Nonlinear Partial Differential Equations I — University of Bonn
Winter term 2017/2018 — Prof. Dr. J. J. L. Velázquez
Problem Sheet 10 (due Monday 18.12.2017) — Dr. M. Bonacini

Problem 1 (Nonlinearities and weak convergence, 2+2 points + 2 extra credit*).
The aim of the exercise is to show that the weak convergence of a sequence $f_n \rightharpoonup f$ in L^2 does **not** imply $a(f_n) \rightharpoonup a(f)$ for any **nonlinear**, real-valued function a.

a) *(Weak convergence of highly-oscillating functions)* Let $u : \mathbb{R} \to \mathbb{R}$ be a 1-periodic function in $L^\infty(\mathbb{R})$, and define $u_n(x) := u(nx)$ for $n \in \mathbb{N}$. Show that, as $n \to \infty$,

$$u_n \rightharpoonup m = \int_{(0,1)} u(x)\,\mathrm{d}x \qquad \text{weakly in } L^2(A), \text{ for every open, bounded set } A \subset \mathbb{R}.$$

Hint: by considering the functions $U(x) = \int_0^x (u(t) - m)\,\mathrm{d}t$, $U_n(x) = U(nx)$, *and integrating by parts, show that* $\lim_{n\to\infty} \int_{\mathbb{R}} (u_n(x) - m)\varphi(x)\,\mathrm{d}x = 0$ *for every* $\varphi \in C^1_{\mathrm{c}}(\mathbb{R})$.

b) Let $a : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $a(f_n) \rightharpoonup a(f)$ weakly in $L^2(0,1)$ whenever $f_n \rightharpoonup f$ weakly in $L^2(0,1)$. Prove that a is affine:

$$a(z) = \alpha z + \beta\,,$$

for some constants α, β.
Hint: use the result in part a) to prove that for every $z_1, z_2 \in \mathbb{R}$ *and* $\lambda \in (0,1)$ *we have* $a(\lambda z_1 + (1-\lambda) z_2) = \lambda a(z_1) + (1-\lambda) a(z_2)$.

c*) (Bonus) Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that for every $p \in \mathbb{R}$ there is a sequence $u_n \in L^\infty(0,1)$ such that $u_n \rightharpoonup 0$ and $f(u_n) \rightharpoonup p$ weakly in $L^2(0,1)$.

Problem 2 (Method of subsolutions and supersolutions, 4 points).
Let $\Omega \subset \mathbb{R}^n$ be open, bounded and connected with smooth boundary, and let $h : \mathbb{R} \to \mathbb{R}$ satisfy the following assumptions:

a) h is Lipschitz continuous and bounded, $h(0) = 0$;

b) h is differentiable at the origin with $h'(0) > \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ in Ω (with Dirichlet boundary conditions).

Use the sub-supersolution method to prove the existence of a weak solution $u \in H^1_0(\Omega)$ to

$$\begin{cases} -\Delta u = h(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega. \end{cases}$$

Hint: use the fact that the first eigenfunction of the Laplacian, that is the function $u_1 \in H^1_0(\Omega)$ *solving* $-\Delta u_1 = \lambda_1 u_1$ *in* Ω, *is bounded and strictly positive in* Ω.

Please turn over.

Problem 3 (Schauder's fixed point theorem - second version, 4 points).
The goal of this exercise is to prove the following version of Schauder's fixed point theorem:

> **Theorem** (Schauder)**.** *Let X be a Banach space. Let $F : X \to X$ satisfy the following assumptions:*
>
> *a) F is continuous;*
>
> *b) F is compact;*
>
> *c) there is a convex, bounded and closed set $B \subset X$ such that $F(B) \subset B$.*
>
> *Then F has a fixed point in B.*

To prove the theorem, argue as follows:

a) Show that, if $A \subset X$ is relatively compact (that is, its closure $\overline{A}$ is compact), then the convex hull of A is relatively compact.

   Hint: you can use the following property: in a complete metric space, a subset A is relatively compact if and only if for every $\varepsilon > 0$ there is a finite number of points $x_1, \dots, x_k \in A$ such that $A \subset \bigcup_{i=1}^{k} B(x_i, \varepsilon)$.

b) Use part a) to find a compact, convex set $K \subset X$ such that $F(K) \subset K$, and invoke Schauder's fixed point theorem.

Problem 4 (An application of Schauder's fixed point theorem, 4 points).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $f \in L^2(\Omega)$, and let $a : \mathbb{R} \to \mathbb{R}$ be continuous and such that $\alpha_1 \leq a(s) \leq \alpha_2$ for every $s \in \mathbb{R}$, where $0 < \alpha_1 < \alpha_2 < \infty$. Use the formulation of Schauder's fixed point theorem in Problem 3 to show that there exists a weak solution $u \in H_0^1(\Omega)$ to the boundary value problem

$$\begin{cases} -\mathrm{div}\big(a(u)\nabla u\big) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

that is,

$$\int_\Omega a(u)\nabla u \cdot \nabla\varphi \, \mathrm{d}x = \int_\Omega f\varphi \, \mathrm{d}x \qquad \text{for every } \varphi \in H_0^1(\Omega).$$

Total: 16 points, extra credit 2 points