Nonlinear Partial Differential Equations I University of Bonn
Winter term 2015/2016 Prof. Dr. J. J. L. Velázquez
Problem Sheet 06 (due Thursday 03.12.15)

Problem 1 (Calderón-Zygmund, 4 points)
Let Q_0 be a cube in $\mathbb{R}^n$ and $u \in L^1(Q_0)$, $u > 0$ such that

$$u_{Q_0} := \frac{1}{|Q_0|}\int_{Q_0} u \,\mathrm{d}x \leq L$$

for some $L > 0$.

1. Show that there exists a (possibly finite) sequence of pairwise disjoint cubes $Q_j \subset Q_0$ with faces parallel to those of Q_0 such that

   $$L < u_{Q_j} \leq 2^n L$$

   for all $j \in \mathbb{N}$ as well as

   $$u \leq L \qquad \text{a. e. in } Q_0 \setminus \cup Q_k.$$

   Hint: Successively decompose Q_0.

2. Conclude that u can be composed as $u = g + b$ where the *good* part g is bounded,

   $$|g(x)| \leq CL \qquad \text{for a. a. } x \in Q_0,$$

   and the *bad* part b satisfies

   $$\operatorname{spt} b \subset \cup Q_k, \qquad b_{Q_j} = 0 \quad \text{and} \quad \int_{Q_j} |b(x)| \,\mathrm{d}x \leq CL|Q_j| \quad \forall j \in \mathbb{N}.$$

   Here $C > 0$ is a constant depending only on the dimension n.

Problem 2 (BMO, 3 points)
Let Q_0 be a cube in $\mathbb{R}^n$ and suppose that $u \in L^1(Q_0; \mathbb{R}^N)$ has bounded mean oscillation

$$[u]_* := \sup_Q \frac{1}{|Q|}\int_Q |u - u_Q| \,\mathrm{d}x$$

where the supremum is taken over all cubes $Q \subset Q_0$ with sides parallel to those of Q_0. Show that $u \in L^p(Q_0; \mathbb{R}^N)$ for every $p \geq 1$ and that

$$\frac{1}{|Q|}\int_Q |u - u_Q|^p \,\mathrm{d}x \leq C[u]_*^p$$

for every cube $Q \subset Q_0$ whose sides are parallel to Q_0.
Hint: Recall Problem 3.1 on Sheet 5.

Please turn over.

Problem 3 (Maximum Principle, 9 points)
Let $n \geq 2$ and consider the differential operator

$$Lu = \sum_{i,j=1}^{n} a^{ij} D_{ij} u, \qquad a^{ij} = \delta^{ij} + g(r)\frac{x_i x_j}{r^2}, \quad i,j = 1, \ldots, n,$$

where $\delta^{ij} = 1$ if $i = j$, $\delta^{ij} = 0$ otherwise, $r = |x|$ and $g : [0, \infty) \to \mathbb{R}$ is a continuous function.

1. Show that a radially symmetric function $u = u(r)$ is a solution to $Lu = 0$ in some punctured ball $B_{r_0}(0) \setminus \{0\}$ if and only if it satisfies the ODE

$$\frac{u''}{u'} = \frac{1-n}{r(1+g)}$$

   for $0 < r < r_0$, provided that $g \neq -1$ in $B_{r_0}(0)$.

2. If $n = 2$ and $g(r) = -2/(2 + \ln r)$, show that L is uniformly elliptic in the disk $D = \{0 \leq r \leq r_0 = e^{-3}\}$ and has continuous coefficients in D. Moreover, show that $Lu = 0$ has bounded solutions $a + b/\ln r$ in the punctured disk $D \setminus \{0\}$ that do not satisfy

$$\limsup_{x \to 0} u(x) \leq \sup_{|x| = r_0} u(x). \qquad (1)$$

3. If $n > 2$ and $g(r) = -[1 + (n-1)\ln r]^{-1}$ show that L is uniformly elliptic and has continuous coefficients in $B = \{0 \leq r \leq r_0 = e^{-1}\}$. Moroever, show that there are solutions $u = u(r)$ to $Lu = 0$ in $B \setminus \{0\}$ that satisfy $u = o(r^{2-n})$ as $r \to 0$ but not (1).

Problem 4 (Maximum Principle continued, EXTRA CREDIT 4 points)
In the setting of Problem 3 for $n > 2$, determine a function $g(r)$ such that L is uniformly elliptic in some ball B and such that $Lu = 0$ has a bounded solution $u = u(r)$ in $B \setminus \{0\}$, which is continuous at $r = 0$ and does not satisfy (1).

Total: 16 points, extra credit 4 points

Extra cedit counts towards your personal score but not towards the total marks.