Nonlinear Partial Differential Equations II — University of Bonn
Summer term 2018 — Prof. Dr. J.J.L. Velázquez
Problem Sheet 13 (due Monday 16.07.2018)

Problem 1 (Symmetries in Boltzmann equation, 3+3 points).
Let $f = f(t, x, \xi)$ be a solution to the Boltzmann equation.

a) Let $R \in SO(3)$ be a rotation matrix and define $\tilde{f}(t, x, \xi) := f(t, Rx, R\xi)$. Show that $\tilde{f}$ is also a solution to Boltzmann equation.

b) Let $\lambda > 0$ and define $\tilde{f}(t, x, \xi) := \lambda^{\alpha} f(\lambda^{-\beta} t, \lambda^{-\gamma} x, \lambda^{-\delta} \xi)$. Under which conditions on the parameters $\alpha, \beta, \gamma, \delta$ is $\tilde{f}$ a solution to Boltzmann equation?

Problem 2 (Characterization of collision invariants, 6 points).
Let $\varphi : \mathbb{R}^3 \to \mathbb{R}$ be a C^2 function such that

$$\varphi + \varphi_* = \varphi' + \varphi'_*$$

for all $\xi, \xi_* \in \mathbb{R}^3$ and unit vector n, where

$$\varphi = \varphi(\xi), \quad \varphi_* = \varphi(\xi_*), \quad \varphi' = \varphi(\xi'), \quad \varphi'_* = \varphi(\xi'_*),$$

$$\xi' = \xi - n((\xi - \xi_*) \cdot n), \quad \xi'_* = \xi + n((\xi - \xi_*) \cdot n)\,.$$

Show that there exist $a, c \in \mathbb{R}$ and $b \in \mathbb{R}^3$ such that $\varphi(\xi) = a + b \cdot \xi + c|\xi|^2$.

Hint: consider $\psi(\xi) = \varphi(\xi) - a - b \cdot \xi - c|\xi|^2$, with $a = \varphi(0)$, $b = \nabla\varphi(0)$, $c = \frac{1}{6}\Delta\varphi(0)$. Notice that ψ is a collision invariant.

a) *With a suitable choice of ξ, ξ_*, n show that ψ is radially symmetric, that is, $\psi(\xi) = \theta(|\xi|)$.*

b) *Then, again choosing ξ, ξ_*, n conveniently, show that $\theta' \equiv 0$ and hence $\theta \equiv 0$ (use in particular the fact that $\Delta\psi(0) = 0$).*

Problem 3 (Collision invariants, 4 points).
Show that the condition

$$\varphi + \varphi_* = \varphi' + \varphi'_*$$

defining collision invariants can be written as follows:

$$\varphi(\xi + u + v) + \varphi(\xi) = \varphi(\xi + u) + \varphi(\xi + v)$$

provided u and v are two vectors such that $u \cdot v = 0$.

Total: 16 points