Nonlinear Partial Differential Equations II — University of Bonn
Summer term 2018 — Prof. Dr. J.J.L. Velázquez
Problem Sheet 8 (due Monday 11.06.2018)

Problem 1 (N-wave, 4 points).
Let u be the entropy solution to

$$\begin{cases} u_t + \partial_x\left(\frac{|u|^p}{p}\right) = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ u(x,0) = u_0(x), \end{cases}$$

where $p \in (2,\infty)$ and

$$u_0(x) = \begin{cases} -1 & \text{for } -1 \le x < 0, \\ 1 & \text{for } 0 \le x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find the asymptotic shape $u(\cdot,t)$ as $t \to \infty$.

Problem 2 (Vanishing viscosity, 8 points).
Let F be smooth and uniformly convex, $g \in L^\infty(\mathbb{R})$ be compactly supported, $T > 0$. Let $\varepsilon > 0$, set $M = \varepsilon^{-1}$, and suppose that $u^\varepsilon \in C^\infty([-M,M]\times[0,T])$ is a solution to the viscous conservation law

$$\begin{cases} u^\varepsilon_t + F(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx} & \text{for } (x,t) \in (-M,M)\times(0,T], \\ u^\varepsilon_x = 0 & \text{for } (x,t) \in \{\pm M\}\times(0,T], \\ u^\varepsilon = g^\varepsilon & \text{for } (x,t) \in [-M,M]\times\{0\}, \end{cases} \tag{1}$$

where $g^\varepsilon = \eta^\varepsilon * g$ and η^ε is the standard mollifier.

a) Show that

$$\|u^\varepsilon\|_{L^\infty([-M,M]\times[0,T])} \le \|g\|_{L^\infty(\mathbb{R})}\,. \tag{2}$$

Hint: Use a maximum principle argument. For instance, consider the function $w(x,t) := u^\varepsilon(x,t) - \delta t$, for $\delta > 0$, and show that the maximum of w is attained for $t = 0$.

b) Since F is uniformly convex, there exists $\theta > 0$ such that $F'' \ge \theta$. Show that

$$u^\varepsilon_x(x,t) \le \frac{1}{\theta t} \qquad \text{for all } (x,t) \in [-M,M]\times(0,T]. \tag{3}$$

Hint: Set $U(x,t) := u^\varepsilon_x(x,t)$ and find a PDE solved by U by differentiating (1). Show that $v(x,t) := \frac{1}{\theta t}$ is a supersolution of this PDE and conclude by a comparison principle argument (similar to the one used in Problem 2 in the Problem Sheet 5).

c) Assume that

$$\lim_{\varepsilon\to 0^+} u^\varepsilon(x,t) = u(x,t) \qquad \text{for almost every } (x,t) \in \mathbb{R}\times[0,T].$$

Show that the limit function u is the entropy solution to the conservation law

$$\begin{cases} u_t + F(u)_x = 0 & \text{for } (x,t) \in \mathbb{R} \times (0,T), \\ u = g & \text{for } (x,t) \in \mathbb{R} \times \{0\}. \end{cases}$$

Hint: Check that u is an integral solution by multiplying (1) *by a test function and passing to the limit as* $\varepsilon \to 0$. *Moreover, use* (3) *to show that* u *satisfies the entropy condition.*

Problem 3 (A linear system, 4 points).
Consider the system of conservation laws

$$\partial_t u + \partial_x (Au) = 0, \qquad (x,t) \in \mathbb{R} \times (0,\infty), \tag{4}$$

where $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ is a diagonalizable matrix with real eigenvalues.

a) Find the Rankine-Hugonoit condition along a shock curve. What does it mean?

b) By a linear transformation, show that (4) decouples into a system of n independent equations.

Total: 16 points