**Nonlinear Partial Differential Equations I** — University of Bonn
Winter term 2015/2016 — Prof. Dr. J. J. L. Velázquez
Problem Sheet 10 (due Thursday 14.01.16)

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**Problem 1 (4 points)**
Let $\Omega$ be open and bounded with smooth boundary, and for $\varepsilon > 0$ define $\beta_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ by

$$\beta_\varepsilon(z) = \begin{cases} 0 & \text{if } z \geq 0, \\ \dfrac{z}{\varepsilon} & \text{if } z < 0. \end{cases}$$

Given $f \in L^2(\Omega)$, let $u_\varepsilon \in H_0^1(\Omega)$ be the weak solution of

$$\begin{aligned} -\Delta u_\varepsilon + \beta_\varepsilon(u_\varepsilon) &= f \qquad \text{in } \Omega, \\ u_\varepsilon &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Show that $u_\varepsilon$ converges weakly in $H_0^1(\Omega)$ as $\varepsilon \to 0$ to a solution $u \in M$ of the variational inequality

$$\int_\Omega \nabla u \cdot \nabla(v - u) \, \mathrm{d}x \geq \int_\Omega f(v-u) \, \mathrm{d}x$$

for all $v \in M$, where $M = \left\{ v \in H_0^1(\Omega) : v \geq 0 \text{ a.e. in } \Omega \right\}$.

**Problem 2 (4 points)**
Let $\Omega \subset \mathbb{R}^n$ be open and assume that $\Phi \colon \Omega \to \mathbb{R}^n$ is a bi-Lipschitz $C^1$-diffeomorphism from $\Omega$ to $\Phi(\Omega)$, that is, there is a constant $L > 0$ such that

$$\frac{1}{L}|x-y| \leq |\Phi(x) - \Phi(y)| \leq L|x-y| \qquad \text{for all} \qquad x, y \in \Omega.$$

Let $0 < \lambda < n+2$. Show that $u \in \mathcal{L}^{2,\lambda}(\Phi(\Omega))$ if and only if $u \circ \Phi \in \mathcal{L}^{2,\lambda}(\Omega)$.

**Problem 3 (4 points)**
Let $\Omega = B_{1/2}(0) \subset \mathbb{R}^2$ and

$$u(x) = (x_1^2 - x_2^2)(-\ln|x|)^{1/2}.$$

Show that $\Delta u = f$ in $\Omega$ for some function $f \in C(\overline{\Omega})$, but that $u \notin C^{1,1}(\Omega)$.

**Problem 4 (EXTRA CREDIT: 4 points)**
If $\Delta u = f$ for some $f \in C^{0,1}(\overline{\Omega})$, do we have $u \in C^{2,1}(\Omega)$?

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Total: 12 points, extra credit 4 points