**Winter Semester 2023/24**
**V5B2 - Selected Topics in Analysis and PDE**

# Introduction to the Boltzmann equation

**Théophile Dolmaire**
dolmaire@iam.uni-bonn.de

Following the pioneering approach of Maxwell [13], and relying on a microscopic description of matter, Ludwig Boltzmann established for the first time the equation satisfied by the distribution function of particles of a sufficiently dilute gas [3], [4]. More precisely, denoting by $f = f(t, x, v)$ the number of particles of a gas lying in a small volume $x + dx$ around $x \in \mathbb{R}^d$, moving with a velocity in $v + dv \subset \mathbb{R}^d$ at time $t$, the Boltzmann equation for the distribution $f$ writes:

$$\partial_t f + v \cdot \nabla_x f = Q(f, f), \tag{1}$$

where the *collision term* $Q(f, f)$ is defined as

$$Q(f,f) = \int_{v_* \in \mathbb{R}^d} \int_{\omega \in \mathbb{S}^{d-1}} b(v - v_*, \omega) \left[ f(v')f(v'_*) - f(v)f(v_*) \right] d\omega dv_*, \tag{2}$$

for $v' = v - [(v - v_*) \cdot \omega] \, \omega$, and $v'_* = v_* + [(v - v_*) \cdot \omega] \, \omega$.

This integro-partial differential equation turned out to be particularly efficient to model the behaviour of dilute gases, such as the high layers of the atmosphere, and its wide range of applications stretches from thermodynamics to aerodynamics of rockets, and can also be applied to study opinion dynamics in populations.

One of the main features of the equation is to encode an irreversible behaviour: the solutions tend to equilibrium by minimizing an associated functional: the entropy. Even though the Boltzmann equation was the object of intensive studies, it remains a very active field of research, where (most of the) central questions remain open.

After deriving the equation (1) and studying its main formal properties, we will focus on the Cauchy problem and the long time behaviour of the solutions, first in the space homogeneous case, and then close to equilibrium. We will then consider the rigorous derivation of the equation from a finite system of particles subject to the Newton's laws (Lanford's theorem [12], [11]), including its generalization to domains with boundary [9]. Finally we will discuss the connection of (1) with other equations (hydrodynamic limits [10], grazing collisions). If time allows it, we will mention the case of inelastically interacting particles [5].
Besides original articles (such as [6], [1], [2]), we will refer to the classical references [7], [8] and [14] on the topic.

**Prerequisites: basic PDE, basic Functional Analysis**

**Time slot and room**

**Thursdays 10:00–12:00, N 0.008 (Neubau)**

## Bibliography


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[10] François Golse, "The Boltzmann Equation and Its Hydrodynamic Limits", *Handbook of Differential Equations: Evolutionary Equations*, **2**, Elsevier, 159-301 (2009).

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[14] Cédric Villani, "A review of mathematical topics in collisional kinetic theory", *Handbook of Mathematical Fluid Dynamics*, **1**, North-Holland, Amsterdam, 71-305 (2002).