“Strange terms coming from nowhere”
Periodic and stochastic homogenization in perforated domains

Graduate seminar on Analysis (S4B2), Summer Term 2018

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Abstract

This seminar focusses on periodic and stochastic homogenization of PDEs solved in domains having many small holes. The main interesting feature of these problems is that in the homogenization limit the holes in the domain become negligible, but new terms may appear in the limit equation.

A simple example which exhibits this feature is the following [1]: For a given bounded domain $\Omega \subseteq \mathbb{R}^3$ and $f \in L^2(\Omega)$, we consider for $\varepsilon > 0$ the solution to the Poisson problem

$$\begin{cases} -\Delta u_\varepsilon = f & \text{in } \Omega_\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

where the perforated domain Ω_ε is defined as

$$\Omega_\varepsilon = \Omega \backslash \Big(\bigcup_{z \in \varepsilon\mathbb{Z}^3} B_{\varepsilon^3}(z) \Big). \tag{0.1}$$

Then $u_\varepsilon \rightharpoonup u$ in $H^1_0(\Omega)$ for $\varepsilon \downarrow 0^+$, with u solving

$$\begin{cases} -\Delta u + 4\pi u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.2}$$

On the one hand, the holes in Ω_ε, which decrease in size and increase in number for smaller ε, disappear in the limit and (0.2) for u is solved in the whole domain Ω. On the other hand, and this is the interesting phenomenon, a new term arises in the equation (0.2), the zero-order term $4\pi u$, which encodes some geometric properties of the holes. In this example, the constant 4π is related to the density of the harmonic capacity of the holes $\bigcup_{z \in \varepsilon\mathbb{Z}^3} B_{\varepsilon^3}(z)$.

The aim of the seminar is to study phenomena such as the one above also for other PDEs, for example for Stokes or Navier-Stokes equations in porous media [2], and to further investigate the settings in which the perforated domains Ω_ε have a more complicated structure. In this latter case, we will focus in particular on problems where Ω_ε is not anymore deterministic, but is for instance obtained by removing from Ω a random number of random small compact sets [3, 4].

Prerequisites: Basic knowledge of Functional Analysis and PDEs. Some parts may also require basic notions of Probability Theory.

There will be a preliminary meeting on 31st January 2018 at 4pm in room N0.003.

Main references

1. D. Cioranescu and F. Murat, *A strange term coming from nowhere*, Topics in the Mathematical Modelling of Composite Materials. Progress in Nonlinear Differential Equations and Their Applications. **31** (1997), 45–93.
2. G. Allaire, *Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. I. Abstract framework, a volume distribution of holes*, Arch. Rational Mech. Anal. **113** (1990), no. 3, 209–259. MR 1079189 (91k:35031a)
3. G. C. Papanicolaou and S. R. S. Varadhan, *Diffusion in regions with many small holes*, pp. 190–206, Springer Berlin Heidelberg, Berlin, Heidelberg, 1980.
4. A. Yu. Beliaev and S. M. Kozlov, *Darcy equation for random porous media*, Communications on Pure and Applied Mathematics **49** (1996), no. 1, 1–34.

For a further reading

- L. A. Caffarelli and A. Mellet, *Random homogenization of an obstacle problem*, Annales de l'I.H.P. Analyse non linre **26** (2009), no. 2, 375–395 (eng).

- G. Allaire, *Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. II. Noncritical sizes of the holes for a volume distribution and a surface distribution of holes*, Arch. Rational Mech. Anal. **113** (1990), no. 3, 261–298. MR 1079190 (91k:35031b)
- V. V. Zhikov, S. M. Kozlov, and O. A. Oleĭnik, *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, Berlin, 1994.