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PDE and Modelling

Exercise sheet 7

Problem 1 ($1 + 2 + 2 = 5$ **points)**

Let X be a Banach space with norm $\|.\|_X$, and let $I \subset \mathbb{R}$ be an interval. Denote by $C^0(I; X)$ the space of all continuous functions $f : I \to X$ such that

$$\|f\|_{C^0(I;X)} = \sup_{t \in I} \|f(t)\|_X$$

is finite.

(a) Show that $C^0(I; X)$ is a Banach space.

(b) Show that $C^0(I; C_b^0(\mathbb{R}^n))$ is isometrically isomorphic to a subspace of $C_b^0(\mathbb{R}^n \times I)$.

(c) Let $m \in \mathbb{N}$, $1 \leq p \leq \infty$, and $f \in C^0(I; W^{m,p}(\mathbb{R}^n))$. Prove that f can be identified with a measurable map $f : \mathbb{R}^n \times I \to \mathbb{R}$.

Hint: Assume first that I is compact, and show that f is uniformly approximated by a map $f_k : I \to C_b^0(\mathbb{R}^n)$ that is piecewise constant and hence measurable as a function on $\mathbb{R}^n \times I$.

Problem 2 ($1 + 1 + 1 + 1 + 2 + 1 = 7$ **points)**

Denote by $\Phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ the heat kernel:

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$

For $t > 0$, define $T(t)u = \Phi(t) * u$, and let $T(0)u = u$. It was shown in Introduction to PDE that $T(t)$ has the following properties (these can be used without proof):

(i) Smoothing: $(x,t) \mapsto T(t)u(x) \in C^\infty(\mathbb{R}^n \times (0, \infty))$.

(ii) Solution of the heat equation: $(\partial_t - \Delta)T(t)u = 0$ for $t > 0$.

(iii) Continuity: $t \mapsto T(t)u$ is continuous in $L^p(\mathbb{R}^n)$ whenever $u \in L^p(\mathbb{R}^n)$ and continuous in C^0 if $u \in C_c^0(\mathbb{R}^n)$.

(iv) Boundedness: $\|T(t)u\|_X \leq \|u\|_X$ if $X = L^p$ or $X = C_b^0$.

(v) Semigroup: $T(t + s) = T(t)T(s)$.

(a) Let $t > 0$. Show that $\|D^\alpha T(t)u\|_{C^0} \leq C_k t^{-\frac{k}{2}} \|u\|_{C^0}$ for some constant C_k, if $u \in C_b^0(\mathbb{R}^n)$ and $|\alpha| = k$.

Hint: It can be used without proof that $\|D^\alpha \Phi(t)\|_{L^1} \leq C_k t^{-\frac{k}{2}}$.

Fix $t^* > 0$, write $I := [0, t^*]$, and suppose that $f \in C^1(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$ with $f(0) = 0$. Given $u \in C^0(I; C^0_b(\mathbb{R}^n))$, define

$$g(s) := f(u(s)), \qquad v(x,t) = \int_0^t [T(t-s)g(s)](x)\,\mathrm{d}s$$

We will now prove that v is a continuous function on $\mathbb{R}^n \times I$.

(b) Fix $t \geq s > 0$. Show that $g(s)$ and $T(t-s)g(s)$ are continuous functions on $\mathbb{R}^n$, and estimate their $C^0(\mathbb{R}^n)$ norm in terms of $|f(0)|$, $\mathrm{Lip}(f)$ and $\|u\|_{C^0(I;C^0_b(\mathbb{R}^n))}$.

(c) Show that $T(t-s)g(s)$ is Lipschitz if $0 \leq s < t$, and use this to prove that $x \mapsto v(x,t)$ is Lipschitz with Lipschitz constant independent from $t \leq t^*$.

(d) Show that $g \in C^0(I; C^0_b(\mathbb{R}^n))$ and $s \mapsto T(t-s)g(s) \in C^0([0,t]; C^0_b(\mathbb{R}^n))$ for fixed $t \in I$.

(e) Show that $v \in C^0(I; C^0_b(\mathbb{R}^n))$.

From these results and property (i), we know that G, defined by

$$(Gu)(t) = T(t)u_0 + \int_0^t T(t-s)f(u(s))\,\mathrm{d}s$$

maps $C^0(I; C^0_b(\mathbb{R}^n))$ to itself.

(f) Suppose that $u \in C^0([0,t^*]; C^0_b(\mathbb{R}^n))$. Show that Gu is Lipschitz, and

$$\mathrm{Lip}((Gu)(t)) \leq \frac{C}{\sqrt{t}}$$

for some constant C, $t \in [0, t^*]$.

Problem 3 (Bonus: $2 + 2 + 1 + 1 = 6$ points)

This exercise deals with basic properties of $L^q(I; X)$ and $L^q(I; W^{m,p}(\mathbb{R}^n))$. Doing this exercise is optional, but the results may be used in other exercises.

Let X be a Banach space with norm $\|.\|_X$, and let $I \subset \mathbb{R}$ be an interval, and $q \geq 1$. For $f \in C^0(I; X)$, denote

$$\|f\|_{L^q(I;X)} = \left(\int_I \|f(t)\|_X^q \,\mathrm{d}t\right)^{\frac{1}{p}}$$

Define $L^q(I; X)$ to be the space of all maps $f : I \to X$ such that there exists a sequence $\{f_k\}_{k\in\mathbb{N}}$ in $C^0(I; X)$ with $\|f_k - f\|_{L^q(I;X)} \to 0$ as $k \to \infty$. Furthermore, $m \in \mathbb{N}$, $1 \leq p \leq \infty$, and consider $L^q(I; W^{m,p}(\mathbb{R}^n))$.

(a) Show that $L^q(I; X)$ is a Banach space.

Hint: Use Egorov's theorem.

(b) Prove that every $f \in L^q(I; W^{m,p}(\mathbb{R}^n))$ can be identified with a measurable map $f : \mathbb{R}^n \times I \to \mathbb{R}$.

(c) Prove that $C^\infty_c(\mathbb{R}^n \times I)$ is dense in $C^0(I, W^{m,p})$ if I is compact.

(d) Prove that $C^\infty_c(\mathbb{R}^n \times I)$ is dense in $L^q(I, W^{m,p})$ if I is compact.

Problem 4 ($1.5 + 1.5 + 2 + 1 = 6$ **points)**

Let I be an interval, and let $m \in \mathbb{N}$, $1 \le p \le \infty$, $1 \le q < \infty$, and suppose that $f : \mathbb{R}^n \times I$ is measurable.

(a) Show that f can be identified with an element of $L^q(I; L^p(\mathbb{R}^n))$ if $\|f\|_{L^q(I;L^p(\mathbb{R}^n))}$ is finite.

(b) Suppose that all distributional derivatives $D^\alpha f$ for $\alpha \le m$ exist in $L^q(I; L^p(\mathbb{R}^n))$. Show that $f \in L^q(I; W^{m,p}(\mathbb{R}^n))$.

(c) Suppose that I is compact and $f \in L^1(I; L^p(\mathbb{R}^n))$, and define

$$g(x) = \int_I f(x,t)\,\mathrm{d}t.$$

Show that $g \in L^p(\mathbb{R}^n)$ with

$$\|g\|_{L^p} \le \int_I \|f(t)\|_{L^p}\,\mathrm{d}t$$

(d) Suppose that I is compact and $f \in L^1(I; W^{m,p}(\mathbb{R}^n))$, and define

$$g(x) = \int_I f(x,t)\,\mathrm{d}t.$$

Show that $g \in W^{m,p}(\mathbb{R}^n)$ with

$$D^\alpha g = D^\alpha \int_I f(x,t)\,\mathrm{d}t = \int_I D^\alpha f(x,t)\,\mathrm{d}t, \qquad \|g\|_{W^{m,p}(\mathbb{R}^n)} \le \int_I \|f(t)\|_{W^{m,p}}\,\mathrm{d}t.$$

for $|\alpha| \le m$.

Due: Friday, June 19 at the end of the lecture