p} (2^{-j_x}R)^{s-n} \mu(B_{j_x}(x)). \tag{10.11}$$

Indeed, otherwise we get

$$1 \le \sum_{j=0}^{\infty} \varepsilon_0^{1/p} (2^{-j})^{(s-n+p)/p)} \le C(s,n,p)\varepsilon_0^{1/p}$$

since $s > n - p$.

By the basic covering theorem there exist countably many disjoint balls $B_k = B(x_k, r_k R)$ such that $E \subset \bigcup_k 5B_k$ and

$$\mu(B_k)(r_k R)^{s-n} \le C R^{s+p-n} \int_{B_k} \rho^p, d\mu.$$

Hence

$$\lambda R^{s-n} \mu(B_R) \le \mathcal{H}^s_\infty(E) \le \sum_k (5 r_k R)^s \le C \sum_k (r_k R)^{s-n} (r_k R)^n$$

$$\le C \sum_k (r_k R)^{s-n} \mu(B_k) \le C R^{s+p-n} \int_{C' B_R} \rho^p \, d\mu,$$

as desired. ☐

[26.11. 2019, Lecture 13]

[3.12. 2019, Lecture 14]

We now turn to the implication that Löwner spaces have a $(1,n)$-Poincaré inequality.

Theorem 10.15 ([9], Thm. 5.12)**.** *Suppose that X is a locally compact, n-regular Löwner space. Then X admits a weak $(1,n)$-Poincaré inequality.*

In view of Proposition 9.13 the assumption 'n-regular' can be replaced by an upper mass bound.

Proof. The results follows from the implication (3) $\Longrightarrow$ (4) in Theorem 10.8 and Lemma 10.16 below. $\square$

Lemma 10.16 ([9], Lemma 5.17)**.** *Suppose that X is an n-regular Löwner space. Then there exist constants $C_4 > 0$, $C_5 \geq 1$ and $C_6 \geq 1$ such that for all $x, y \in (4C_6)^{-1}B$ we have*

$$|u(x)-u(y)|^n \leq C_4|x-y|^n((M_{C_5|x-y|}\rho^n)(x)+(M_{C_5|x-y|}\rho^n)(y) \tag{10.12}$$

Lemma 10.17 (Variant of [9], Lemma 3.17)**.** *Let X be an n-regular, n-Löwner space . For $M_1 \geq 1$ and $M_2 > 0$ there exist $M_4 \geq 1$, $M_5 \geq 1$ and $\rho > 0$, depending only on M_1, M_2 and the data of X (i.e. the Löwner function ϕ_n and $M := \sup\{\mu(B(x,R)/R^n, R^n/\mu B(x,R) : x \in X, R > 0\}$) with the following property.*

If $x \in X$, $r > 0$ and if E and F be continua in $B(x, M_1r)\backslash B(x, r/M_1)$ with $\operatorname{diam} E \geq M_2 r$, $\operatorname{diam} F \geq M_2 r$. Then the families of curves

$$\Gamma' = \{\gamma : \ \gamma \textit{ rectifiable curve joining } E \textit{ and } F \textit{ in } B(x, M_4r) \setminus B(x, r/M_4)\ \}$$

and

$$\Gamma'' = \{\gamma \in \Gamma' : \operatorname{length}(\gamma) \leq M_5M_4r\}$$

satisfy

$$\operatorname{mod}_n\Gamma' = \operatorname{mod}_n(E, F; B(x, M_4r) \setminus B(x, r/M_4)) \geq 2\rho, \tag{10.13}$$

$$\operatorname{mod}_n\Gamma'' \geq \rho \tag{10.14}$$

The same conclusion holds if we replace $B(x, M_1r) \setminus B(x, M_1/r)$ by $B(x, M_1R)$ and $B(x, M_4r) \setminus B(x, r/M_4)$ by $B(x, M_4r)$

Proof of Lemma 10.17. Let

$$\Gamma := \{\gamma : \ \gamma \text{ rectifiable curve joining } E \text{ and } F.\}$$

Note that $\operatorname{dist}(E, F) \leq \operatorname{diam} B(x, M_1r) \leq 2M_1r$. Thus

$$\Delta(E, F) \leq \frac{2M_1}{M_2}.$$

Hence the Löwner property implies that

$$\operatorname{mod}_n\Gamma \geq \phi_n(\frac{2M_1}{M_2}).$$

Set

$$\rho = \frac{1}{4}\phi_n(\frac{2M_1}{M_2}).$$

Let

$$\Gamma_+ = \{\gamma : \ \gamma \text{ rectifiable curve joining } E \text{ and } F,\ \gamma \setminus B(x, M_1r) \neq \emptyset.\}$$

$$\Gamma_- = \{\gamma : \ \gamma \text{ rectifiable curve joining } E \text{ and } F,\ \gamma \cap \bar{B}(x, M_1r) \neq \emptyset.\}$$

Then $\Gamma \subset \Gamma' \cup \Gamma_+ \cup \Gamma_-$ and hence

$$4\rho \leq \mathrm{mod}_n\Gamma \leq \mathrm{mod}_n\Gamma' + \mathrm{mod}_n\Gamma_+ + \mathrm{mod}_n\Gamma_.$$

Now every curve in Γ_+ contains a subcurve which connects $\partial B(x, M_1 r)$ and $\partial B(x, M_4 r)$ and thus

$$\mathrm{mod}_n\Gamma_+ \leq \mathrm{mod}_n\Gamma(\partial B(x, M_1 r), \partial B(x, M_4 r); X) \leq C \ln^{1-n} \frac{M_4}{M_1}.$$

Choose M_4 so large that the right hand side is $\leq \rho$. Together with the analogous bound for $\mathrm{mod}_n\Gamma_-$ we get (10.13).

To prove (10.14) it suffices to show that we can choose M_5 so large that $\mathrm{mod}_n(\Gamma'' \setminus \Gamma') \leq \rho$. consists of curve of length at least $M_5 M_4 r$. Thus $\rho = (M_5 M_4 r)^{-1}$ is admissible of $\Gamma'' \setminus \Gamma'$. Hence

$$\mathrm{mod}_n(\Gamma'' \setminus \Gamma') \leq M_5^{-n}(M_4 r)^{-n}\mu(B(x, M_4 r) \leq C M_5^{-n}$$

since X is n-regular. Choose M_5 so large that the right hand side does not exceed ρ. This completes the proof. □

Lemma 10.18 ([9], Thm. 3.13)**.** *Let X be an n-Löwner space of Hausdorff dimension $n > 1$ which satisifes*

$$\mu(B_R) \leq C R^n$$

for all balls B_R of radius R. Then X is quasiconvex.

Even more is true: two points $x, y \in X \setminus B(z, r)$ can be connected by a rectiable curve in $X \setminus B(z, r/C)$.

Proof. Let $x, y \in X$ and $r = d(x, y)$. One inductively constructs a family of 2^k balls of radius 4^{-k} which includes $B(x, 4^{-k})$ and $B(y, 4^{-k})$ such that the balls a connected by curves whose total length does not exceed $C \sum_{j=0}^{k-1} 2^j 4^{-j} \leq C$, see the proof of Thm 3.13 in [9]. By passage to the limit one finds the desired curve connecting x and y.

Here is a sketch of the constrruction. For first step $r_1 = \frac{1}{4} r$, $E_1 = \overline{B}(x, r_1)$ and $F_1 = \overline{B}(y, r_1)$. It follows from Lemma 10.17] that the family of curves which connect E_1 and F_1 in $B(x, 8M_4 r_1)$ and has length not exceeding $8M_5 M_4 r_1$ has positive modulus. In particular there exists such at curve Γ. Now take $x_2 \in E_1 \cap \Gamma$ and $y_2 \in F_1 \cap \Gamma$. Then $d(x_1, x_2) \leq r_1$ and $d(y_1, y_2) \leq r_1$

In the second repeat the argument for the pairs (x_1, x_2) and (y_1, y_2) and $r_2 = \frac{1}{4} r_1$. The proceed by induction. □

Proof of Lemma 10.16. See the proof of Lemma 5.17 in [9]. I am not quite sure how to derive (5.21) from a '(the proof of) Lemma 3.17 in

[9]' by choosing C_2 surfficiently large, so in class I used the following slightly different definition of the annuli A_j, namely

$$A_j = B(x, 2^{-2j}d) \setminus B(x, 2^{-2j-1}d).$$

Setting $B_j = B(x, 2^{-2j}d)$ we can apply Lemma 10.17 and we get (5.21) in [9] where Γ_j now consists of all curves in the family $(\gamma_j, \gamma_{j+1}, M_4 B_j)$ whose length does not exceed $M_5 M_4 2^{-2j} d$. □

[3.12. 2019, Lecture 14]

[10.12. 2019, Lecture 15]

10.4. **The Poincaré inequality in the first Heisenberg group.** This is based on [11], but simplified for the first Heisenberg group. A alternative short and elegant proof of the $1-1$ Poincaré inequality for the first Heisenberg group is due to Varopoulos [18], see also [7], Proposition 11.17, pp. 68–69 or [15], p. 461.

Change in notation: our X^R is $-X^R$ in [11].

Convolution in Lie groups. Let G be a group. We define left- and right translation by

$$\ell_x y = xy, \quad r_x y = yx \quad \forall x, y \in G.$$

Let M be a smooth manifold. As usual we identify tangent vectors with first order differential operators. The commutator of two smooth vectorfields is defined by

$$[X, Y]f = XYf - YXf$$

for all smooth function $f : G \to \mathbb{R}$. The push-forward of a smooth tangent vector field is given by

$$\varphi : M \to M, \quad [\varphi_* X](\varphi(y)) = d\varphi(y) X(y).$$

The pull-back of a smooth k-form is defined as

$$(\varphi^* \theta)(y)(Y_1, \ldots, Y_k) = \theta(\varphi(y), \varphi_* Y_1, \ldots, \varphi_* Y_k).$$

Definition 10.19. *A group G is a (finite-dimensional) Lie group if it is a smooth (finite-dimensional) manifold and the map*

$$G \times G \ni (x, y) \mapsto xy^{-1} \in G$$

is smooth. We denote by

$$\mathfrak{g} := T_e G$$

the tangent space of G at the identity.

A vectorfield X on a Lie group is left-invariant or right-invariant if $(\ell_x)_* X = X$ or $(r_x)_* X = X$ for all $x \in G$. Similarly a k-form on G is left-invariant or right-invariant if $(\ell_x)^* \alpha = \alpha$ or $(r_x)^* \alpha = \alpha$.

A Lie group is unimodular if it has a nontrivial volume form which is left- and right-invariant.

Note that for a Lie group the maps ℓ_x and r_x are smooth diffeomormphisms.

Proposition 10.20. *For each $V \in \mathfrak{g}$ there exists a unique left-invariant vectorfield X_V and a unique right-invariant vector-field Y_V such that*

$$X(e) = Y(e) = V.$$

Similarly for a (constant) k-form $\alpha \in \Lambda^k \mathfrak{g}$ there exist unique left- and right-invariant k-forms $\beta, \gamma \in \Omega^k(G)$ such that

$$\beta(e) = \gamma(e) = \alpha.$$

A left-invariant vectorfield is generated by a family of right-translations. More precisely let $\eta : (-a, a) \to G$ be a smooth curve with $\eta(0) = e$ and $\eta'(0) = V$. Then the vectorfield defined by

$$X(y) = \frac{d}{dt}_{t=0} f(r_{\eta(t)} y)$$

is left-invariant.

If X is a left-invariant vectorfield and Y is a right-invariant vectorfield then

$$[X, Y] = 0.$$

Define reflection R given by $Rf(x) = f(x^{-1})$. If X is left-invariant then the vectorfield X^R defined by

$$X^R F = -RXRf$$

is right-invariant and $X^R(e) = X(e)$.

From now on we assume that G is unimodular and we will always equip G with left- and right-invariant (Haar) measure μ. Note that μ is also invariant under the inversion map $I(x) = x^{-1}$.

Definition 10.21. *For a continuous function $f : G \to \mathbb{R}$ and a continuous function $g : G \to \mathbb{R}$ with compact support we define the convolution*

$$(f * g)(x) = \int_G f(xy^{-1}) g(y)\, \mu(dy). \tag{10.15}$$

Proposition 10.22. *If f and g are in addition C^1 then for all left-invariant vectorfields X and all right-invariant vectorfields Y*

$$X(f * g) = f * Xg, \quad Y(f * g) = Yf * g$$

and

$$f * X^R g = (Xf) * g.$$

Moreover Young's inequality holds for all $p \in [1, \infty]$

$$\|f * g\|_{L^p} \leq \|f\|_{L^p} \|g\|_{L^1}.$$

The first Heisenberg group $\mathbb{H}^1$. We recall from the introduction the exponential map

$$\exp : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$$

given by

$$\exp x = \exp \begin{pmatrix} 0 & x_1 & x_3 \\ 0 & 0 & x_2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & x_3 + \frac{1}{2}x_1 x_2 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}$$

Note exp is a diffeomorphism from $\mathbb{R}^3$ to a 3×3 matrices of the form Id + upper triangular matrices.

Then the first Heisenberg group $\mathbb{H}^1$ can be identified with $\mathbb{R}^3$ together with the group operation $*$ defined by

$$\exp(x * y) = \exp x \exp y.$$

A short calculation shows that

$$x * y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + \frac{1}{2}(x_1 y_2 - y_1 x_2)).$$

Note that $0 * x = x * 0 = x$ and

$$x * (-x) = 0.$$

Thus 0 is the neutral element in the group and inverse x^{-1} agrees with $-x$. As before the left-translation and right-translation are defined by $l_x y = x * y = r_y x$ for all $x, y \in \mathbb{H}^1$. A basis of left-invariant vector-field is given by

$$X_1(x) = \frac{\partial}{\partial x_1} - \frac{1}{2}x_2 \frac{\partial}{\partial x_3}, \quad X_2(x) = \frac{\partial}{\partial x_2} + \frac{1}{2}x_1 \frac{\partial}{\partial x_3}, \quad X_3(x) = \frac{\partial}{\partial x_3}.$$

The dual basis of differential forms is given by

$$\theta_1 = dx^1, \quad \theta_2 = dx^2, \quad \theta_3 = dx^3 + \frac{1}{2}(x^2 dx^1 - x^2 dx^1).$$

Then

$$[X_1, X_2] = X_3, \quad d\theta_3 = -\theta_1 \wedge \theta_2.$$

For the above basis we obtain

$$X_1^R(x) = \frac{\partial}{\partial x_1} + \frac{1}{2}x_2 \frac{\partial}{\partial x_3}, \quad X_2^R(x) = -\frac{\partial}{\partial x_2} - \frac{1}{2}x_1 \frac{\partial}{\partial x_3}, \quad X_3^R(x) = \frac{\partial}{\partial x_3}.$$

In particular we have

$$X_1^R = X_1 + x_2 X_3, \quad X_2^R = X_2 - x_1 X_3, \quad X_3^R = X_3$$

and

$$[X_1^R, X_2^R] = -X_3^R.$$

Moreover

$$[X_j, X_k^R] = 0 \quad \text{for all } j \text{ and } k.$$

We have already seen that the Lebesgue measure $\mu = \mathcal{L}^3$ is biinvariant. Alternative view point The volume form is given by

$$\omega = \theta_1 \wedge \theta_2 \wedge \theta_3 = \theta_1^R \wedge \dots$$

is left- and right-invariant.

We also reacall the definition of dilation

$$\delta_t(x) = (tx_1, tx_2, t^2 x_3)$$

and the fact that $\delta_t(x * y) = \delta_t x * \delta_t y$. The Jacobian of δ_t with respect to Lebesgue measure is $J\delta_t = t^4$. We define

$$I_t \varphi := t^{-4} \varphi \circ \delta_t.$$

Then

$$\|I_t \varphi\|_{L^1} = \|\varphi\|_{L^1}.$$

Theorem 10.23. *There exists a constant C such that for all smooth $f : \mathbb{H}^1 \to \mathbb{R}$ we have*

$$\min_{c \in \mathbb{R}} \int_{B_{x,r/2}} |f - c|^2 \leq Cr \int_{B_{x,2r}} \sum_{i=1}^{2} |X_i f|^2 \, d\mu.$$

Overview of the proof. By translation and dilation by δ_r we may assume without loss of generality that $x = 0$ and $r = 1$.

The proof is based three ideas

- The quantity

$$\inf_{\gamma \in \mathbb{R}} \int_{B_{\frac{1}{2}}} |f * \varphi - \gamma|^2 \, d\mu$$

  can be estimated by the usual Poincaré inequality in $\mathbb{R}^3$ which involves the full gradient of $f * \varphi$.
- The full gradient of $f * \varphi$ can be estimated in terms of $\|X_1 f\|_{L^2(B_2)} + \|X_2 f\|_{L^2(B_2)}$.
- We have

$$f - f * \varphi = \int_0^1 \frac{d}{dt} (f * I_t \varphi) \, dt.$$

  This is a continuous version of the telescoping sum trick $f(x) - f_{B_1} = \sum_{i=0}^{\infty} u_{B_{2^{-i}}}$.

$\square$

[10.12. 2019, Lecture 15]
[11.12. 2019, Lecture 16]

To carry out this argument we use the following result which will be proved below.

Lemma 10.24 (Control of all derivatives by horizontal derivatives, [11], Lemma 3.1.)**.** *For every $j \in \{1,2,3\}$ and every $i \in \{1,2\}$ there exists a differential operators D_{ij} such that*

$$X_j(f * \varphi) = \sum_{i=1}^{2} X_i f * D_{ji}\varphi \quad \forall f \in C^\infty(\mathbb{H}),\ \forall \varphi \in C_c^\infty(\mathbb{H}) \tag{10.16}$$

For each $i \in \{1,2\}$ there exists a differential operator $D^{(i)}$ such that for any $\varphi \in C_c^\infty(\mathbb{H})$ the functions $\varphi^{(i)} := D^{(i)}\varphi$ satisfy

$$\frac{\partial}{\partial t} f * I_t\varphi = \sum_{i=1}^{2} (X_i f) * I_t\varphi^{(i)}. \tag{10.17}$$

Proof of Theorem10.23 . Using left translation and dilation it suffices to show that

$$\min_{\gamma\in\mathbb{R}} \int_{B_{1/2}} |f(x) - \gamma|\, dx \le C \int_{B_2} \sum_{i=1}^{2} |Y_i f(x)|^2\, dx \tag{10.18}$$

where $B_{1/2}$ and B_2 are balls with center 0.

Let $\varphi \in C_c^\infty(B_1)$ we a standard mollifying kernel.For a continuous function $f : \mathbb{H} \to \mathbb{R}$ we have

$$\lim_{t\to 0} f * I_t\varphi = f$$

Thus we get

$$\|f - f * \varphi\|_{L^2(B_{\frac{1}{2}})} \le \int_0^1 \|\frac{d}{dt} f * I_t\varphi\|_{L^2(B_{\frac{1}{2}})}\, dt.$$

The second assertion in Lemma 10.24 and Young's inequality imply that

$$\|f - f * \varphi\|^2_{L^2(B_{\frac{1}{2}})} \le \sum_{i=1}^{2} \|X_i f\|^2_{L^2(B_2)}.$$

On the other hand we use the usual Poincaré inequality to estimate $\|f * \varphi - (f * \varphi)_{B_{1/2}}\|_{L^2(B_{1/2})}$ in terms of $\sum_{j=1}^{3} \|X_j(f * \varphi)\|_{B_{\frac{1}{2}}}$. Now the assertion follows from the first assertion in in Lemma 10.24 and Young's inequality. $\square$

Proof of Lemma 10.24. In view of the identity $f * X^R g = Xf * g$ which holds for left-invariant vectorfields the result follows directly from the following lemma. $\square$

Lemma 10.25. *(1) For $j = 1,2,3$ and $i = 1,2$ there exists differential operators D_{ij} such that*

$$X_j\varphi = \sum_{i=1}^{2} X_i^R D_{ij}\varphi \quad \forall \varphi \in C_c^\infty(G). \tag{10.19}$$

(2) For $i = 1,2$ there exists differential operators $D^{(i)}$ such that for any $\varphi \in C_c^\infty(G)$

$$\frac{\partial}{\partial t} I_t\varphi = \sum_{i=1}^{2} X_i^R I_t \varphi^{(i)}$$

where $\varphi^{(i)} = D^{(i)}\varphi$.

Proof.

$$\begin{aligned}
&(\frac{d}{dt} I_t\varphi)(x) = \frac{d}{dt} t^{-4}\varphi(t^{-1}x_1, t^{-1}x_2, t^{-2}x_3)\\
=&-4t^{-5}\varphi(tx_1, tx_2, t^2x_3)\\
&-\sum_{i=1}^{2} t^{-6}x_i(\partial_i\varphi)(t^{-1}x_1, t^{-1}x_2, t^{-2}x_3) - 2t^{-7}x_3(\partial_3\varphi)(t^{-1}x_1, t^{-1}x_2, t^{-2}x_3)\\
=&-4t^{-5}\varphi(tx_1, tx_2, t^2x_3)\\
&-\sum_{i=1}^{2} t^{-5}x_i\frac{\partial}{\partial x_i}\varphi(t^{-1}x_1, t^{-1}x_2, t^{-2}x_3) - 2t^{-5}x_3\frac{\partial}{\partial x_i}\varphi(t^{-1}x_1, t^{-1}x_2, t^{-2}x_3)\\
=&-\sum_{i=1}^{2} t^{-4}\frac{\partial}{\partial x_i}(t^{-1}x_i\varphi(t^{-1}x_1, t^{-1}x_2, t^{-2}x_3)) - 2t^{-3}\frac{\partial}{\partial x_3}(t^{-2}x_3\varphi(t^{-1}x_1, t^{-1}x_2, t^{-2}x_3))
\end{aligned}$$

Now

$$\frac{\partial}{\partial x_1} = X_1^R(x) - \frac{1}{2}x_2\frac{\partial}{\partial x_3}, \quad \frac{\partial}{\partial x_2} = X_2^R(x) + \frac{1}{2}x_1\frac{\partial}{\partial x_3}, \quad \frac{\partial}{\partial x_3} = -[X_1^R, X_2^R].$$

Setting $\eta_i = y_i\varphi(y)$ and using that $x_i\partial_3 h = \partial_3(x_i h)$ for $i = 1,2$ we get

$$\begin{aligned}
&(\frac{d}{dt} I_t\varphi)(x)\\
=&-X_1^R I_t\eta_1 - X_2^R I_t\eta_2 + 2t(X_1^R X_2^R - X_2^R X_1^R) I_t\eta_3\\
=&-X_1^R I_t\eta_1 - X_2^R I_t\eta_2 + 2(X_1^R I_t(X_2^R\eta_3) - X_2^R I_t(X_1^R I_t\eta_3)).
\end{aligned}$$

Here we used that fact for $i = 1, 2$ we have $X_i^R(h \circ \delta_r) = r(X_i^R h) \circ \delta_r$ Thus the assertion holds with

$$\varphi^{(1)} = -\eta_1 + 2X_2^R \eta_3, \quad \varphi^{(2)} = -\eta_2 - 2X_1^R \eta_3.$$

□

10.5. **No Poincaré inequality in hyperbolic space.** Let u be a C^1 map on a Riemannian manifold (equipped with the inner metric). Then the smallest upper gradient is given by

$$|\nabla u|_g = \left(\sum g_{ij}^{-1} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} \right)^{1/2}.$$

[11.12. 2019, Lecture 16]

[17.12. 2019, Lecture 17]

11. Quasisymmetric maps

This section very closely follows Chapter 10 of [8].

11.1. **Definition and elementary properties.** Recall: C^1 map $f : \mathbb{R}^n \to \mathbb{R}^n$. Linear maps map balls to ellipsoids. Extends to small neighbourhood. More global picture? Example $f(x) = |x|^{p-1}x$.

General assumption: X and Y will always denote metric spaces. When necessary we write the metrics explicitly as d_X and d_Y.

Definition 11.1. *(1) We say that $f : X \to Y$ is an embedding if f is a homeomorphism from X to $f(X)$.*

(2) We say that an embedding is quasisymmetric if there exists a homemorphism $\eta : [0, \infty) \to [0, \infty)$ such for all triples $x, a, b \in X$ and all $t > 0$

(11.1)
$$d_X(a, x) \leq t d_X(b, x) \quad \Longrightarrow \quad d_Y(f(a), f(x)) \leq \eta(t) d_Y(f(b), f(x)).$$

(3) An embedding is called bi-Lipschitz if there exists an $L \geq 1$ such that both f and f^{-1} are L-Lipschitz.

We use also use the terms η-quasisymmetric and L-Lipschitz. We often just write d instead of d_X and d_Y.

One easily verifies that every L-bi-Lipschitz map is η quasisymmetric with $\eta(t) = L^2 t$. If $0 < \varepsilon \leq 1$ and

$$X = Y, \quad d_Y(x, y) = d^\varepsilon(x, y)$$

then the identity map is alway quasisymmetric but rarely bi-Lipschitz.

Example: $X = Y = \mathbb{R}^n$, $f(x) = \lambda x$.

Example: $X = Y = \mathbb{R}^n$, $f(x) = |x|^{p-1}x$, $p > 0$.

Definition 11.2. *An embedding $f : X \to Y$ is called weakly $(H-)$quasisymmetric if there exists a constant H such that*
(11.2)
$$d_X(a,x) \le d_X(b,x) \quad \Longrightarrow \quad d_Y(f(a), f(x)) \le H d_Y(f(b), f(x)).$$

Weakly quasisymmetric maps need not be quasisymmetric.

Example $X = \mathbb{N} \times \{0, -1\}$, $f(n,0) = (n,0)$, $f(n, -\frac{1}{4}) = (n, -\frac{1}{4n})$. Consider $x = (n,0)$, $b = (n, -\frac{1}{4})$, $a = (n-1, 1)$. Then $d(x,a) \le 4d(x,b)$ and $d(f(x), f(a))/d(f(x), f(b)) \to \infty$ as $n \to \infty$.

We will show below that in many interesting spaces weak quasisymmetry implies quasisymmetry. First we develop the basic theory.

Proposition 11.3. *If $f : X \to Y$ is η-quasisymmetric, then $f^{-1} : f(X) \to X$ is $\tilde\eta$ quasisymmetric with*
$$\tilde\eta(t) = \frac{1}{\eta^{-1}(t^{-1})} \quad \textit{for } t > 0.$$
If $f : X \to Y$ and $g : Y \to Z$ are η_f and η_g quasisymmetric, the $g \circ f : X \to Z$ is $\eta_g \circ \eta_f$ quasisymmetric.

Proof. Let $x', a', b' \in Y$ and let $x = f^{-1}(x')$, $a = f^{-1}(a')$, $b = f^{-1}(b')$. Let $t > 0$ and assume that
$$\frac{d(a', x')}{d(b', x')} \le t, \quad s := \frac{d(a,x)}{d(b,x)}.$$
Then $d(b,x)/d(a,x) = s^{-1}$ and hence
$$\frac{1}{t} \le \frac{d(b', x')}{d(a', x')} \le \eta\big(\frac{1}{s}\big)$$
Note that η is strictly increasing since η is a homemorphism from $[0, \infty)$ to itself. Thus
$$\eta^{-1}\big(\frac{1}{t}\big) \le \frac{1}{s}$$
which implies that $s \le \tilde\eta(t)$ as desired.

The proof of the second assertion is straightforward. □

Proposition 11.4. *The restriction to a subset of an η-quasisymmetric map is η-quasisymmetric.*

Proposition 11.5. *Quasisymmetric embeddings map bounded spaces into bounded spaces. More quantitatively, if $f : X \to Y$ is η-quasisymmetric and if $A \subset B \subset X$ are such that $0 < \operatorname{diam} A \le \operatorname{diam} B < \infty$, then* $\operatorname{diam} f(B)$ *is finite and*

(11.3)
$$\frac{1}{2\eta\left(\frac{\operatorname{diam} B}{\operatorname{diam} A}\right)} \le \frac{\operatorname{diam} f(A)}{\operatorname{diam} f(B)} \le \eta\left(\frac{2 \operatorname{diam} A}{\operatorname{diam} B}\right).$$

Proof. To see that $\operatorname{diam} B$ is finite choose $b_n \in B$ and $b'_n \in B$ such that

$$\frac{1}{2}\operatorname{diam} B \le d(b_n, b'_n) \to \operatorname{diam} B, \quad \text{as } n \to \infty.$$

Then for any $b \in B$ we have

$$d(b, b_1) \le \operatorname{diam} B \le 2d(b'_1, b_1).$$

Thus

$$d(f(b), f(b_1)) \le \eta(2) d(f(b'_1), f(b_1)).$$

Thus $\operatorname{diam} B \le 2\eta(2) d(f(b'_1), f(b_1))$.

To prove (11.3) let $x, a \in A$. Then

$$d(b_n, b'_n) \le d(b_n, a) + d(b'_n, a).$$

By symmetry we may assume that

$$d(b_n, a) \ge \frac{1}{2} d(b_n, b'_n).$$

This implies

$$d(f(x), f(a)) \le \eta\left(\frac{d(x,a)}{d(b_n,a)}\right) d(f(b_n), f(a))$$

Since $d(b_n, b'_n) \to \operatorname{diam} B$ we get the second inequality in (11.3).

The first inequality follows by applying the second inequality to f^{-1} and using Proposition 11.3. Indeed we have

$$\frac{\operatorname{diam} A}{\operatorname{diam} B} \le \tilde{\eta}\left(\frac{2\operatorname{diam} f(A)}{\operatorname{diam} f(B)}\right).$$

Thus

$$\frac{\operatorname{diam} A}{\operatorname{diam} B} \le 1/\eta^{-1}\left(\frac{\operatorname{diam} f(B)}{2\operatorname{diam} f(A)}\right),$$

$$\frac{\operatorname{diam} B}{\operatorname{diam} A} \ge \eta^{-1}\left(\frac{\operatorname{diam} f(B)}{2\operatorname{diam} f(A)}\right),$$

$$\eta\left(\frac{\operatorname{diam} B}{\operatorname{diam} A}\right) \ge \frac{\operatorname{diam} f(B)}{2\operatorname{diam} f(A)}$$

and the last inequality is equivalent to the first inequality in (11.3). ∎

Two useful consequences.

Proposition 11.6. *Quasisymmetric maps take Cauchy sequences to Cauchy sequences.*

Proof. Let (a_n) be a Cauchy sequence and set $A_n = \{a_k : k \ge n\}$. Then $B := A_1$ is bounded and by assumption $\operatorname{diam} A_n \to 0$ as $n \to \infty$. It follows from (11.3) that $\operatorname{diam} f(A_n) \to 0$. ∎

Proposition 11.7. *An η-quasisymmetric embedding of one space in another extends to an η-quasisymmetric embedding of the completions.*

Proof. Use that Cauchy sequences are mapped to Cauchy sequences and the following fact. If a_n and a'_n are Cauchy sequences and $d(a_n, a'_n) \to 0$ as $n \to \infty$ then $d(f(a_n), f(a'_n)) \to 0$. $\square$

11.2. **Doubling spaces.**

Definition 11.8. *A metric space is called doubling if there is a constant $C_1 \geq 1$ such that every set of diameter d can be covered by at most C_1 sets of diameter $d/2$.*

If X is doubling there exists a function $C_1 : (0, \frac{1}{2}]$ such that each set of diameter d can be covered by at most $C_1(\varepsilon)$ sets of diameter εd. The function C_1 be chosen of the form

$$C_1(\varepsilon) = C\varepsilon^{-\beta} \tag{11.4}$$

for some $C \geq 1$ and $\beta > 0$.

Definition 11.9. *Given a doubling metric space, the infimum of all numbers $\beta > 0$ such that a covering function of the form* (11.4) *can be found is the Assouad dimension*[1] *of X.*

Example: the Assouad dimension of $\mathbb{R}^n$ is n.

Example: show that the Hausdorff dimension does not exceed the Assouad dimension. Show that the Assouad dimension of the compact set $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\} \subset \mathbb{R}$ is 1. Hint: cover the set $[\frac{1}{2l}, \frac{1l}{]} \cap X$.

[17.12. 2019, Lecture 17]

[18.12. 2019, Lecture 18]

Proposition 11.10. *If X carries a doubling measure then X is doubling.*

The converse implication is in general false, but one can show that every *complete* doubling space carries a doubling measure [12].

Proof. It suffices to show that any closed ball $B(x, d)$ can be covered by a fixed number of closed balls of radius $d/4$.

To do so we use the following observation. Let $\rho > 0$ and let $A \subset X$ be a set with $\operatorname{diam} A > \rho$. Define

$$M := \sup\{k : \exists x_1, \ldots, x_k \in A : d(x_i, x_j) \geq \rho \text{ if } i \neq j.\}$$

[1]The terms *(metric) covering dimension* or *uniform metric dimension* are also used for this concept

If $M < \infty$ then A can be covered by M closed balls of radius ρ. Indeed let $x_1, \ldots, x_M$ points as in the definition of M. Then $\bigcup_{i=1}^{M} B(x_i, r) \subset A$ as otherwise we get a contradiction to the definition of M.

Now we apply the observation with $A = B(x, d)$. Assume that there exist k points $x_1, \ldots x_k \in B(x, d)$ with $d(x_i, x_j) \geq d/4$ whenever $i \neq j$. Then the balls $B(x_i, d/12)$ are disjoint. Moreover $B(x_i, d/12) \subset B(x, 2d) \subset B(x_i, 3d)$. Since μ is doubling we have

$$\mu(B(x, 2d)) \leq C\mu(B(x_i, d/12).$$

Summing over i we get

$$k\mu(B(x, 2d)) \leq C\mu(B(x, 2d))$$

and thus $k \leq \lfloor C \rfloor$. Hence $B(x, d)$ can be covered by $\lfloor C \rfloor$ balls of radius $d/4$ and diameter $d/2$. □

Theorem 11.11. *A quasisymmetric image of a doubling space is doubling, quantitatively.*

Proof. Let $f : X \to Y$ be an η-quasisymmetric homeomorphism. It suffices to show that every ball B of diameter d in Y can be covered by at most some fixed number C_2 of sets of diameter at most $d/4$. Let $B = B(y, d)$ and let

$$L = \sup_{z \in B} d(f^{-1}(y), f^{-1}(z))$$

(recall that the image of a bounded set under the quasisymmetric map f^{-1} is bounded). Then we can cover $f^{-1}(B)$ by at most $C_1(\varepsilon)$ sets of diameter at most $\varepsilon 2L$ for any $\varepsilon \leq \frac{1}{2}$ where C_1 is a covering function of X. Let $A_1, \ldots, A_p$ be such sets, so that $p = p(\varepsilon) \leq C_1(\varepsilon)$. We may clearly assume that $A_i \in f^{-1}(B)$ for all $i = 1, \ldots, p$. Thus $f(A_1), \ldots f(A_p)$ cover B and are contained in B so by Proposition 11.5 we compute that their diameters satisfy

$$\begin{aligned}\operatorname{diam} f(A_i) &\leq \operatorname{diam} B\ \eta\left(\frac{2 \operatorname{diam} A_i}{\operatorname{diam} f^{-1}(B)}\right) \\ &\leq d\eta\left(\frac{4\varepsilon L}{L}\right) \leq d\eta(4\varepsilon).\end{aligned}$$

Now choose ε so small that $\eta(\varepsilon) \leq \frac{1}{4}$. □

Theorem 11.12. *A weakly quasisymmetric embedding of a connected doubling space in a doubling space is quasisymmetric, quantitatively.*

Let X be a metric space and let $x_0, \ldots, x_N \in X$ and let $\varepsilon > 0$. We say that the $(N+1)$-tuple $(x_0, \ldots, x_N)$ is an ε-chain if joining x_0 and

x_N if $d(x_i, x_{i+1}) \leq \varepsilon$. If $(x_0, \dots x_N)$ and $(y_0, \dots y_M)$ are ε-chains and $x_N = y_0$ then $(x_0, \dots x_N, y_1, \dots y_M)$ is an ε-chain.

Lemma 11.13. *A connected metric space X is ε-chainable for all $\varepsilon > 0$, that is for each $\varepsilon > 0$ and each pair of points $a, b \in X$ there exist an ε-chain joining a and b.*

Note that in general for connected metric space X there is no path (=continuous curve) joining a and b. Example: $X = \{(x, y) \in \mathbb{R}^2 : x \in (0, 1], y = \sin x^{-1}\} \cup \{(0, 0)\} \subset \mathbb{R}^2$, $a = (0, 0)$, $b = (1, \sin 1)$. There exist metric spaces which are ε-chainable for all $\varepsilon > 0$ but not connected. The simplest example is $X = (-1, 0) \cap (0, 1) \subset \mathbb{R}$.

Proof. Let A be a the set of points which can be joined to a by an ε-chain. We show that A is open and closed. Since $a \in A$ the set A is also non-empty and thus $A = X$ as X is connected. To see that A is open not that $b \in A$ implies that $B(b, \varepsilon) \subset A$ since every point in $x \in B(b, \varepsilon)$ can be joined to b by the chain (b, x). To see that A is closed consider $b_n \in A$ with $b_n \to b$. Then there exist a k such that $d(b_k, b) \leq \varepsilon$. Thus (b_k, b) is an ε-chain and hence $b \in A$. $\square$

Proof of Theorem 11.12. Let $f : X \to Y$ be weakly H-quasisymmetric, where both X and Y are doubling, and in addition X is connected. We may clearly assume that X is not a singleton. Pick three distinct points a, b, x from X and write

$$t = \frac{d(a, x)}{d(b, x)}, \quad t' = \frac{d(f(a), f(x))}{d(f(b), f(x))}.$$

We need to show that $t' \leq \eta(t)$ where $\eta(t) \to 0$ as $t \to 0$.

First assume that $t > 1$. It suffices to show that there exist $s \leq C(t)$ and points $a_0, \dots a_{s+1}$ such that

$$a_0 = x,$$

$$d(a_0, a_1) \leq r, \tag{11.5}$$

$$d(a_i, a_{i+1}) \leq d(a_{i-1}, a_i) \quad \text{for } i = 1, \dots, s, \tag{11.6}$$

$$d(x, a) \leq d(x, a_{s+1}). \tag{11.7}$$

Then weak $H-$quasisymmetry implies that

$$\begin{aligned} d(f(a_0), f(a_1) &\leq H d(f(x), f(b)), \\ d(f(a_i), f(a_{i+1})) &\leq H d(f(a_{i-1}, f(a_i)) \quad \text{for } i = 1, \dots, s, \\ d(f(x), f(a)) &\leq H d(f(x), f(a_{s+1})). \end{aligned}$$

From this we conclude that

$$\begin{aligned} Hd(f(x), f(a_{s+1})) &\leq H \sum_{i=0}^{s} d(f(a_i), f(a_{i+1}) \\ &\leq H \sum_{i=0}^{s} H^{i+1} d(f(x), f(b)). \end{aligned}$$

Hence we may take

$$\eta(t) = \frac{H^2}{H-1} H^{C(t)+1}.$$

To explain the construction of the points a_i we first make the additional assumption that X is path-connected. Let $\gamma : [0,1] \to X$ be a (continuous) curve with $\gamma(0) = x$ and $\gamma(1) = a$. Let

$$r := d(x, b).$$

Set $\sigma_0 = 0$ and define inductively

$$\sigma_{i+1} = \sup\{\sigma \in [\sigma_i, 1] : d(\gamma(\sigma), \gamma(\sigma_i)) \leq r\}.$$

Case 1: $d(x, \gamma(\sigma_{i+1})) < d(x, a)$
Continue the iteration.
Case 2: $d(x, \gamma(\sigma_{i+1})) \geq d(x, a)$
Set $s = i$ and stop the iteration.
Set

$$a_i = \gamma(\sigma_i).$$

In Case 1 we have $\sigma_{i+1} \in (0, 1)$ (since $\gamma(1) = a$ so that $d(x, \gamma(1)) = d(x, a)$) and thus

$$d(a_i, a_{i+1}) = r. \tag{11.8}$$

The key point is to show that Case 2 indeed occurs and that s can be bounded in terms of t. To see this assume that $d(x, a_i) < d(x, a)$ for $i = 0, \dots, k$. We will derive an upper bound for k from the doubling property of X. It follows from the definition of σ_j that for $j = 0, \dots k - 2$.

$$d(\gamma(\sigma), \gamma(\sigma_j)) > r \quad \text{if } \sigma > \sigma_{j+1}.$$

Thus $d(a_i, a_j) > r$ if $i \geq j + 2$. Together with (11.8) and the symmetry of d we see that

$$d(a_i, a_j) \geq r \quad \text{for } i \neq j \text{ and } i, j \leq k. \tag{11.9}$$

Now $tr = d(x, a)$ and thus $\{a_0, \dots, a_k\} \subset B(x, tr)$. The ball $B(x, tr)$ can be covered by $C_1(1/4t)$ sets of diameter $r/2$ where C_1 is a covering function of X. In view of (11.9) none of those sets can contain more than one of the points $a_0, \dots a_k$. Thus $k+1 \le C_1(1/4t)$. It follows that $s+1 \le C_1(1/4t)$.

Now we carry out the proof using only the fact that X is connected. In this case there may not be a curve which joins a and x, but for each $\varepsilon > 0$ there is an ε-chain $(x_0, \dots, x_N)$ which joins a and x. We will select the points a_i from the set $\{x_0, \dots, x_N\}$. In this case we will in general not be able to choose the points a_i such that $d(a_{i+1}, a_i) = r$ but we have to allow for an error of ε. To ensure that $i \mapsto d(a_{i+1}, a_i)$ is decreasing we will choose the points a_i such that

$$r - (i+1)\varepsilon < d(a_{i+1}, a_i) \le r - i\varepsilon.$$

where

$$r = d(b, x) \quad \text{and} \quad \varepsilon = \min\big(\frac{r}{2C_1(1/6t)}, \frac{r}{2}\big).$$

where C_1 denotes the covering function of X.

More precisely we define indices k_i and points a_i inductively as follows. Set $k_0 = 0$ and define

$$k_{i+1} = \max\{k \in [k_i, N] \cap \mathbb{N} : d(x_k, x_{k_i}) \le r - i\varepsilon.$$

$$a_{i+1} = x_{k_{i+1}}.$$

Case 1 : $d(x, a_{i+1}) < d(x, a)$.
Continue the iteration.

Case 2: $d(x, a_{i+1}) \ge d(x, a)$. Set $s = i$ and stop the iteration.

In Case 1 we have in particular $k_{i+1} < N$ (since $d(x, x_N) = d(x, a)$) and

$$d(a_i, x_{k_{i+1}+m}) > r - i\varepsilon. \tag{11.10}$$

for all $m \in \mathbb{N} \cap [1, N - k_{i+1}]$ by the maximality of k_{i+1}. Taking $m = 1$ and using that $(x_0, \dots, x_N)$ is an ε-chain we deduce that

$$r - i\varepsilon \ge d(a_i, a_{i+1}) > r - (i+1)\varepsilon. \tag{11.11}$$

We next show Case 2 must occur and that $s + 1 \le \lfloor C(1/6t) \rfloor =: K$. To see this assume that either Case 2 does not occur or $s + 1 \ge K + 1$. Since $tr = d(x, a)$ there then exist $a_0, \dots a_K$ as above such that $a_i \in B(a, tr)$. Moreover applying (11.11) for $i \le K - 1$ and (11.10) for $i \ge K - 2$ we see that

$$d(a_j, a_i) \ge r - (i+1)\varepsilon \ge r - K\varepsilon \ge \frac{1}{2}r \quad \text{if } K \ge j > i.$$

On the other hand we have $\{a_0, \dots a_K\} \subset B(a, tr)$. The ball $B(a, tr)$ can be covered by at most $C(1/6t)$ sets of diameter $r/3$. Thus each point $a_0, \dots a_K$ can be contained in at most one of the covering sets.

Hence $K+1 \leq C(1/6t)$. This contradicts the definition of K, namely $K = \lfloor C(1/6t) \rfloor$.

Now (11.5)–(11.7) follow from (11.11) and the definition in Case 2.

Now consider the case $t \leq 1$. We need to find $\eta(t)$ such that $t' \leq \eta(t)$ and $\eta(t) \to 0$ as $t \to 0$. We also need to make η a homemorphism but this is easily achieved. Weak quasisymmetry implies that $t' \leq H$ for $t \leq 1$. Hence we may assume $t < \frac{1}{3}$.

Let s be the smallest integer such that

$$3^{-s} \leq t \quad \text{or, equivalently} \quad 3^{-s} d(b,x) \leq d(a,x).$$

Then $s \geq 2$ and

$$\frac{\log 1/t}{\log 3} \leq s. \tag{11.12}$$

Since X is connected there exists points $b = b_0, b_1, \ldots b_{s-1}$ such that

$$d(b_i, x) = 3^{-i} d(b,x) \quad \text{for } i = 0, \ldots, s-1.$$

Now for $0 \leq i < j \leq s-1$ we have

$$d(b_i, b_j) \geq d(b_i, x) - d(b_j, x) \geq \frac{2}{3} 3^{-i}$$

and thus

$$d(a, b_j) \leq d(a,x) + d(b_j, x) \leq 2\, 3^{-j} \leq d(b_i, b_j).$$

Weak h-quasisymmetry implies that

$$d(f(a), f(b_j)) \leq H d(f(b_i), f(b_j)).$$

Moreover $d(x, b_j) \leq d(b_i, b_j)$ which yields

$$d(f(x), f(b_j)) \leq H d(f(b_i), f(b_j)).$$

Hence

$$t' d(f(b), f(x)) = d(f(a), f(x)) \leq 2H d(f(b_i), f(b_j)). \tag{11.13}$$

Since $d(b_i, x) \leq d(b, x)$ all s the points $f(b_0), \ldots f(b_{s-1})$ lie in the closed ball $B(f(x), Hd(f(b), f(x)))$. By (11.13) these points are separated by $(t'/2H) d(f(b), f(x))$. Thus the doubling property of Y implies that

$$s \leq C'(t'/(4H^2)) \leq C 4^{\beta'} H^{2\beta'} t'^{-\beta'}$$

where C' is the covering function of Y. Combining this with (11.12) we see that

$$t' \leq \tilde{C} H^2 (\log 1/t)^{-1/\beta'}$$

and the right and side goes to zero as $t \to 0$. □

[18.12. 2019, Lecture 18]
[21.1. 2020, Lecture 19]

11.3. **Compactness of families of quasisymmetric maps.** A family of η-quasisymmetric maps need not be compact. For example, for $X = \mathbb{R}^n$ the family $\mathcal{F} = \{x \mapsto \lambda x : \lambda > 0\}$ is not compact. Under mild normalizing condition we do, however, obtain compactness and compactness is a key property of quasisymmetric maps.

The first key ingredient an equicontinuity result.

Definition 11.14. *Let X and Y be metric spaces. A family $\mathcal{F}$ of maps from X to Y is equicontinuous at point $x_0 \in X$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in X$*

$$d(x, x_0) < \delta \textit{ and } f \in \mathcal{F} \quad \Longrightarrow \quad d(f(x), f(x_0)) < \varepsilon.$$

The familiy is called (pointwise) equicontinuous if it is equicontinuous at every $x_0 \in X$. The family is called uniformly equicontinuous if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in X$

$$d(x, y) < \delta \textit{ and } f \in F \quad \Longrightarrow \quad d(f(x), f(y)) < \varepsilon.$$

There is also version of equicontinuity at x_0 from maps from a topological space X into a metric space Y: for each $\varepsilon > 0$ there exists a neighbourhood U_{x_0} of x_0 such that $f(U_{x_0}) \subset B(f(x_0), \varepsilon)$ for all $f \in \mathcal{F}$.

Proposition 11.15. *Given metric spaces X and Y, two points $a, b \in X$ and a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ with $\eta(0) = 0$ the family*

$$\mathcal{F} := \{f : X \to Y : f \textit{ is } \eta\textit{-quasisymmetric and } |f(a) - f(b)| \leq M\}$$

is equicontinuous

Proof. Fix $x_0 \in X$. Exchanging the roles of a and b if necessary we can assume that $x_0 \neq a$. Then

$$\begin{aligned} d(f(x), f(x_0) &\leq \eta\left(\frac{d(x, x_0)}{d(a, x_0)}\right) d(f(a), f(x_0)) \\ &\leq \eta\left(\frac{d(x, x_0)}{d(a, x_0)}\right) \eta\left(\frac{d(a, x_0)}{d(a, b)}\right) d(f(a), f(b)) \\ &\leq \eta\left(\frac{d(x, x_0)}{d(a, x_0)}\right) \eta\left(\frac{d(a, x_0)}{d(a, b)}\right) M. \end{aligned}$$

This shows equicontinuity at x_0. ☐

Corollary 11.16. *If Y is bounded then the family of all η-quasisymmetric embeddings of X in Y is equicontinuous.*

Definition 11.17. *Let X be a topological space and let Y be a metric space. A family F of maps from X to Y is normal if every sequence from the family subconverges uniformly on compact sets.*

Theorem 11.18 (Arzela-Ascoli theorem)**.** *An equicontinuous family $\mathcal{F}$ of maps from a separable topological space X to a metric space Y is normal if for each $x \in X$ of the set*

$$\{f(x) : f \in \mathcal{F}\}$$

is precompact.

Sketch of proof. Let (f_k) be a sequence in $\mathcal{F}$. Let D be a countable dense subset. By diagonalization we find a subsequence (still denoted f_k) such that $f_k(x)$ converges for every $x \in D$. This sequence will not be changed. We do not take further subsequences.

For an arbitrary point $a \in X$ use equicontinuity at a and the fact that every neighbourhood U_a intersects D to show that $f_k(a)$ is a Cauchy sequence. Since $\{f_k(a) : k \in \mathbb{N}\}$ is precompact, a subsequence converges. Thus the whole sequence converges. Hence there exists $f : X \to Y$ such that

$$f_k(x) \to f(x) \quad \forall x \in X.$$

Equicontinuity is inherited. In particular f is continuous.

Uniform convergence on compact sets follows by equicontinuity and finite subcovers. ☐

The second key ingredient is the stability of the qs-condition under pointwise convergence.

Proposition 11.19. *Suppose that a sequence (f_k) of η-quasisymmetric maps from X to Y converges pointwise to a (not necessarily continuous) map f. Then f is either constant or η-quasisymmetric and the convergence is uniform on compact sets.*

Proof. If $f(a) = f(b)$ for some $a \neq b$ then $d(f_n(a), f_n(b)) \to 0$. By qs $d(f_n(a), f_n(x)) \to 0$ for any x. Hence f constant.

Generally: let $a, b, x \in X$. Then

$$d(f_n(a), f_n(x)) \leq \eta\left(\frac{d(a,x)}{d(b,x)}\right) d(f_n(b), f_n(x))$$

Pass to pointwise limit. Thus f has the same property.

It follows that f is continuous (fix b, take $a \to x$). If f is not injective then f is constant (there exist b, x s.t. $f(b) = f(x)$)).

If f is injective and qs then f^{-1} is $\tilde{\eta}$ qs (same proof as before; continuity of the inverse was not used). Thus f^{-1} is continuous. Hence f is an embedding.

For uniform convergence on compact sets use a finite cover by small balls and η-quasisymmetry. $\square$

Recall that a metric space is proper if all closed and bounded sets are compact (or, equivalently all closed balls are compact).

Corollary 11.20. *Let X be a separable metric space. Let $\mathcal{F}$ be a family of η-quasisymmetric embeddings of X in Y. Assume that*

(1) Y is compact or;
(2) Y is proper and there is $x_0 \in X$ and $y_0 \in Y$ such that $f(x_0) = y_0$ for all $f \in \mathcal{F}$.

Assume in addition that there exists a $C \geq 1$ and $a, b \in X$ such that

$$\forall f \in \mathcal{F} \quad C^{-1} \leq d(f(a), f(b)) \leq C. \tag{11.14}$$

Then $\mathcal{F}$ is a (sequentially) compact family of embeddings, i.e., every sequence (f_n) in $\mathcal{F}$ subconverges (uniformly on compacta) to an element of $\mathcal{F}$.

Proof. To show subconvergence we use the Arzela-Ascoli theorem, Theorem 11.18. Equicontinuity follows from (11.14) and Proposition 11.15.

Precompactness of $\{f(x) : f \in \mathcal{F}\}$ for fixed x is clear under assumption (i). Under assumption (ii) we use that by qs $f(x)$ lies in the closed ball $f(y_0, \eta(d(x, x_0))$. This ball is compact since Y is proper.

Quasisymmetry of the limit follows from Proposition 11.19. $\square$

For maps from $\mathbb{R}^n$ to $\mathbb{R}^p$ there is a characterisation of qs embeddings based on compactness. Fix $H \geq 1$, integers $1 \leq p \leq n$ and unit vector $e \in \mathbb{R}^n \subset \mathbb{R}^p$. Let

$$\mathcal{Q}_H := \{f : \mathbb{R}^p \to \mathbb{R}^n : f \text{ weakly } H\text{-qs embedding, } f(0) = 0,\ f(e) = e\ \}.$$

Note that in this case weak quasisymmetry implies quasisymmetry, quantitatively by Theorem 11.12.

Recall that a similarity of $\mathbb{R}^n$ is a composition of translations, dilations and rotations. For an arbitrary embedding $f : \mathbb{R}^n \to \mathbb{R}^p$ let

$$\mathcal{W}_f := \{g = \alpha \circ f \circ \beta : \alpha,\, \beta \text{ similarities, } g(0) = 0,\ g(e) = e\ \}.$$

Theorem 11.21. *The family $\mathcal{Q}_H$ is (sequentially) compact. An embedding $f : \mathbb{R}^p \to \mathbb{R}^n$ is quasisymmetric if and only if $\mathcal{W}_f$ belongs to some (sequentially) compact family of embeddings.*

Proof. Compactness of $\mathcal{Q}_H$ follows from Corollary 11.20. Note that if f is η-quasisymmetric then all elements of $\mathcal{W}_f$ are η-quasisymmetric and hence W_f is contained a a set Q_H for a suitable H.

If remains to show that f is (weakly) quasisymmetric if $\mathcal{W}_f$ belongs to some compact family of embeddings. Pick three points $a, b, x \in \mathbb{R}^p$ so that

$$d(a, x) \leq d(b, x).$$

By pre- and postcomposing with appropriate similarities we can assume $x = 0 = f(x)$ and $b = e = f(b)$ without changing the family $\mathcal{W}_f$. It suffices to show that $f(a)$ lies in some fixed ball whose radius in independent of f. But this is clear from the fact that a lies in the closed unit ball of $\mathbb{R}^p$ and $\mathcal{W}_f$ lies in a sequentially compact family of embeddings. $\square$

[21.1. 2020, Lecture 19]

[22.1. 2020, Lecture 20]

12. Quasisymmetric maps II

This section very closely follows Chapter 11 of [8].

12.1. **Hölder continuity of quasisymmetric maps.** We have seen that under a mild normalizing condition quasisymmetric maps are equicontinuous. Now we want to investigate whether they are even Hölder continuous. The key condition is a new condition on the domain of definition X which is weaker than connectness. In the following definition $B(x, r)$ denotes the open ball or radius r.

Definition 12.1. *A metric space X is called uniformly perfect if there exists a constant $C > 1$ such that for every $x \in X$ and for each $r > 0$ the set $B(x, r) \setminus B(x, r/C)$ is non-empty whenever $X \setminus B(x, r)$ is non-empty.*

The condition forbids small islands in the set. It can be reformulated as follows. We say that $B(x, R) \setminus B(x, r)$ is an annulus if $0 < r < R$ and $X \setminus B(x, R)$ is non-empty. At set is uniformly perfect if and only if for every empty annulus the ratio R/r is bounded from above.

Connected spaces are uniformly perfect, as well as many fractals. For example, the classical ternary Cantor set is uniformly perfect (Exercise !).

Proposition 12.2. *Uniform perfectness is a quasisymmetric invariant, quantitatively.*

Proof. Exercise. $\square$

Theorem 12.3. *A quasisymmetric embedding f of a uniformly perfect space X is η-quasisymmetric with η of the form*

$$\eta(t) = C \max(t^\alpha, t^{1/\alpha})$$

where $C \geq 1$ and $\alpha \in (0, 1]$ depend on the data associated with X and f.

Corollary 12.4. *Quasisymmetric embeddings on uniformly perfect spaces are Hölder continuous on bounded sets.*

Proof. This follows from the second inequality in (11.3) with B a bounded set and $A = \{x, y\} \subset B$. □

For general sets there are maps which are quasisymmetric, but not Hölder continuous. As an example consider the compact space $X = \{0\} \cup \{e^{-n!} : n = 2, 3, \dots\} \subset \mathbb{R}$ and the map $f : X \to \mathbb{R}$ given by $f(x) = -(\log x)^{-1}$ for $x \neq 0$ and $f(0) = 0$. Note that f cannot be extended to a quasisymmetric from $\mathbb{R} \to \mathbb{R}$ or from $[0, 1] \to \mathbb{R}$ (otherwise Theorem 12.3 could be applied to the extension).

For the proof of Theorem 12.3 it is convenient to use the following equivalent characterisation of uniformly perfect spaces.

Lemma 12.5. *A metric space X is uniformly perfect if and only if there are numbers $0 < \lambda_1 \leq \lambda_2 < 1$ so that for each pair of points $a, b, \in X$ there exists a point $x \in X$ with*

$$\lambda_1 d(a, b) \leq d(a, x) \leq \lambda_2 d(a, b). \tag{12.1}$$

Proof. If X is uniformly perfect we set $R = d(a, b)/2$. The set $X \setminus B(a, R)$ is non-empty because it contains the point b. Thus there is a point $x \in X$ which satisfies

$$R/C \leq d(x, a) < R.$$

It follows that the desired assertion holds with $\lambda_2 = \frac{1}{2}$ and $\lambda_1 = 1/(2C)$.

To prove the converse implication, let $B(a, r)$ be a ball such that $X \setminus B(a, r)$ contains a point b. We will show that the annulus $B(a, r) \setminus B(a, \lambda_1 r)$ is not empty, so that X is uniformly perfect with constant $C = \lambda_1^{-1}$. This is achieved by iterating condition (12.1): pick a point $x_0 \in X$ such that

$$\lambda_1 d(a, b) \leq d(a, x_0) \leq \lambda_2 d(a, b).$$

If $d(a, x_0) < r$ then x_0 lies in the annulus $B(a, r) \setminus B(a, \lambda_1 r)$ and we are done. If $d(a, x_0) \geq r$ we repeat the preceeding reasoning with b replaced with x_0. Thus we find a point x_1 such that

$$\lambda d(a, x_0) \leq d(a, x_1) \leq \lambda_2 d(a, x_0) \leq \lambda_2^2 d(a, b).$$

Now as long as $d(a, x_{k-1}) \geq r$ we inductively pick points x_k such that

$$\lambda_1 r \leq \lambda_1 d(a, x_{k-1}) \leq d(a, x_k) \leq \lambda_2 d(a, x_{k-1}) \leq \lambda_2^{k+1} d(a, b).$$

Since the right hand side converges to zero there must exist a k such that $d(a, x_k) < r$ and $d(a, x_{k-1}) \geq r$. This shows that $B(a, r) \backslash B(a, \lambda_1 r)$ is non-empty and the proof is finished. □

Proof of Theorem 12.3. Pick three distinct points $x, a, b \in X$ and set

$$t = \frac{d(x, a)}{d(x, b)}.$$

Case 1: $t \geq 1$.

Using condition (12.1) we find points $x_0 = a$, x_1, $x_2, \ldots$ such that

$$\lambda_1 d(a, x) \leq d(x_1, x) \leq \lambda_2 d(a, x)$$

and

$$\lambda_1 d(x_i, x) \leq d(x_{i+1}, x) \leq \lambda_2 d(x_i, x) \leq \lambda_2^{i+1} d(a, x).$$

Let s be the largest integer such that $d(b, x) \leq d(x_s, x)$. Then $d(b, x) > d(x_{s+1}, x) \geq \lambda_1 d(x_s, x)$. Thus

$$\lambda_1 d(x_s, x) \leq d(b, x) \leq d(x_s, x) \leq \lambda_2^s d(a, x).$$

This implies that

$$d(f(a), f(x)) \leq H^{s+1} d(f(b), f(x)), \quad \text{where } H = \eta(\lambda_1^{-1}).$$

On the other hand,

$$d(b, x) \leq \lambda_2^s d(a, x) = \lambda_2^s t d(b, x).$$

Hence

$$t \geq \lambda_2^{-s}$$

and thus

$$d(f(a), f(x)) \leq H t^{\beta} d(f(b), f(x))$$

where $\beta = (\log H)/(\log \lambda_2^{-1})$.

Case 2: $t < 1$.

We first note that we may assume without loss of generality that $\eta(\lambda_2) \leq \frac{1}{2}$. Indeed, we know that there exist $0 < \lambda_1 \leq \lambda_2 < 1$ such that for each pair of points p, q there is a point y_1 which satisfies $\lambda_1 d(p, q) \leq d(p, y_1) \leq \lambda_2 d(p, q)$. Applying the condition iteratively to the pairs p, y_{i-1} we find that there exists points y_i such that $\lambda_1^i d(p, q) \leq d(p, y_i) \leq$

$\lambda_2^i d(p,q)$. Thus in condition (12.1) we may replace λ_1, λ_2 by λ_1^i and λ_2^i. Since $\eta(\lambda_2^i) \to 0$ as $i \to \infty$ we may in particular assume that $\eta(\lambda_2) \leq \frac{1}{2}$.

Now we can proceed as in the case $t \geq 1$. Exchanging the roles of a and b we find points $x_0 = b$, $x_1, \ldots, x_s$ such that

$$\lambda_1 d(x_i, x) \leq d(x_{i+1}, x) \leq \lambda_2 d(x_i, x) \leq \lambda_2^{i+1} d(b, x).$$

and

$$\lambda_1 d(x_s, x) \leq d(a, x) \leq d(x_s, x) \leq \lambda_2^s d(b, x).$$

Thus

$$d(f(x_s), f(x)) \leq 2^{-s} d(f(b), f(x)), \quad d(f(a), f(x)) \leq H d(f(x_s), f(x))$$

where $H = \eta(1)$. Hence

$$t \geq \lambda_1^{s+1}$$

and

$$d(f(a), f(x)) \leq \eta(1) 2^{-s} d(f(b), f(x)) \leq 2\eta(1) t^\alpha d(f(b), f(x))$$

with $\alpha = \log 2 / \log \lambda_1^{-1}$. $\square$

12.2. **Quasisymmetry in Euclidean domains.**

Theorem 12.6. *Let D and D' be domains in $\mathbb{R}^n$, $n \geq 2$. A homemorphism $f : D \to D'$ is K-quasiconformal (in the sense of Definition 8.3) if and only if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ with $\eta(0) = 0$ such f is an η-quasisymmetric map in each ball $B(x, \frac{1}{2}\operatorname{dist}(x, \partial D))$ for $x \in D$. The statement is quantitative involving K, η, and the dimension n.*

Proof. For the time being we only show necessity. Let $B = B(x, r)$ for some $x \in D$ where $r = \frac{1}{2}\operatorname{dist}(x, \partial D)$. By Theorem 11.12 it suffices to show that f is weakly quasisymmetric in B. Thus pick three distinct points $a, z, w \in B$ such that $d(a, z) \leq d(a, w)$. We need to show that there exists an $H = H(n, K)$ such that

$$d(f(a), f(z)) \leq H d(f(a), f(w)). \tag{12.2}$$

In the following H will denote any positive constant that depends on K and n only.

First we will show that there is a path γ' joining $f(a)$ and $f(w)$ in D' such that

$$\operatorname{diam} \gamma' \leq H d(f(a), f(w)). \tag{12.3}$$

Indeed, if the line segment $[f(a), f(w)]$ from $f(a)$ to $f(w)$ does not meet $\partial D'$ there is nothing to show. Otherwise let σ' be the closed subsegment of $[f(a), f(w)]$ that connects $f(a)$ to $\partial D'$ inside D' (we ignore that f^{-1} may not be defined on $\partial D'$. This can be cured by a simple limiting argument). The preimage of σ of σ' connects a to ∂D.

Let y' be a point on $\gamma' := f([a,w])$ such that $d(y', f(a)) > 2d(f(w), f(a))$. If no such point exists then the desired condition (12.3) holds with $H = 4$. Consider the half-line from $f(a)$ that passes through y' and let L' be the part of that line that lies in D' but outside the ball $B(f(a), d(f(a), y'))$ and connects y' to $\partial D'$. Note that L' may not meet any finite part of $\partial D'$.

By the standard modulus estimate, see (8.5), we find that the curve family Γ' that joins σ' to L' inside D' has modulus at most

$$C\left(\log \frac{d(y', f(a))}{d(f(w), f(a))}\right)^{1-n}. \tag{12.4}$$

On the other hand, in the domain D, both preimages $\sigma := f^{-1}(\sigma')$ and $L = f^{-1}(L')$ connect the ball $B(x,r)$ to the complement of $B(x,2r)$. Thus $\Delta(\sigma \cap B(x, \frac{3}{2}r), L \cap B(x, \frac{3}{2}r) \leq \frac{1}{2}$. Because $B(x,2r)$ lies in D, the modulus of the preimage family $\Gamma = f^{-1}\Gamma'$ is bounded from below by a fixed constant, see Theorem 9.5 and Lemma 10.17. [2]

The quasi-invariance of the modulus thus implies, in view of expression (12.4) That $d(y', f(a)) \leq H d(f(w), f(a))$, and the estimate (12.3) follows.

The proof of the estimate (12.2) is very similar. Essentially one replaces σ' by γ' and uses the point $f(z)$ rather than $f(w)$ for the definition of L'. First note that if $d(f(a), f(z)) \leq 2 \operatorname{diam} \gamma'$ were are done. Otherwise consider the half-line from $f(a)$ through $f(z)$ and let L' be the segment of the line that lies in D' by outside the ball $B(f(a), d(f(a), f(z))$. Let Γ' denote the family of curves which join γ'

[2] The main technical point which was not discussed in class is that we need a lower bound for the family of curves which joint σ to L and *in addition* stay inside D. If we apply Lemma 10.17 with $M_1 = \frac{3}{2}$, $E = \sigma \cap B(x, \frac{3}{2}r)$, $F = L \cap B(x, \frac{3}{2}r)$ and use the version of the lemma with $B(x, M_1 r)$ rather than $B(x_1, M_1 r) \setminus B(x, r/M_1)$ then we get a lower bound for the modulus for curves which join E to F and stay inside $B(x, M_4 r)$. We would need $M_4 = 2$ but this is not guaranteed by the lemma. We instead obtain a slightly weaker result, namely that f is quasisymmetric in a smaller set $B(x, \frac{1}{C} \operatorname{dist}(x, \partial D)$ for some large C. To get the full strength of the result we can argue as follows. There exists a K_1-quasiconformal map $g : B(x, M_4 r) \to B(x, 2r)$ which is the identity on $B(x, \frac{3}{2}r)$, see Proposition12.7 and take $\rho = 2r$, $\lambda = \frac{3}{4}$ and $L = M_4/2$. Let $\tilde{\Gamma}$ be the family of curve which joint E to F and stay inside $B(x, M_4 r)$. We know from Lemma 10.17 that $\operatorname{mod}_n \tilde{\Gamma} \geq c > 0$. Thus $\operatorname{mod}_n g(\tilde{\Gamma}) \geq c/K_1$. Now g is the identity on E and F. Thus $g(\Gamma)$ consists of curve which joint E and stay inside $B(x, 2r)$. Then $\Gamma \supset g(\tilde{\Gamma})$ which implies the desired lower bound for $\operatorname{mod}_n \Gamma$.

and L' inside D'. Again by (8.5) we have

$$\operatorname{mod}_n(\Gamma') \leq C \left(\log \frac{d(f(z), f(a))}{\operatorname{diam} \gamma'} \right)^{1-n} . \tag{12.5}$$

Let $R = d(a, w)$ and note that L joins $B(a, R)$ and $B(x, r)$ to the complement of $B(x, 2r)$. If $R \geq r/2$ we can use that $E := B(x, \frac{3}{2}) \cap \gamma$ and $F =: B(x, \frac{3}{2}) \cap L$ both have diameter not smaller than $r/2$. Thus there is a lower bound on the modulus of the family Γ of curves which join E and F in $B(x, 2r)$. Then $\operatorname{mod}_n \Gamma' \geq \operatorname{mod}_n f(\Gamma) \geq c/K$ and the desired estimate follows.

If $R < r/2$ then $B(a, 2R) \subset B(x, 2r)$. Thus $E = \gamma \subset B(a, \frac{3}{2}R)$ and $F = L \cap B(a, \frac{3}{2}R)$ have diameter not smaller than $R/2$. Hence there is a lower bound on the modulus of the family Γ of curves which join E and F inside $B(a, 2R)$. Moreover $\Gamma' \supset f(\Gamma)$ and we conclude as before. □

Proposition 12.7. *Let $L > 1$, $\lambda \in (\frac{1}{2}, 1)$. Then there exists a K_1 (depending on L and λ) such that for all $\rho > 0$ there exists a K_1-quasiconformal map $g : \mathbb{R}^n \supset B(x, L\rho) \to B(\rho)$ and g is the identity on $B(x, \lambda\rho)$.*

Proof. By translation and dilation we may assume $x = 0$ and $\rho = 1$. Let $h : [0, \infty) \to [0, \infty)$ be smooth with $0 < c \leq h' \leq C$ and define

$$g(y) = h(|y|) \frac{y}{|y|}$$

for $y \neq 0$ and $g(0) = 0$. The g is a C^1 map and the eigenvalues of $(Dg)^T Dg$ are given by a single eigenvalue $\lambda_r = h'(|y|)$ and an $(n-1)$-fold eigenvalue $\lambda_t = h(|y|)/|y|$. Choose h such that $h(t) = t$ for $t \leq \lambda$, $h(L) = 1$ and $\frac{1-\lambda}{2L} \leq h'(t) \leq 1$ for $t \in (\lambda, L)$. Then λ_t/λ_r is bounded from above and below in terms of λ and L. Hence g is K_1-quasiconformal where K_1 depends only on λ and L. □

[22.1. 2020, Lecture 20]

[28.1. 2020, Lecture 21]

References

[1] J. Cheeger, Differentiability of Lipschitz functions on metric spaces. Geom. Funct. Anal. **9** (1999), 428–517.

[2] Herbert Federer, Geometric measure theory, Springer Grundlehren der mathematischen Wissenschaften, 1969.

[3] Mikhael Gromov, Structures métriques pour les variétés riemanniennes, Textes Mathématiques [Mathematical Texts], 1, Paris: CEDIC, 1981.

[4] Mischa Gromov, Metric Structures for Riemannian and Non-Riemannian Spaces (with appendics by M. Katz, P. Pansu, and S. Semmes), 2001.

[5] P. Hajlasz, Geometric approach to Sobolev spaces and badly degenerated el- liptic equations. In Nonlinear Analysis and Applications (Warsaw, 1994). Volume 7 of GAKUTO International Series Mathematical Sciences and Applications. Gakkotosho, Tokyo, 1996, pp. 141-168.

[6] P. Hajlasz, Sobolev spaces on an arbitrary metric space. Potential Anal. **5** (1996), 403–415.

[7] P. Hajlasz and P. Koskela, Sobolev met Poincaré, Mem. Amer. Math. Soc. **145** (2000).

[8] Juha Heinonen, Lectures on Analysis on Metric Spaces, Springer, 2001.

[9] Juha Heinonen, Pekka Koskela, Quasiconformal maps in spaces with controlled geometry, Acta Math. **181** (1998), 1–61.

[10] Juha Heinonen, Pekka Koskela, Nageswari Shanmugalingam and Jeremy T. Tyson Sobolev spaces on metric measure spaces. An approach based on upper gradients. New Mathematical Monographs, 27. Cambridge University Press, Cambridge, 2015.

[11] D. Jerison, The Poincaré inequality for vector fields satisfying Hörmander' condition, Duke Mathematical Journal **53**, 503–523.

[12] J. Luukkainen and E. Saksman (1998). Every complete doubling metric space carries a doubling measure. Proceedings of the American Mathematical Society, 126(2), 531–534.

[13] Charles Löwner, On conformal capacity in space, J. Math. Mech. **8** (1959), 411–414.

[14] A. Korány and H.M. Reimann, Foundations for the theory of quasiconformal mappings on the Heisenberg group, Adv. Math. **111** (1995), 1–87.

[15] L. Saloff-Coste, Parabolic Harnack inequality for divergence form second order differential operators, Potential Anal. **4** (1995), 429–467.

[16] Nageswari Shanmugalingam, Newtonian spaces: an extension of Sobolev spaces to metric measure spaces. Rev. Mat. Iberoamericana **16** (2000), 243–279.

[17] J. Väisälä, Lectures on n-dimensional quasiconformal mappings, Springer Lecture Notes in Mathematics 229, Springer, 1971.

[18] T. Varopoulos, Fonctions harmoniques sur les groupes Lie, C. R. Acad. Sci. Paris Ser. I Math. **309** (1987), 519–521.

[19] William P. Zimmer, Weakly differentiable functions, Springer, 1989.