Nonlinear Partial Differential Equations II University of Bonn
Summer term 2018 Prof. Dr. J.J.L. Velázquez
Problem Sheet 6 (due Monday 28.05.2018)

Problem 1 (Conservation, 4 points).
Let $F \in C^\infty(\mathbb{R})$, $F(0) = 0$, and $g \in C^0_c(\mathbb{R})$. Assume that u is a continuous integral solution to the conservation law

$$\begin{cases} u_t + F(u)_x = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u = g & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

and $u(t, \cdot)$ has compact support in $\mathbb{R}$ for all $t \in [0, \infty)$. Prove that

$$\int_{-\infty}^{+\infty} u(x, t)\, \mathrm{d}x = \int_{-\infty}^{+\infty} g(x)\, \mathrm{d}x \qquad \text{for all } t > 0.$$

Problem 2 (Shock curves, 4 points).
Compute explicitly the unique entropy solution to

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u = g & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

with

$$g(x) = \begin{cases} 1 & \text{for } x < 0, \\ -1 & \text{for } 0 < x < 1, \\ 0 & \text{for } x > 1. \end{cases}$$

Sketch the characteristics diagram, including all shocks curves.

Problem 3 (Unbounded entropy solution to Burgers equation, 4 points).
Show that

$$u(x, t) := \begin{cases} -\frac{2}{3}\left(t + \sqrt{3x + t^2}\right) & \text{if } 4x + t^2 > 0, \\ 0 & \text{if } 4x + t^2 < 0 \end{cases}$$

is an (unbounded) entropy solution to Burgers equation $u_t + (\frac{u^2}{2})_x = 0$. Verify the Rankine-Hugoniot condition and sketch the projected characteristics in the (x, t)-plane.

Problem 4 (Travelling waves as solutions of regularized Burgers equation, 4 points).
Let $\sigma_0 \in \mathbb{R}$ and let

$$U_0(z) := \frac{1}{1 + e^{\frac{z}{2}}}\,.$$

a) Under what conditions on a curve $\sigma(t)$, $\sigma(0) = \sigma_0$, is

$$u_\varepsilon(x,t) := U_0\Big(\frac{x-\sigma(t)}{\varepsilon}\Big), \qquad 0 < \varepsilon < 1,$$

a solution to the regularized Burgers' equation

$$\partial_t u + u\partial_x u = \varepsilon\partial^2_{xx} u?$$

b) Find the limit

$$u(x,t) = \lim_{\varepsilon\to 0} u_\varepsilon(x,t)$$

and the asymptotic problem that is solved by u. In what sense does u solve the problem?

Total: 16 points