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PDE and Modelling

Exercise sheet 8

Denote by $\Phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ the heat kernel:

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$

For $t > 0$, define $T(t)u = \Phi(t) * u$, and let $T(0)u = u$. It was shown in Introduction to PDE that $T(t)$ has the following properties (these can be used without proof):

(i) Smoothing: $(x,t) \mapsto T(t)u(x) \in C^\infty(\mathbb{R}^n \times (0,\infty))$.

(ii) Solution of the heat equation: $(\partial_t - \Delta)T(t)u = 0$ for $t > 0$.

(iii) Continuity: $t \mapsto T(t)u$ is continuous in $L^p(\mathbb{R}^n)$ whenever $u \in L^p(\mathbb{R}^n)$ and continuous in C^0 if $u \in C_c^0(\mathbb{R}^n)$.

(iv) Boundedness: $\|T(t)u\|_X \leq \|u\|_X$ if $X = L^p$ or $X = C_b^0$.

(v) Semigroup: $T(t+s) = T(t)T(s)$.

Problem 1 ($1+1+1+2+2=7$ points)

(a) Argue that property (iii) also holds in H^m ($m \in \mathbb{N}$), that is, $t \mapsto T(t)u$ is continuous in H^m whenever $u \in H^m$.

(b) Prove that property (iv) also holds if $X = H^m$, that is,

$$\|T(t)u\|_{H^m} \leq \|u\|_{H^m}$$

for any $u \in H^m$, $m \in \mathbb{N}$.

(c) Show that for $m \in \mathbb{N}$, $u \in H^{m+2}$ and $t > 0$,

$$\|T(t)u - u\|_{H^m} \leq t\|\Delta u\|_{H^m}$$

Hint: Use that $\frac{\partial}{\partial \tau}(T(\tau)u) = \Delta T(\tau)u$ for $\tau > 0$, and estimate $\|T(t)u - T(\varepsilon)u\|$ for $0 < \varepsilon < t$ first.

(d) Under the same assumptions, show that

$$\lim_{h \downarrow 0} \frac{\|T(h)u - u - h\Delta u\|_{H^m}}{h} = 0,$$

that is, as a map taking values in H^m, $t \mapsto T(t)u$ has right sided derivative Δu at $t = 0$. *Hint:* Estimate first $\|T(h)u - T(\varepsilon)u - (h-\varepsilon)\Delta u\|$, and use property (iv).

(e) Prove that $t \mapsto T(t)u$, taking values in H^m is continuously differentiable.

Problem 2 ($1+2+2=5$ points)

Let $m \geq 1$, and $t^* > 0$. Given $f \in C^0([0,t^*];H^m)$ ($m \geq 1$), set

$$v(t) = \int_0^t T(t-s)f(s)\,\mathrm{d}s.$$

(a) Show that

$$\sup_{t\in[0,t^*]} \|v(t)\|_{H^{m+1}}.$$

is finite *Hint:* Use that $\|T(\tau)f\|_{H^{m+1}} \leq \frac{C}{\sqrt{\tau}}\|f\|_{H^m}$.

(b) Show that $v \in C^0([0,t^*];H^{m+1})$, that is, $t \mapsto v(t)$ is continuous as a map from $[0,t^*]$ to H^{m+1}.

Hint: You will need dominated convergence.

(c) Show that $v \in C^1([0,t^*];H^{m-1})$, that is, there exists a map $v' \in C^0([0,t^*];H^{m-1})$ such that

$$\lim_{h\to 0} \frac{\|v(t+h)-v(t)-hv'(t)\|}{|h|} = 0,$$

and $\frac{\partial v}{\partial t} = \Delta v + f$. *Hint:* Estimate first $\|T(h)u - T(\varepsilon)u - (h-\varepsilon)\Delta u\|$, and use property (iv).

Due: Friday, June 26 at the end of the lecture