Nonlinear Partial Differential Equations I — University of Bonn
Winter term 2017/2018 — Prof. Dr. J. J. L. Velázquez
Problem Sheet 9 (due Monday 11.12.2017) — Dr. M. Bonacini

Problem 1 (Constant mean curvature, 4 points).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded, with smooth boundary. Assume that $u : \Omega \to \mathbb{R}$ is a smooth minimizer of the area functional

$$I[w] = \int_\Omega \sqrt{1 + |Dw|^2}\,\mathrm{d}x$$

subject to given boundary conditions $u = g$ on $\partial\Omega$ and an integral constraint

$$J[w] = \int_\Omega w\,\mathrm{d}x = 1\,.$$

Prove that the graph of u is a surface of constant mean curvature.
(Recall that the mean curvature of the graph of u has the expression $\frac{1}{n}\mathrm{div}\big(\frac{Du}{(1+|Du|^2)^{1/2}}\big)$.)

Problem 2 (Minimization with a pointwise gradient constraint, 4 points).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary and let $f \in L^2(\Omega)$. Show that there is a unique minimizer u of

$$F(v) = \int_\Omega \Big(\frac{1}{2}|\nabla v|^2 - fv\Big)\,\mathrm{d}x$$

in

$$M = \{v \in H^1_0(\Omega) \,:\, |\nabla v| \le 1 \text{ a.e.}\}\,,$$

and find the variational inequality satisfied by u.

Problem 3 (An obstacle problem in dimension 1, 4 points).
Let $U = (-1,1)$ and $h(x) = 3 - 4x^2$. Consider the minimization problem

$$\min_{w\in\mathcal{A}} \ \frac{1}{2}\int_{-1}^{1} |w'|^2\,\mathrm{d}x$$

over the admissible class $\mathcal{A} := \{w \in H^1_0(-1,1) \,:\, w \ge h\}$. Let $u \in \mathcal{A}$ be the unique minimizer, according to the general theory discussed in class.

a) Show that the minimizer u satisfies the conditions

$$\begin{cases} -u'' = 0 & \text{in the set } \{x \,:\, u(x) > h(x)\}\,, \\ \displaystyle\int_{-1}^{1} u'w'\,\mathrm{d}x \ge 0 & \text{for every } w \in H^1_0(-1,1),\ w \ge 0. \end{cases} \tag{1}$$

b) Compute explicitly the minimizer.
Hint: use the second condition in (1) *to obtain a tangency condition at the contact points: the graph of u is tangent to the graph of h at the points where they touch.*

Please turn over.

Problem 4 (Variational inequality, 4 points).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded, with smooth boundary. For $\varepsilon > 0$ let $\beta_\varepsilon : \mathbb{R} \to \mathbb{R}$ be given by

$$\beta_\varepsilon(t) = \begin{cases} 0 & \text{if } t \geq 0, \\ \frac{t}{\varepsilon} & \text{if } t \leq 0. \end{cases}$$

Let $u_\varepsilon \in H_0^1(\Omega)$ be the solution to

$$\int_\Omega Du_\varepsilon \cdot Dv \,\mathrm{d}x + \int_\Omega \beta_\varepsilon(u_\varepsilon) v \,\mathrm{d}x = \int_\Omega fv \,\mathrm{d}x \qquad \text{for all } v \in H_0^1(\Omega),$$

where $f \in L^2(\Omega)$ is given, and let $M = \{v \in H_0^1(\Omega) : v \geq 0 \text{ a.e.}\}$. Show that for $\varepsilon \to 0$ the sequence u_ε converges weakly in $H_0^1(\Omega)$ to the unique solution $u \in M$ of the variational inequality

$$\int_\Omega Du \cdot D(v - u) \,\mathrm{d}x \geq \int_\Omega f(v - u) \,\mathrm{d}x \qquad \text{for all } v \in M.$$

Total: 16 points