Nonlinear Partial Differential Equations I University of Bonn
Winter term 2017/2018 Prof. Dr. J. J. L. Velázquez
Problem Sheet 12 (due Monday 15.01.2018) Dr. M. Bonacini

Problem 1 (Uniqueness for various boundary conditions, 2+2+2 points).
Let $\Omega \subset \mathbb{R}^n$ be open, bounded, and connected, and suppose that Ω satisfies an interior ball condition at every point on $\partial\Omega$. If $x_0 \in \partial\Omega$, denote by $\nu(x_0)$ the exterior normal to an interior ball tangent to $\partial\Omega$ at x_0.
Consider $Lu := -\sum_{i,j=1}^n a_{ij}u_{x_ix_j} + \sum_{i=1}^n b_i u_{x_i} + cu$, where $a_{ij}, b_i, c \in C^0(\overline{\Omega})$, $c \geq 0$, and a_{ij} are uniformly elliptic. Assume that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution to $Lu = 0$ in Ω. Prove:

a) If the normal derivative $\frac{\partial u}{\partial \nu}$ is defined everywhere on $\partial\Omega$ and $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, then u is constant in Ω. If furthermore $c > 0$ at some point in Ω, then $u \equiv 0$.

b) Assume that $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega$, with $\partial_D\Omega \neq \emptyset$, and that $u \in C^1(\Omega \cup \partial_N\Omega)$ satisfies the mixed boundary condition

$$u = 0 \text{ on } \partial_D\Omega, \qquad \sum_{i=1}^n \beta_i(x)u_{x_i} = 0 \text{ on } \partial_N\Omega,$$

where $\beta(x) = (\beta_1(x), \dots, \beta_n(x))$ has a non-zero normal component (to the interior ball) at each point $x \in \partial_N\Omega$. Then $u \equiv 0$.

c) Assume that $u \in C^1(\overline{\Omega})$ satisfies the regular oblique derivative boundary condition

$$\alpha(x)u + \sum_{i=1}^n \beta_i(x)u_{x_i} = 0 \qquad \text{on } \partial\Omega,$$

where $\alpha(\beta \cdot \nu) > 0$ on $\partial\Omega$. Then $u \equiv 0$.

Problem 2 (Maximum Principle in a narrow domain, 4 points).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $c \in L^\infty(\Omega)$. Prove that there exists $d_0 > 0$, depending only on the L^∞-norm of c, such that if

$$\Omega \subset \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \,:\, 0 < x_1 < d_0\}$$

then the maximum principle holds in Ω: if $u \in C^2(\overline{\Omega})$ satisfies

$$\begin{cases} -\Delta u + cu \geq 0 & \text{in } \Omega, \\ u \geq 0 & \text{on } \partial\Omega, \end{cases}$$

then either $u \equiv 0$ or $u > 0$ in Ω.

Hints: suppose first by contradiction that $\inf_\Omega u < 0$. *Then for* d_0 *small enough consider the function* $w(x) = \sin(\frac{\pi x_1}{d_0})$ *and let*

$$\lambda_0 = \inf\{\lambda > 0 : \lambda w + u > 0 \text{ in } \Omega\}.$$

Obtain a contradiction using the refinement of Hopf's Lemma in [Evans, Sect. 9.5, Lemma 1]. Once you have proved that $u \geq 0$ in Ω, prove the dichotomy $u \equiv 0$ or $u > 0$ by applying again the refined version of Hopf's Lemma.

Problem 3 (Sliding method, 6 points).
Consider a rectangle $R = (-1,1) \times (0,1) \subset \mathbb{R}^2$ and let $u \in C^2(\overline{R})$ solve the boundary value problem

$$\begin{cases} -\Delta u = f(u) & \text{in } R, \\ u(x) = g(x) & \text{on } \partial R, \end{cases}$$

where the Dirichlet data satisfy $g(-1,y) = 0$, $g(1,y) = 1$ on the vertical boundaries, and on the horizontal boundaries the functions $g_0(x) := g(x,0)$, $g_1(x) := g(x,1)$ are strictly monotone increasing. Suppose further that f is Lipschitz continuous and

$$f(s) \geq 0 \quad \text{for } s \leq 0\,, \qquad\qquad f(s) \leq 0 \quad \text{for } s \geq 1\,.$$

Prove that

$$u(x+\tau, y) > u(x,y) \qquad \text{for all } (x,y), (x+\tau, y) \in R \text{ and } \tau > 0,$$

that is, $u(x,y)$ is monotone in x.

Hints: we use a method similar to the moving plane technique. For $\tau \in (0,2)$ let $R_\tau = R - \tau e_1$, $D_\tau = R \cap R_\tau$ and

$$w_\tau(x,y) = u(x+\tau, y) - u(x,y) \quad \textit{for } (x,y) \in D_\tau.$$

a) First observe that $0 < u < 1$ in R.

b) For τ close to the largest value $\tau = 2$, use the Maximum Principle in narrow domains (Problem 2) to prove that $w_\tau > 0$ in D_τ.

c) As in the moving plane method, decrease τ (sliding the domain R_τ to the right), and show (by contradiction) that you can go all the way to $\tau = 0$ keeping the condition $w_\tau > 0$ in D_τ enforced.

Total: 16 points