Nonlinear Partial Differential Equations II University of Bonn
Summer term 2018 Prof. Dr. J.J.L. Velázquez
Problem Sheet 7 (due Monday 04.06.2018)

Problem 1 (Riemann's problem, 4 points).
Find the entropy solution to the Riemann's problem

$$\begin{cases} u_t + (e^u)_x = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u = g & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

with

$$g(x) = \begin{cases} 2 & \text{for } x < 0, \\ 1 & \text{for } x > 0. \end{cases}$$

Problem 2 (Weak vs. strong solution, 4 points).
Show that every smooth solution u to the conservation law

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0$$

also solves

$$\partial_t (u^2) + \frac{2}{3} \partial_x (u^3) = 0,$$

which is a conservation law for u^2 with flux $F(u) = \frac{2}{3} u^{\frac{3}{2}}$.
By considering a Riemann problem with $u_l > u_r$ or otherwise, show that the two equations have different weak solutions.

Problem 3 (Uniqueness of entropy solutions, 4 points).
Consider the Riemann's problem

$$\begin{cases} u_t + \left(\frac{u^2}{2} \right)_x = 0 & (x, t) \in \mathbb{R} \times (0, \infty), \\ u(x, 0) = 0 & \text{for } x < 0, \\ u(x, 0) = 1 & \text{for } x > 0. \end{cases}$$

Let u and v be defined by

$$u(x, t) = \begin{cases} 0 & \text{if } x < \frac{1}{2} t, \\ 1 & \text{if } x > \frac{1}{2} t, \end{cases} \qquad v(x, t) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{x}{t} & \text{if } 0 < x < t, \\ 1 & \text{if } x > t. \end{cases}$$

Sketch u and v and the characteristics. Are both u and v integral solutions to the initial value problem? Are they entropy solutions?

Problem 4 (Long time asymptotics for periodic initial data, 4 points).
Let u be the entropy solution to the initial value problem

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}, \end{cases}$$

where g is the periodic extension of

$$g(x) = \begin{cases} 0 & \text{if } -1 < x < 0, \\ 1 & \text{for } 0 < x < 1. \end{cases}$$

Show that

$$\|u(\cdot, t) - \bar{g}\|_\infty \to 0 \qquad \text{as } t \to \infty,$$

where $\bar{g} = \frac{1}{2} \int_{-1}^{1} g(x)\, \mathrm{d}x = \frac{1}{2}$, and determine the rate of convergence in t.

Total: 16 points