# Coagulation-fragmentation equations

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Graduate seminar on analysis (S4B1), Winter term 2025/2026

Coagulation-fragmentation equations are a class of integro-differential equations that can be used to describe different types of coagulation fragmentation processes. These type of processes are relevant in a wide range of applications, for instance in polymerization, animal grouping, atmospheric science and many others.

Coagulation-fragmentation equations are equations of the following form

$$\partial_t f = \mathbb{K}[f] + \mathbb{F}[f]$$

where $\mathbb{K}$ is the coagulation operator

$$\mathbb{K}[f](t,x) := \frac{1}{2}\int_0^x K(x-y,y) f(t,y) f(t,x-y) dy - f(t,x)\int_0^\infty K(x,y) f(t,y) dy$$

while $\mathbb{F}$ is the fragmentation operator given by

$$\mathbb{F}[f](t,x) := \frac{1}{2}\int_x^\infty F(y-x,y) f(t,y) dy - f(t,x)\int_0^x F(x-y,y) dy.$$

The solution $f(t,x)$ is the density of particles of size $x > 0$ at time $t > 0$. The coagulation kernel $K(x,y)$ describes the rate at which two particles coalesce while the fragmentation kernel $F(x,y)$ describes the rate at which one particle of size $x+y$ fragment in two particles, one of size $x$ and the other of size $y$.

During this seminar, we will study the well posedness of a class of coagulation-fragmentation equations, with particular focus on *gelation* and *shattering*. These two phenomena occur when the solution of the coagulation-fragmentation equation does not conserve mass at all time. In the case of gelation the reason why mass conservation does not hold is the formation of a particle of infinite size in finite time, in the case of shattering mass conservation is lost due to the formation of particles of size zero in finite time.

We will also study the existence of self-similar solution for the coagulation equation $\partial_t f = \mathbb{K}[f]$. These are solutions of the form

$$f(t,x) = t^\alpha \Phi\left(\frac{x}{t^\beta}\right),$$

where the function $\Phi$ satisfies a suitable equation and where $\alpha$ and $\beta$ are two parameters that depend on the coagulation kernel. Self-similar solutions are expected to describe the long time behaviour of the solution to the coagulation equations.

We will also study the stability of the steady states of the coagulation-fragmentation equation under the so-called *detailed balance* assumption between the coagulation rate and the fragmentation rate.

**Prerequisites:** Good knowledge of functional analysis.

**Preliminary meeting**: Monday 14th July 2025 at 14:15 room 2.040.

# References

[1] J. B. McLeod. On an infinite set of non-linear differential equations. Quart. J. Math. Oxford Ser. 13(2), pp. 119–128, 1962.

[2] G. Menon, R. L. Pego. Approach to self-similarity in Smoluchowski's coagulation equations. Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences, 57(9), pp. 1197-1232, 2004.

[3] J. Banasiak, W. Lamb, and P. Laurençot. Analytic Methods for Coagulation-Fragmentation Models. Chapman and Hall/CRC, 2019.

[4] J. A. Cañizo. Convergence to equilibrium for the discrete coagulation-fragmentation equations with detailed balance. Journal of Statistical Physics, 129, 1-26, 2007.

[5] M. Escobedo, S. Mischler, and B. Perthame. Gelation in coagulation and fragmentation models. Communications in Mathematical Physics 231, pp. 157-188, 2002.