Nonlinear Partial Differential Equations II University of Bonn
Summer term 2018 Prof. Dr. J.J.L. Velázquez
Problem Sheet 3 (due Monday 30.04.2018)

Problem 1 (Method of characteristics, 2 points).
Use the method of characteristics to solve the nonlinear initial value problem

$$\begin{cases} u_t + (u_x)^4 = 0 \quad (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = \frac{3}{4}x^{\frac{4}{3}}. \end{cases}$$

Problem 2 (A nonlinear PDE, 4 points).
Consider the initial value problem

$$\begin{cases} -(u_{x_1})^2 + (u_{x_2})^2 + {x_2}^2 = 0 \quad \text{in } \mathbb{R}^2, \\ u(x_1,0) = g(x_1), \end{cases}$$

where g is a C^1-function satisfying $g'(x_1) > 0$ for every x_1.

a) Find explicitly the projected characteristics $(x_1(s), x_2(s))$ starting from the point $(y,0)$, $y \in \mathbb{R}$, assuming $u_{x_2}(x,0) \geq 0$.

b) In the case $g(x_1) = x_1$, sketch the projected characteristics and write down an explicit solution in a neighborhood of the x_1-axis (your answer may depend on the function $I(s) = \int_0^s \sin^2(s)\,\mathrm{d}s$).

Problem 3 (Blow up in Burgers' equation, 3+3 points).
Let $u_0 \in C^\infty(\mathbb{R})$ be such that u_0' is bounded and has a unique minimizer s_0 with $u_0'(s_0) < 0$, $u_0''(s_0) = 0$, $u_0'''(s_0) > 0$.

a) Show that there is $0 < t^* < \infty$ such that the solution to the initial value problem

$$\begin{cases} \partial_t u + u\partial_x u = 0 \quad \text{for } t \in (0,t^*), x \in \mathbb{R}, \\ u(0,x) = u_0(x) \end{cases} \tag{1}$$

is given by

$$u(t,x) = u_0(\xi), \qquad x = \xi + u_0(\xi)t\,.$$

Determine the largest t^*, which depends on some $\xi^* \in \mathbb{R}$, such that u as a function of (t,x) exists and solves (1). What happens with $\partial_t u, \partial_x u$ and the characteristic curves of (1) as $t \to t^*$?

b) Perform a local analysis (Taylor expansion, term comparison) to argue that

$$u(t^*,x) - u_0(\xi^*) \sim -(x - x^*)^{\frac{1}{3}}$$

for x close to x^*, where $x^* = \xi^* + u_0(\xi^*)t^*$. Here $a \sim b$ means $a = Cb$ for some constant $C > 0$, up to higher order terms.

Problem 4 (Eikonal equation on a strip, 4 points).
Consider the Eikonal equation

$$|Du| = ((\partial_{x_1} u)^2 + (\partial_{x_2} u)^2)^{\frac{1}{2}} = 1$$

in the strip $U = \{0 < x_1 < 1\} \subset \mathbb{R}^2$, with boundary data $u(0, x_2) = u(1, x_2) = 0$ for $x_2 \in \mathbb{R}$.

a) Write the characteristic ODE and determine admissible initial data. (Note that at each point of the boundary you have two possible sets of initial data).

b) Choose the initial data $p_1(0) = 1$ on the left boundary $\{x_1 = 0\}$, and $p_1(0) = -1$ on the right boundary $\{x_1 = 1\}$. Explicitly solve the characteristic ODE.

c) Let $(x_1(s, x_0), x_2(s, x_0))$ be the projected characteristics starting from a boundary point $x_0 \in \partial U$, found in b). Notice that, given $(x_1, x_2) \in U$, you cannot uniquely determine $x_0 \in \partial U$ and s such that $(x_1, x_2) = (x_1(s, x_0), x_2(s, x_0))$. What solution do you construct if you assign to (x_1, x_2) the characteristic starting from the left boundary if $x_1 < \alpha$, and the one starting from the right boundary if $x_1 > \alpha$, where $\alpha \in (0, 1)$ is fixed? Is one solution "better" than the others?

Total: 16 points