Nonlinear Partial Differential Equations II University of Bonn
Summer term 2016 Prof. Dr. J.J.L. Velázquez
Problem Sheet 9 (due Monday 20.06.2016)

Problem 1 (Behavior of eigenvectors, 4 points).
For $z \in \mathbb{R}$, $z \neq 0$, define the matrix function

$$B(z) := e^{-1/z^2} \begin{pmatrix} \cos(2/z) & \sin(2/z) \\ \sin(2/z) & -\cos(2/z) \end{pmatrix},$$

and set $B(0) = 0$. Show that B is of class C^1 and has real eigenvalues, but we cannot find unit-length right eigenvectors $\{\mathbf{r_1}(z), \mathbf{r_2}(z)\}$ depending continuously on z near 0. What happens to the eigenspaces as $z \to 0$?

Problem 2 (Shallow water, 4 points).
Consider the shallow water equations

$$\begin{cases} \phi_t + (v\phi)_x = 0, \\ v_t + \left(\frac{v^2}{2} + \phi\right)_x = 0. \end{cases} \tag{1}$$

Recall that (1) is a system of conservation laws with flux $\mathbf{F}(z_1, z_2) = \left(z_1 z_2, \frac{z_2^2}{2} + z_1\right)$. Check that the system is strictly hyperbolic, provided $\phi > 0$. Compute the eigenvalues λ_k and the right eigenvectors $\mathbf{r}_k$, $k = 1, 2$. Is any of the pairs $(\lambda_k, \mathbf{r}_k)$ genuinely nonlinear or linearly degenerate?

Problem 3 (Chromotography, 8 points).
Consider the following system of conservation laws

$$\begin{cases} \partial_t u_1 + \partial_x \left(\frac{u_1}{1+u_1+u_2}\right) = 0, \\ \partial_t u_2 + \partial_x \left(\frac{u_2}{1+u_1+u_2}\right) = 0, \end{cases} \tag{2}$$

with $u_1, u_2 > 0$.

a) Show that the system is strictly hyperbolic and determine the eigenvalues $\lambda_k(u_1, u_2)$ ($\lambda_1 < \lambda_2$) and the corresponding right eigenvectors $r_k(u_1, u_2)$, $k = 1, 2$.

b) Determine whether the pairs (λ_k, r_k) are genuinely nonlinear or linearly degenerate.

c) Given $\mathbf{u}^0 = (u_1^0, u_2^0)$, $u_1^0, u_2^0 > 0$, find the rarefaction curves $R_k(\mathbf{u}^0)$ and the shock sets $S_k(\mathbf{u}^0)$, $k = 1, 2$, and draw them in the (u_1, u_2)-plane.

d) Given two initial states $\mathbf{u}^l$ and $\mathbf{u}^r$ with $\mathbf{u}^r \in R_2(\mathbf{u}^l)$, construct an integral solution to the Riemann problem corresponding to (2) with initial condition

$$u(x, 0) = \begin{cases} \mathbf{u}^l & \text{if } x < 0, \\ \mathbf{u}^r & \text{if } x > 0. \end{cases}$$

Total: 16 points