Nonlinear Partial Differential Equations II University of Bonn
Summer term 2018 Prof. Dr. J.J.L. Velázquez
Problem Sheet 12 (due Monday 09.07.2018)

Problem 1 (Barenblatt solutions $0 < m < 1$, 6 points).
Find $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}^+$ to be a positive self-similar solution $u(t,x) = t^{-\alpha}\Phi(\frac{x}{t^\beta})$ to the fast diffusion equation

$$\partial_t u - \partial_x^2(u^m) = 0 \text{ in } (0,\infty) \times \mathbb{R} \text{ and } u(0,\cdot) = \delta_0 \text{ in the distributional sense.}$$

Problem 2 (Quasi-stationary, 6 Points).
Consider $1 < m < \infty$ and the following Ansatz $u(t,x) = \frac{U(x)}{(T-t)^\alpha}$ to be a solution to

$$\partial_t u - \partial_x^2(u^m) = 0 \text{ in } (0,T) \times \mathbb{R}.$$

a) Find an $\alpha \in \mathbb{R}$ and the ODE that U has to satisfy.

b) Transfer the ODE into the form $\Phi''(x) = c\Phi^p(x)$ for $p \in (0,1)$ and $c \in \mathbb{R}$.

c) Use the ODE to find a solution that satisfies $U(x) = 0$ for $x \in [0,1]$ and $U(x) > 0$ for $x \in [0,1]^c$. *Hint: Multiply the ODE $\Phi''(x) = c\Phi^p(x)$ with $\Phi'(x)$ and use the fact that solutions of the respective ODE are not unique.*

Problem 3 (Self-similar solutions with boundary values, 4 points).
Find a self-similar positive solution $u(t,x) = t^\alpha\Phi(\frac{x}{t^\beta})$ to the problem

$$\partial_t u - \partial_x^2 u = 0 \text{ in } (0,\infty) \times (0,\infty),\ u(0,\cdot) = 0 \text{ in } [0,\infty) \text{ and } u(t,0) = t \text{ for } t \in [0,\infty).$$

Total: 16 points