Nonlinear Partial Differential Equations I University of Bonn
Winter term 2015/2016 Prof. Dr. J. J. L. Velázquez
Problem Sheet 13 (due Thursday 04.02.16 if you want it marked)

Problem 1 (Uniqueness for Neumann problem)
Let $\Omega \subset \mathbb{R}^n$ be open, connected and bounded with smooth boundary. Show that the only smooth solutions to the Neumann problem

$$\begin{aligned} -\Delta u &= 0 \qquad \text{in } \Omega, \\ \nabla u \cdot \nu &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

are $u = \text{const}$.

Problem 2 (Galerkin approximation for elliptic problems)
Let V be a real Hilbert space, $f \in V$ and $B\colon V \times V \to \mathbb{R}$ a bilinear form that is bounded and coercive, that is

$$B(u,v) \le M\|u\|\|v\| \qquad \text{and} \qquad B(u,u) \ge \alpha\|u\|^2 \qquad \text{for all } u,v \in V$$

Moreover, let (V_n) a sequence of finite-dimensional subspaces such that

$$V_1 \subset V_2 \subset \ldots \subset V_n \subset V_{n+1} \subset \ldots \subset \overline{\cup_{n=1}^{\infty} V_n} = V.$$

1. Show that for any $n \in \mathbb{N}$ the variational equation

   $$B(u,v) = (f,v) \qquad \text{for all } v \in V_n$$

   has a unique solution $u_n \in V_n$. Furthermore, show that the sequence (u_n) converges weakly in V to the unique solution u of $B(u,v) = (f,v)$ for all $v \in V$ as $n \to \infty$.

2. Show that $B(u_n - u, v) = 0$ for all $v \in V_n$, that is, $u_n - u$ is B-orthogonal to V_n.

3. Deduce that

   $$\|u_n - u\| \le \frac{M}{\alpha}\,\text{dist}(u, V_n).$$

Problem 3 (A nonlinear parabolic equation)
Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary, $g \in H_0^1(\Omega)$ and $f\colon \mathbb{R} \to \mathbb{R}$ Lipschitz-continuous. Consider the problem

$$\begin{aligned} \partial_t u - \Delta u &= f(u) \qquad \text{in } \Omega \times (0,T), \\ u &= 0 \qquad \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) &= g \qquad \text{on } \Omega \end{aligned}$$

and show that a suitable weak solution exists for any $T > 0$.
Hint: Use a contraction principle and linear theory from the lecture or Evans Chapter 7.
