**Nonlinear Partial Differential Equations I** University of Bonn
Winter term 2017/2018 Prof. Dr. J. J. L. Velázquez
Problem Sheet 13 (optional) Dr. M. Bonacini

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**Problem 1 (Galerkin method and Céa's Lemma).**
Let $H$ be a real Hilbert space, $F : H \to \mathbb{R}$ a linear and continuous functional, and let $B : H \times H \to \mathbb{R}$ be a symmetric, bounded and coercive bilinear form:

$$B(u,v) \leq C\|u\|\|v\|, \qquad B(u,u) \geq \alpha\|u\|^2 \qquad \text{for every } u,v \in H.$$

Let $(H_n)$ be a sequence of finite dimensional subspaces such that $H_n \subset H_{n+1}$ for every $n$ and $\overline{\cup_{n=1}^{\infty} H_n} = H$.

a) Show that for every $n$ the equation

$$B(u,v) = F(v) \qquad \text{for every } v \in H_n$$

has a unique solution $u_n \in H_n$. Moreover, show that the sequence $(u_n)_n$ converges weakly in $H$ to the unique solution $u$ to

$$B(u,v) = F(v) \qquad \text{for every } v \in H.$$

b) Prove the following estimate for the error between the real solution $u$ and the approximate solution $u_n$:

$$\|u_n - u\| \leq \frac{C}{\alpha} \inf\{\|u - v_n\| \,:\, v_n \in H_n\}$$

($u_n$ is, up to a constant, as close to $u$ as the best approximation in $H_n$).
*Hint: show that $u_n - u$ is orthogonal to $H_n$ with respect to $B$.*

**Problem 2 (Exponential decay).**
Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary, $g \in H_0^1(\Omega)$. Assume that $u$ is a smooth solution to

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times [0,\infty), \\ u = g & \text{on } \Omega \times \{0\}. \end{cases}$$

Prove the exponential decay estimate

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leq e^{-\lambda_1 t}\|g\|_{L^2(\Omega)}\,,$$

where $\lambda_1 > 0$ is the principal eigenvalue of $-\Delta$ in $\Omega$ (with zero boundary conditions).

**Problem 3 (A nonlinear parabolic equation).**
Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary, $g \in H_0^1(\Omega)$ and $f : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous. Consider the initial/boundary value problem

$$\begin{cases} u_t - \Delta u = f(u) & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u = g & \text{on } \Omega \times \{0\}. \end{cases}$$

Show that a weak solution exists for every $T > 0$.
*Hint: used the linear theory discussed in class, combined with Banach's fixed point theorem.*

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