Nonlinear Partial Differential Equations II University of Bonn
Summer term 2018 Prof. Dr. J.J.L. Velázquez
Problem Sheet 4 (due Monday 07.05.2018)

Problem 1 (Legendre transform, 2+2+2 points).
For a function $H : \mathbb{R}^n \to \mathbb{R}$, denote by $L = H^*$ its Legendre transform.

a) Let $H(p) := \frac{1}{r}|p|^r$, for $1 < r < \infty$. Show that $L(q) = \frac{1}{r'}|q|^{r'}$, where $\frac{1}{r} + \frac{1}{r'} = 1$.

b) Let $H(p) := \frac{1}{2}p \cdot Ap + b \cdot p$, where $A \in \mathbb{R}^{n\times n}$ is a symmetric, positive definite matrix, and $b \in \mathbb{R}^n$. Compute L.

c) Let $H(p) = \sqrt{1 + p^2}$, $p \in \mathbb{R}$. Compute L on its domain of definition.

Problem 2 (Subdifferential and Hopf-Lax formula, 2+2 points).
Let $H : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $L = H^*$ is the Legendre transform of H.

a) We say that $q \in \mathbb{R}^n$ belongs to the subdifferential of H at p, written $q \in \partial H(p)$, if

$$H(r) \geq H(p) + q \cdot (r - p) \qquad \text{for all } r \in \mathbb{R}^n.$$

Prove that

$$q \in \partial H(p) \quad \Longleftrightarrow \quad p \in \partial L(q) \quad \Longleftrightarrow \quad p \cdot q = H(p) + L(q).$$

b) Assume that $g \in C^1(\mathbb{R}^n)$ is globally Lipschitz continuous and that $H \in C^1(\mathbb{R}^n)$ satisfies $\lim_{|p|\to\infty} \frac{H(p)}{|p|} = \infty$. Recall, the Hopf-Lax formula:

$$u(x,t) = \min_{y\in\mathbb{R}^n} \left\{ tL\Big(\frac{x-y}{t}\Big) + g(y) \right\}.$$

Let $R = \sup_{y\in\mathbb{R}^n} |DH(\nabla g(y))|$, where $H = L^*$. Prove that

$$u(x,t) = \min_{y\in B(x,Rt)} \left\{ tL\Big(\frac{x-y}{t}\Big) + g(y) \right\},$$

where $B(x,Rt) = \{y \in \mathbb{R}^n : |y - x| < Rt\}$. (This proves *finite propagation speed* for the Hamilton-Jacobi equation).
Hint: you can use the fact that, for a C^1 function H, the subdifferential $\partial H(p)$ is actually its derivative. In other words $\partial H(p)$ contains precisely one element, $DH(p)$.

Problem 3 (Hopf-Lax formula, 2 points).
Use Hopf-Lax formula to solve explicitly the PDE

$$\begin{cases} u_t + \frac{3}{4}u_x^{\frac{4}{3}} = 0 & \text{for } (x,t) \in \mathbb{R}\times(0,\infty), \\ u(x,0) = x. \end{cases}$$

Problem 4 (L^∞-contraction property of Hamilton-Jacobi equations, 4 points). Let $H \in C^2(\mathbb{R}^n)$ be uniformly convex with $\lim_{|p|\to\infty} \frac{H(p)}{|p|} = \infty$, and let $g_1, g_2 : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz continuous. Assume u^1, u^2 are weak solutions of the initial-value problem

$$\begin{cases} u^i_t + H(\nabla u^i) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u^i = g^i & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

$i = 1, 2$. Prove the inequality

$$\|u^1(\cdot,t) - u^2(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le \|g^1 - g^2\|_{L^\infty(\mathbb{R}^n)} \qquad \text{for all } t > 0.$$

Total: 16 points