

PDE and Modelling
Exercise sheet 9

Problem 1 (1.5 + 2.5 + 1 = 5 points)

Consider the Lagrangian $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^{n \times n}$ given by

$$f(x, z, P) = \det P.$$

- (a) Derive the Euler-Lagrange equation corresponding to f .
- (b) Show that f is a null Lagrangian, that is, every $u \in C^2(\Omega)$ solves the Euler-Lagrange equation corresponding to f .
- (c) Show that, for $u \in C^2$, the volume of $u(\Omega)$ only depends on the boundary values of u at $\partial\Omega$.

Problem 2 (4 points)

Suppose that $\Omega \subset \mathbb{R}^n$ is open and bounded $T > 0$, and let $\varepsilon > 0$. Find $f : \Omega \times (0, T) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that the Euler-Lagrange equation for the functional

$$I[u] = \int_0^T \int_{\Omega} f(x, t, u, u_t, Du) \, dx \, dt$$

is

$$u_t - \Delta u - \varepsilon u_{tt} = 0 \quad \text{in } \Omega \times (0, T).$$

Hint: Try a term containing $e^{-t/\varepsilon}$.

Problem 3 (1 + 2 + 1 + 1.5 + 1.5 = 7 points)

According to Fermat's principle, a ray of light traveling between two points will follow the path that can be traversed in the shortest time.

Let $\gamma : [-1, 1]^2 \rightarrow (0, \infty)$ be a function denoting the speed of light at a point $(x, y) \in [-1, 1]^2$, and consider a piecewise C^1 curve $x \mapsto (x, u(x))$.

- (a) Show that the time needed to traverse the path is given by

$$I[u] = \int_{-1}^1 \frac{\sqrt{1 + u'(x)^2}}{\gamma(x, u(x))} \, dx$$

- (b) Suppose that γ is smooth. Compute the Euler-Lagrange equation corresponding to I .

- (c) Suppose that $\gamma(x, y) = \gamma_0$ for $x < 0$ and $\gamma(x, y) = \gamma_1$ for $x \geq 0$. Show that a continuous minimizer u of I satisfies $u''(x) = 0$ for $x \neq 0$.
- (d) Under the same assumptions, derive an equation for the left and right side derivatives of u at $x = 0$ in terms of γ_0 and γ_1 .
- (e) Derive Snell's law:

$$\frac{\sin \alpha_1}{\sin \alpha_0} = \frac{\gamma_1}{\gamma_0}$$

where α_0 denotes the angle of incidence and α_1 the angle of refraction of u , that is, α_0 and α_1 are the angles that u makes with the line $y = u(0)$ on the left and right, respectively.

Due: Friday, July 3 at the end of the lecture

<http://www.iam.uni-bonn.de/afa/teaching/15s/pdgm0d/>