Nonlinear Partial Differential Equations I — University of Bonn
Winter term 2015/2016 — Prof. Dr. J. J. L. Velázquez
Problem Sheet 04 (due Thursday 19.11.15)

Problem 1 (Brouwer's FPT, 3 points)
Recall that Brouwer's fixed point theorem states that a continuous map $T : \overline{B_1(0)} \subset \mathbb{R}^n \to \overline{B_1(0)}$ has a fixed point.

1. Use Brouwer's FPT to show that there exists no continuous map $\omega \colon \overline{B_1(0)} \to \partial B_1(0)$ such that $\omega(x) = x$ for all $x \in \partial B_1(0)$.
2. Use the following result from the lecture to prove Brouwer's FPT: If $g \colon \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies $g(x) \cdot x \geq 0$ for all $x \in \partial B_R(0)$ for some $R > 0$, then there exists $x_0 \in \overline{B_R(0)}$ such that $g(x_0) = 0$.

Problem 2 (Weak convergence and nonlinear functions, 6 points)

1. Let $f \colon \mathbb{R} \to \mathbb{R}$ be continuous and assume that
$$f(u_n) \overset{*}{\rightharpoonup} f(u) \quad \text{in } L^\infty(0,1) \qquad \text{whenever} \qquad u_n \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,1).$$
Show that f is affine.
2. Find a continuous $f \colon \mathbb{R} \to \mathbb{R}$ such that for any $p \in \mathbb{R}$ there is a sequence $(u_n) \in L^\infty(0,1)$ such that $u_n \overset{*}{\rightharpoonup} 0$ and $f(u_n) \overset{*}{\rightharpoonup} p$.

Problem 3 (Variational inequalities with a monotone operator, 7 points)
Let V be a reflexive, separable Banach space and let $M \subset V$ be a nonempty, closed and convex set. Assume that $A : M \to V^*$ satisfies

- monotonicity:
$$\langle Au - Av, u - v \rangle \geq 0 \qquad \text{for all } u, v \in M,$$
- hemi-continuity:
$$t \mapsto \langle A(tu + (1-t)v), w \rangle \qquad \text{is continuous on } [0,1] \text{ for } u, v \in M, w \in V$$
- coerciveness: there is $u_0 \in M$ such that
$$\frac{\langle Au, u - u_0 \rangle}{\|u - u_0\|} \to \infty \qquad \text{as} \qquad \|u\| \to \infty, u \in M,$$
- boundedness:
A maps bounded sets into bounded sets.

We want to find a solution $u \in M$ of the variational inequality
$$\langle Au, u - v \rangle \leq 0 \qquad \text{for all } v \in M. \tag{1}$$

Please turn over.

1. (Minty's trick) Show that for $u \in M$ and $u^* \in V^*$ we have

$$\langle Au - u^*, u - v\rangle \leq 0 \qquad \text{for all } v \in M$$

if and only if

$$\langle Av - u^*, u - v\rangle \leq 0 \qquad \text{for all } v \in M.$$

2. (Continuity) Let $(u_n) \subset M$, $u \in M$ and $u^* \in V^*$ such that $u_n \rightharpoonup u$ in V and $Au_n \overset{*}{\rightharpoonup} u^*$ in V^*. Show that

   (a) $\langle u^*, u\rangle \leq \liminf_{n\to\infty}\langle Au_n, u_n\rangle$,

   (b) if $\langle u^*, u\rangle = \liminf_{n\to\infty}\langle Au_n, u_n\rangle$ then $\langle Au - u^*, u - v\rangle \leq 0$ for all $v \in M$.

3. (Galerkin approximation) Assume that for any finite-dimensional subspace X of V the variational inequality

$$\langle Au, u - v\rangle \leq 0 \qquad \text{for all } v \in X \cap M$$

has a solution $u \in X \cap M$ provided that the latter is non-empty. Show that a solution to (1) exists.

Total: 16 points