Nonlinear Partial Differential Equations II University of Bonn
Summer term 2018 Prof. Dr. J.J.L. Velázquez
Problem Sheet 1 (due Monday 16.04.2018)

Problem 1 (Uniqueness by energy methods, 4 points).
Show that there is at most one smooth solution $u(x,t)$ to the equation

$$u_{tt} + cu_t - u_{xx} = f \qquad (x,t) \in (0,1) \times (0,\infty),$$

$c \in \mathbb{R}$, with initial conditions $u(x,0) = g(x)$, $u_t(x,0) = h(x)$, and boundary conditions $u(0,t) = u(1,t) = 0$.
Hint: study the function $E(t) := \frac{1}{2}\int_0^1 (u_t^2(x,t) + u_x^2(x,t))\,\mathrm{d}x$ *(you may distinguish the cases* $c \geq 0$ *and* $c < 0$*).*

Problem 2 (Picone's example, 4 points).
Let $u \in C^1(\overline{\Omega})$ be a solution to

$$a(x,y)u_x + b(x,y)u_y = -u,$$

where $\Omega = B_1(0)$ is the unit ball in $\mathbb{R}^2$. Assume that $a, b \in C^\infty(\mathbb{R}^2)$ and $a(x,y)x + b(x,y)y > 0$ on $\partial\Omega$. Prove that u vanishes identically in Ω.
Hint: show that $\max_{\overline{\Omega}} u \leq 0$, $\min_{\overline{\Omega}} u \geq 0$.

Problem 3 (Viscous Burgers' equation, 4 points).
Consider the viscous Burgers' equation

$$u_t + uu_x = \nu u_{xx},$$

with $\nu \in \mathbb{R}$.

a) Identify the exponents n, m such that self-similar solutions of the form $u(x,t) = t^m f(xt^n)$ can be obtained. Write down the resulting ODE for the function f.

b) Let $s(x,t)$ be a solution to the heat equation $s_t = \nu s_{xx}$. Show that, if we make the transformation $u(x,t) = -2\nu s_x/s$, we obtain a solution to the viscous Burgers' equation.

Problem 4 (Korteweg-de Vries equation, 4 points).
Consider the Korteweg-de Vries equation

$$u_t + 6uu_x + u_{xxx} = 0.$$

We look for a traveling wave solution $u(x,t) = f(x - ct)$. Write the resulting ODE for f. Assuming that both f and its derivatives vanish at infinity, integrate this equation once to get a second order ODE for f.

Total: 16 points