**Nonlinear Partial Differential Equations II** University of Bonn
Summer term 2018 Prof. Dr. J.J.L. Velázquez
Problem Sheet 5 (due Monday 14.05.2018)

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**Problem 1 (Hopf-Lax formula, 4 points).**
Let $E$ be a closed subset of $\mathbb{R}^n$. Show that *if* the Hopf-Lax formula could be applied to the initial-value problem

$$\begin{cases} u_t + |Du|^2 = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = u_0 & \text{on } \mathbb{R}^n \times \{t=0\}, \end{cases} \tag{1}$$

where

$$u_0(x) = \begin{cases} 0 & \text{if } x \in E, \\ \infty & \text{if } x \notin E, \end{cases}$$

then it would give the solution

$$u(x,t) = \frac{1}{4t}\mathrm{dist}\,(x,E)^2. \tag{2}$$

Is (2) a reasonable solution to (1)? Notice that, if $\partial E$ is sufficiently smooth, there is a neighborhood $U$ of $E$ such that $\mathrm{dist}\,(x,E) := \inf_{y\in E}|x-y|$ is differentiable almost everywhere in $U\backslash E$ with $|D\mathrm{dist}\,(x,E)| = 1$.

**Problem 2 (Uniqueness for Hamilton-Jacobi equation, 4 points).**
Let $U \subset \mathbb{R}^n$ be a bounded open set, and let $U_T := U \times (0,T]$, $\Gamma_T := \overline{U}_T \backslash U_T$. Let $H : \overline{U}_T \times \mathbb{R}^n \to \mathbb{R}$.

a) Let $u, v \in C^1(U_T) \cap C(\overline{U}_T)$ satisfy

$$\begin{cases} u_t + H(x,Du) \le 0 & \text{in } U_T, \\ u \le g & \text{on } \Gamma_T, \end{cases}$$

$$\begin{cases} v_t + H(x,Dv) \ge 0 & \text{in } U_T, \\ v \ge g & \text{on } \Gamma_T. \end{cases}$$

Show that $u \le v$ in $\overline{U}_T$.

b) Show that there is at most one solution $u \in C^1(U_T) \cap C(\overline{U}_T)$ to

$$\begin{cases} u_t + H(x,Du) = 0 & \text{in } U_T, \\ u = g & \text{on } \Gamma_T. \end{cases}$$

*Hint: show that, for $\varepsilon > 0$, the function $w := u - v - \varepsilon t$ attains its maximum on $\Gamma_T$.*

**Problem 3 (Conservation laws: shock, 4 points).**
Find an integral solution to

$$\begin{cases} u_t + F(u)_x = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u = g & \text{on } \mathbb{R} \times \{t=0\}, \end{cases}$$

with

$$F(u) = u^2 + u, \qquad g(x) = \begin{cases} 1 & \text{for } x < 0, \\ -3 & \text{for } x > 0. \end{cases}$$

**Problem 4 (Conservation laws: rarefaction wave, 4 points).**
Find an integral solution to the initial value problem

$$\begin{cases} u_t + u^2 u_x = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u = g & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

with

$$g(x) = \begin{cases} 0 & \text{for } x < 0, \\ -2 & \text{for } x > 0. \end{cases}$$

*Hint: the solution should involve a rarefaction wave of the form $v(x/t)$.*

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Total: 16 points