Nonlinear Partial Differential Equations I — University of Bonn
Winter term 2017/2018 — Prof. Dr. J. J. L. Velázquez
Problem Sheet 7 (due Monday 27.11.2017) — Dr. M. Bonacini

Problem 1 (Minimal graphs of revolution, 2+2+2 points).
Let $u \in C^1([-1,1])$ such that $u > 0$, $u(-1) = a$, $u(1) = b$ for given $a, b > 0$. The area of the surface obtained by rotating the graph of u about the x-axis is

$$A(u) = 2\pi \int_{-1}^{1} u(x)\sqrt{1 + |u'(x)|^2}\,\mathrm{d}x\,.$$

a) Derive the Euler-Lagrange equation of A.

b) Show that, if u is a (sufficiently regular) minimizer of A in the class of functions with the same boundary values, then

$$u^2 = c^2(1 + |u'|^2) \tag{1}$$

for some constant $c \in \mathbb{R}$.

c) Solve the equation (1). Is the solution always a minimizer of the area among surfaces of revolution with the same boundary values? Is there always a minimizer among smooth graphs with prescribed boundary values? You can assume for simplicity $a = b$.
Hint: look for a solution in the form $u(x) = c\cosh v(x)$.

Problem 2 (Null Lagrangian, 3 points).
Show that

$$L(P) = \mathrm{trace}\,(P^2) - (\mathrm{trace}\,P)^2\,,$$

for $P \in \mathbb{M}^{n\times n}$, is a null Lagrangian.

Problem 3 (Euler-Lagrange equations, 3 points).
Find a Lagrangian $L = L(p, z, x)$ such that the PDE

$$-\Delta u + D\phi \cdot Du = f \qquad \text{in } \Omega$$

is the Euler-Lagrange equation corresponding to the functional $I(u) = \int_\Omega L(Du, u, x)\,\mathrm{d}x$. Here $\phi, f : \overline{\Omega} \to \mathbb{R}$ are given smooth functions.
Hint: look for a Lagrangian with an exponential term.

Please turn over.

Problem 4 (Euler-Lagrange equations and boundary conditions, 2+2 points).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary, let $q \geq 2$ and let $g \in L^q(\Omega)$.

a) Consider the functional

$$I(u) = \int_\Omega |Du|^q \, \mathrm{d}x + \int_\Omega |u - g|^q \, \mathrm{d}x \,.$$

Prove that the minimum problem

$$\min_{u \in W^{1,q}(\Omega)} I(u)$$

has a solution. If the minimizer is smooth, what is the Euler-Lagrange equation that it satisfies? Which boundary condition does u satisfy?

b) Let $h \in C(\partial\Omega)$ with $\int_{\partial\Omega} h \, \mathrm{d}s = 0$ and consider the functional

$$J(u) = \int_\Omega |Du|^2 \, \mathrm{d}x + \int_{\partial\Omega} u \, h \, \mathrm{d}s \,.$$

Write the Euler-Lagrange equation and the boundary condition that a smooth minimizer of J must satisfy.

Total: 16 points