Nonlinear Partial Differential Equations II University of Bonn
Summer term 2018 Prof. Dr. J.J.L. Velázquez
Problem Sheet 14 (optional)

Problem 1 (Macroscopic balance law).
Suppose that $f = f(t,x,\xi)$ is a solution to Boltzmann equation. Define mass and momentum density as follows:

$$\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,\xi)\ \mathrm{d}\xi\,,$$

$$(\rho u)(t,x) = \int_{\mathbb{R}^3} \xi f(t,x,\xi)\ \mathrm{d}\xi\,.$$

Furthermore let $c = \xi - u \in \mathbb{R}^3$ and define the internal energy density and the heat and momentum flow as

$$(\rho e)(t,x) = \frac{1}{2}\int_{\mathbb{R}^3} |c|^2 f(t,x,\xi)\ \mathrm{d}\xi\,,$$

$$q_i(t,x) = \frac{1}{2}\int_{\mathbb{R}^3} c_i|c|^2 f(t,x,\xi)\ \mathrm{d}\xi\,,$$

$$p_{ij}(t,x) = \int_{\mathbb{R}^3} c_i c_j f(t,x,\xi)\ \mathrm{d}\xi\,,$$

$i,j = 1,2,3$. Show that the following equations hold:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, & \\ \partial_t(\rho u_j) + \operatorname{div}(\rho u_j u + p_{\cdot j}) = 0, & j = 1,2,3, \\ \partial_t\Big(\frac{1}{2}\rho|u|^2 + \rho e\Big) + \operatorname{div}\Big(\rho u\Big(\frac{1}{2}|u|^2 + e\Big)\Big) + \operatorname{div}(pu + q) = 0. & \end{cases}$$

What do you obtain if f is independent of x?
Hint: choose suitable collision invariants.

Problem 2 (Maxwell distributions).
Let $f = f(t,\xi)$ be a spatially homogeneous solution to Boltzmann equation (that is, f does not depend on the spatial variable x). Recall that in this case the quantities

$$\rho(t) := \int_{\mathbb{R}^3} f(t,\xi)\,\mathrm{d}\xi, \quad (\rho u)(t) := \int_{\mathbb{R}^3} \xi f(t,\xi)\,\mathrm{d}\xi, \quad (\rho e)(t) := \frac{1}{2}\int_{\mathbb{R}^3} |\xi|^2 f(t,\xi)\,\mathrm{d}\xi$$

are constant in time. Show that, if M is the Maxwellian with the same $\rho, \rho u, \rho e$ as f, then

$$\mathcal{H}(f) \geq \mathcal{H}(M)\,, \qquad \text{where } \mathcal{H}(f) := \int_{\mathbb{R}^3} f\ln f\,\mathrm{d}\xi\,.$$

Hint: first show that $\int_{\mathbb{R}^3} \ln M(f-M)\,\mathrm{d}\xi = 0$*, using the fact that* $\ln M$ *is a collision invariant; moreover, use the inequality* $z\ln z - z\ln y + y - z \geq 0$.
