Nonlinear Partial Differential Equations I University of Bonn
Winter term 2017/2018 Prof. Dr. J. J. L. Velázquez
Problem Sheet 1 (due Monday 16.10.2017) Dr. M. Bonacini

In this sheet we review basic properties of harmonic functions.

Problem 1 (Mean value property, 4 points).
Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in C^2(\Omega)$ be *harmonic* in Ω, that is

$$\Delta u := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0 \qquad \text{in } \Omega.$$

Prove that u has the mean value property: for every $x \in \Omega$ and for every radius $r < \mathrm{d}(x, \partial\Omega)$

$$u(x) = \frac{1}{\omega_n r^n} \int_{B_r(x)} u(y)\,\mathrm{d}y\,, \qquad u(x) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B_r(x)} u\,\mathrm{d}S\,,$$

where $\omega_n = |B_1|$ is the volume of the unit ball in $\mathbb{R}^n$. (Here $\mathrm{d}(x, \partial\Omega)$ denotes the distance of the point x from the boundary of Ω, and $B_r(x) = \{y : |x-y| < r\}$ is the open ball with center x and radius r).

Problem 2 (Regularity of harmonic functions, 4 points).
Let $u \in C^2(\Omega)$ be harmonic in an open set $\Omega \subset \mathbb{R}^n$. Prove that $u \in C^\infty(\Omega)$.
Hint: use the mean value property.

Problem 3 (Maximum principle, 4 points).
Prove the following statements.

a) (*Strong maximum principle*) Let $\Omega \subset \mathbb{R}^n$ be an open, connected set, and let $u \in C^2(\Omega)$ be harmonic in Ω. If u attains its maximum at a point $x_0 \in \Omega$ (that is, there exists $x_0 \in \Omega$ such that $u(x_0) = \sup_{x\in\Omega} u(x)$), then u is constant.

b) (*Weak maximum principle*) Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be harmonic in Ω. Then

$$\max_{x\in\Omega} u(x) = \max_{x\in\partial\Omega} u(x).$$

c) (*Uniqueness for the Dirichlet problem*) There exists at most one solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where $f \in C^0(\Omega)$ and $g \in C^0(\partial\Omega)$ are given functions.

Problem 4 (Convergence of sequences of harmonic functions, 4 points).
Let $(u_k)_{k\in\mathbb{N}}$ be a sequence of harmonic functions in an open set $\Omega \subset \mathbb{R}^n$. Suppose that $u_k \to u$ in $L^1_{\mathrm{loc}}(\Omega)$. Prove that u is harmonic in Ω and $u_k \to u$ uniformly on compact subsets of Ω.

Total: 16 points