Nonlinear Partial Differential Equations II University of Bonn
Summer term 2018 Prof. Dr. J.J.L. Velázquez
Problem Sheet 10 (due Monday 25.06.2018)

Problem 1 (Shallow water, 8 points).
Consider the shallow water equations

$$\begin{cases} \varphi_t + (v\varphi)_x = 0, \\ v_t + \left(\frac{v^2}{2} + \varphi\right)_x = 0, \end{cases} \tag{1}$$

which is a system of conservation laws with flux function $\mathbf{F}(z_1, z_2) = \left(z_1 z_2, \frac{z_2^2}{2} + z_1\right)$. Recall from exercise 2 in Problem Sheet 9 that $\mathbf{r}(z_1, z_2) = (\sqrt{z_1}, 1)$ is a right eigenvector of $D\mathbf{F}$.

a) Solve the ODE $\dot{\mathbf{v}}(s) = \mathbf{r}(\mathbf{v}(s))$ with initial condition $\mathbf{v}(0) = \mathbf{0}$.

b) Compute, for $s > 0$,

$$\Lambda(s) := \int_0^s \lambda(\mathbf{v}(t))\,\mathrm{d}t\,,$$

where $\lambda(z)$ is the eigenvalue of $D\mathbf{F}(z)$ corresponding to $\mathbf{r}(z)$.

c) Let

$$g(x) := \begin{cases} 0 & \text{if } x < 0, \\ 2 & \text{if } x > 0. \end{cases}$$

Find the entropy solution to the Riemann's problem

$$\begin{cases} w_t + \Lambda(w)_x = 0 & (x,t) \in \mathbb{R} \times (0, \infty), \\ w(x, 0) = g(x) & x \in \mathbb{R}, \end{cases}$$

and use the solution to find a continuous integral solution of (1) with initial conditions

$$v(x,0) = \begin{cases} 0 & \text{if } x < 0, \\ 2 & \text{if } x > 0, \end{cases} \qquad \varphi(x,0) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Sketch the graphs of φ and v for a few values of $t > 0$.

Problem 2 (Entropy for shallow water, 4 points).
Suppose that $\Phi, \Psi : \mathbb{R}^2 \to \mathbb{R}$ are an entropy/entropy-flux pair for the shallow water equations (1); meaning that Φ is convex and

$$D\Phi(z_1, z_2) D\mathbf{F}(z_1, z_2) = D\Psi(z_1, z_2).$$

Prove that

$$\frac{\partial^2 \Phi}{\partial v^2} = \varphi \frac{\partial^2 \Phi}{\partial \varphi^2}.$$

Problem 3 (Entropy for barotropic Euler equations, 4 points).
Consider the barotropic Euler equations

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ (\rho v)_t + (\rho v^2 + p)_x = 0, \end{cases}$$

where $p = p(\rho)$ is a smooth function, $p' > 0$. Show that $\Phi = \rho v^2/2 + P(\rho)$ is an entropy for this system, provided $P''(\rho) = p'(\rho)/\rho$, $\rho > 0$. Confirm that Φ is convex in the proper variables. What is the corresponding entropy flux Ψ?

Total: 16 points