
Problem 1 (About Maxwell's equation, 4 points)
Show that for any vector field $m \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ the functional

$$F(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \,\mathrm{d}x + \int_{\mathbb{R}^3} m \cdot \nabla u \,\mathrm{d}x$$

has a unique minimizer in $X = \{u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3;\mathbb{R}^3)\}$. Find the associated Euler-Lagrange equation.

Problem 2 (Deformation of a one-dimensional beam, 4 points)
In a simple one-dimensional model, the deformation of a beam of length $l > 0$ under a load f is described by its vertical deflection $v\colon (0,l) \to \mathbb{R}$. Given a bending rigidity $a \in L^\infty(0,l)$ such that $a(x) \geq a_0 > 0$ for all $x \in (0,l)$ and $f \in L^2(0,l)$, the deflection minimizes

$$E(v) = \int_0^l a(x)|v''(x)|^2 - f(x)v(x)\,\mathrm{d}x$$

among all $v \in H^2(0,l)$ subject to certain boundary conditions.
Assume that the beam is fixed horizontally at the left end $x = 0$ and show that E has a unique minimizer u among deformations in $H^2(0,l)$ with the same boundary conditions. Given sufficient regularity of a, f and u, what boundary value problem does u satisfy?

Problem 3 (Minimization s. t. a pointwise gradient constraint, 4 points)
Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary and $f \in L^2(\Omega)$. Show that there is a unique minimizer u of

$$F(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 - fv\,\mathrm{d}x$$

in

$$M = \{v \in H_0^1(\Omega) : |\nabla v| \leq 1 \text{ a. e.}\},$$

and find the variational inequality satisfied by u.

Problem 4 (Now for something completely different, EXTRA CREDIT 8 points)
Let X be a Banach space, $B \subset X$ non-empty, bounded, closed and convex. Let $T\colon B \to B$ be non-expansive, i. e.

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in B$. Show that:

1. For any $\varepsilon > 0$ there is an ε fixed point $x_\varepsilon \in B$, i. e.
$$\|Tx_\varepsilon - x_\varepsilon\| \leq \varepsilon.$$
2. If $(\mathrm{id} - T)(B) \subset X$ is closed, then T has a fixed point.
3. Let $\Omega \subset \mathbb{R}^n$ be open and smoothly bounded such that $|\Omega| \leq 1$. Show that
$$\begin{aligned} -\Delta u + \frac{1}{1 + |\nabla u|} + u &= 0 \qquad \text{in } \Omega,\\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$
has a weak solution $u \in H_0^1(\Omega)$.

Merry Christmas and a Happy New Year! Total: 12 points, extra credit 8 points